Scale Consideration of Planetary Motions of the Atmosphere[1]

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Abstract

The application of scale considerations to atmospheric motions indicates that the equations hitherto used in numerical prediction are unfit for detailed prognosis on the planetary (or very large) scale.

It appears that the dominant terms of the vorticity equation in this case are the divergence term and the β-term, which means that the vorticity equation here displays the same quasi-stationary character as the undifferentiated equations of motion. An attempt at expressing more than the quasistationary character of the planetary flow leads to undue complication of the equation, even requiring inclusion of the twisting terms.

As for the application of the geostrophic approximation, it is confirmed that the β-term in the divergence equation is important in the representation of the planetary scale. Further, though metrical terms (containing the reciprocal of the earth's radius) are important on this scale in spherical coordinates, it is shown that, in using the geostrophic approximation and a square grid on a conformal plane projection of the earth's surface, no metrical terms appear.

Remarks are also included on a relation of Prandtl, which loses its validity on the planetary scale, and also on the use of the Richardson number to characterize large-scale atmospheric motion.

1. Introduction

In his classical article of 1948, Charney used scale considerations systematically for determining the relative importance of the different terms in the set of hydrodynamical equations which are supposed to govern the behaviour of the large scale motions of the atmosphere over short periods. As is known, this analysis led to a rational way of using the geostrophic approximation in the equations, and served as a basis for much of the further development of dynamical weather prediction. At a later stage Stümke (1953) applied a similar analysis with a view to improving on the geostrophic approximation in medium scale motions. In both these studies the notion of "large scale" is apparently distinct from what may be called the "very large scale", or, more concisely, the planetary scale. However, this planetary scale, in which the characteristic distance is comparable in magnitude to the earth's radius, is known to be very significantly represented in the spectrum of atmospheric motions (cf. White and Cooley, 1956). It is therefore of some importance to investigate to what extent the equations used in dynamical prediction are applicable to this scale of motion. The present paper aims at making a contribution on this subject.

One instance in which the longest waves have received special attention, was in the

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article of CHARNEY and ELIASSEN (1949). In their one-dimensional barotropic model the equation was adjusted to suit the planetary (stationary) waves, by introducing an empirically determined current width; however, this was simply a subterfuge to compensate for the absence of a second space dimension in the model.

Later BOLIN (1955) pointed out that the barotropic model, as used in Stockholm, performs less well on the planetary scale than on the moderately large scale. To remedy this, he proposes the introduction of a divergence term with an empirical coefficient into the barotropic model. The improvement brought about by this means has not yet been thoroughly tested; however, the present analysis will indicate that the planetary scale does require a divergence term to be present in the model and also that it should be supported by a β-term in the divergence equation.

In general we hope to illustrate that the equations currently used in both barotropic and baroclinic models are unsuited to supply reliable specific information on the planetary scale.

The arguments leading to this last statement are based on the fundamental relation (22) between Charney's scale quantities; this and other relations are established in sections 2 and 3 under very unrestrictive assumptions.

In section 4 it is shown how the equations may be simplified with the aid of the above results in the case of the medium-long waves treated by Charney. Here the difficulties attached to the quasi-stationary character of the motion may be overcome by using the vorticity equation.

Section 5 contains the extension of these considerations to the planetary scale, and it appears that in this case even the vorticity equation displays a quasi-stationary character. It turns out that a divergence term, (which may here be geostrophically approximated), is needed in the vorticity equation, while further for good representation the omission of β-terms in the geostrophic formulae for vorticity and divergence is not permitted. Also metrical terms become important in spherical coordinates, but not with the geostrophic approximation and cartesian coordinates on a conformal map. These and other conclusions are listed in the final section.

Included in the material mentioned, are discussions of the direct representation of ordinary and geostrophic divergence (section 4) and of the interrelation of results obtained in the "z-system" and "p-system" of coordinates (section 5). Other matters mentioned in passing are the ceased validity of Prandtl's ratio between horisontal and vertical scales when the planetary scale is considered, and a comparison of the use of the Rossby and Richardson numbers for expressing the quasi-geostrophic character of large-scale motion.

2. Basic assumptions

It is assumed that the short-term behaviour of the atmosphere can be described by the hydrodynamical equations for adiabatic and laminar motion of an ideal fluid, and, following Charney, we shall make use of characteristic scale quantities to represent the relative magnitudes of different terms.

We denote by S and H the respective half-wave-lengths, by V and W the characteristic absolute horizontal and vertical particle speeds and by C the horizontal phase speed of streamline configurations. The use of these quantities to characterize a complicated field of flow, mplies that harmonic components, rather than the full observed motion, are treated; further the use of results based on such simplified phenomena treated in isolation, implies the assumption that the non-linear interaction between motions on different scales is considered negligible.

Although actual computations are usually performed using a square grid on a conformal map projection, we shall use spherical coordinates λ, ϕ and r, so that the Eulerian equations of motion are:

$$\left.\begin{aligned} &\frac{du}{dt} - \frac{uv}{r}\tan\phi + \frac{uw}{r} + jw - fv = -\frac{p_x}{\varrho} \\ &\frac{dv}{dt} + \frac{u^2}{r}\tan\phi + \frac{vw}{r} + fu = -\frac{p_y}{\varrho} \\ &\frac{dw}{dt} - \frac{1}{r}(u^2+v^2) - ju = -\frac{p_z}{\varrho} - g \end{aligned}\right\} \quad (1)$$

in which u, v and w are the eastward, northward and upward velocity components respectively, while x, y and z denote differentiation in the corresponding directions, e.g.

$$p_x = \frac{1}{r\cos\phi}\frac{\partial p}{\partial\lambda},\ p_y = \frac{1}{r}\frac{\partial p}{\partial\phi},\ p_z = \frac{\partial p}{\partial r}.$$

Further, f is the coriolis parameter $2\Omega\sin\phi$ and j is $=2\Omega\cos\phi$, where Ω is the earth's angular speed.

As in Charney's paper, we represent the order of magnitude of velocity derivatives by the scale quantities, as follows:

$$u_x \sim v_x \sim u_y \sim v_y \sim \frac{V}{S}$$

$$u_t \sim v_t \sim C\frac{V}{S}$$

$$u_z \sim v_z \sim \frac{V}{H}$$

$$w_x \sim w_y \sim \frac{W}{S}$$

$$w_t \sim C\frac{W}{S}$$

$$w_z \sim \frac{W}{H}$$

and, since the order of H is again taken to be about the depth of the troposphere, the change of pressure and density with height is of the same order as the variables themselves, so that

$$\frac{p_z}{p} \sim \frac{\varrho_z}{\varrho} \sim \frac{1}{H}. \tag{2}$$

Again the horizontal variation of p and ϱ will need more precise treatment.

We now impose a number of restricions on the possible values of the scale quantities, in conformity with observational evidence. For this purpose attention is focussed on medium and large scale synoptic motions in middle latitudes, and we may then evidently write:

$$\left.\begin{array}{l} H < S \lesssim a \\ W \lesssim V \\ C \lesssim V \end{array}\right\} \tag{3}$$

in which a is the earth's radius. We further note that, in the range of interest (where $S \gtrsim$ $\gtrsim 1{,}000$ km, $V \lesssim 100$ m/sec, and $f \sim 10^{-4}$ sec^{-1}), the Rossby parameter $\frac{V}{fS}$ satisfies

$$\frac{V}{fS} \lesssim 1. \tag{4}$$

If, in addition, the order of H is fixed at 10 km, we may also write

$$\left.\begin{array}{l} \dfrac{V^2}{gH} << 1 \\ \dfrac{fV}{g} \lesssim \dfrac{H}{a} << 1 \end{array}\right\} \tag{5}$$

in which the symbol << indicates that the left-hand side is at most of the order of about one-tenth of the right-hand side.

As far as the variables of state are concerned, restrictions are placed on the vertical variation of temperature and on the horizontal variation of pressure and density. Thus, firstly we require the lapse-rate to be sub-adiabatic, while even a substantial temperature inversion is allowed:

$$-2 \lesssim \frac{\Gamma}{\Gamma_a} < 1 \tag{6}$$

where Γ_a is the adiabatic lapse-rate $\frac{\varkappa g}{R}$; secondly we assume that the pressure force has the same order as the coriolis force:

$$\frac{p_x}{\varrho} \sim \frac{p_y}{\varrho} \sim fV \tag{7}$$

It may be noted that this last assumption, which is well substantiated by observation, is much weaker than the geostrophic approximation, since equality in order of magnitude only is required. It is also in keeping with (4), since the Rossby parameter measures the ratio of horizontal advection terms to coriolis terms in the horizontal equations of motion.

3. Derivation of the basic relations

We may now apply the above relations to the different hydrodynamical equations in order to obtain further relations. Firstly, the third equation of motion reduces at once to the hydrostatic equation, since, by (3) and (5) its entire left-hand side is of lower order than the gravity acceleration g.

Thus

$$\frac{p_z}{\varrho} = -g. \tag{8}$$

Together with (2), this yields

$$RT = \frac{p}{\varrho} \sim \left(\frac{p}{p_z} \cdot \frac{p_z}{\varrho}\right) \sim gH \tag{9}$$

which is consistent with the observation that the absolute temperature is always about 200 or 300 K in the troposphere. Combination of (8) with (2) and the (weak) geostrophic relation (7) yields

$$\frac{p_x}{p} = \frac{p_x}{\varrho} \cdot \frac{\varrho}{p_z} \frac{p_z}{p} \sim \frac{fV}{gH}$$

and similarly for the y-direction. With a little more care we may write, from (9) and (7),

$$\frac{\varrho_x}{\varrho} \sim \frac{1}{\varrho}\left(\frac{p}{gH}\right)_x \sim \frac{1}{gH}\frac{p_x}{\varrho} + \frac{H_x}{H} \sim \frac{1}{gH} \cdot \frac{p_x}{\varrho} \sim \frac{fV}{gH}$$

so that, in summary,

$$(\ln p)_x \sim (\ln p)_y \sim (\ln \varrho)_x \sim (\ln \varrho)_y \sim \frac{fV}{gH}. \tag{10}$$

Our next step is to substitute this into the adiabatic equation, viz. into

$$\frac{1}{\gamma} \cdot \frac{d_h \ln p}{dt} - \frac{d_h \ln \varrho}{dt} = -\sigma w, \tag{11}$$

in which $\gamma = 1.4$ is the ratio of the specific heats, the subscript h indicates that the vertical advection term is to be omitted from the individual time-derivative, while σ is the stability parameter

$$\sigma = \frac{1}{T}\left(T_z + \frac{\varkappa g}{R}\right) = \frac{\varkappa g}{RT}\left(1 - \frac{\Gamma}{\Gamma_a}\right). \tag{12}$$

By (10) the left-hand side of (11) has the order $\frac{fV^2}{gH}$ (unless cancellation of terms occurs), so that we may write

$$\frac{fV^2}{gH} \sim \sigma W. \tag{13}$$

This relation expresses the inhibiting effect of stability on vertical motion. Taking note of (9) and (6) and the value 0.3 for $\varkappa$, the expression for σ may be written as

$$\sigma \sim \frac{\varkappa}{H}\left(1 - \frac{\Gamma}{\Gamma_a}\right) \lesssim \frac{1}{H} \tag{14}$$

so that (13) yields

$$\frac{fV^2}{gH} \lesssim \frac{W}{H}. \tag{15}$$

We can now relate the vertical speed W and the divergence D of the horizontal wind with the aid of the continuity equation

$$\frac{d \ln \varrho}{dt} + D + \frac{2w}{r} + w_z = 0 \tag{16}$$

in which

$$D = u_x + v_y - \frac{v}{r}\tan\phi. \tag{17}$$

From (10) and (15), the first term of (16) satisfies

$$\frac{d \ln \varrho}{dt} \sim \frac{fV^2}{gH} + \frac{W}{H} \sim \frac{W}{H}$$

while w_z is of the same order and $\frac{2w}{r}$ is negligible. Therefore we may write.

$$D \sim \frac{W}{H} \tag{18}$$

This result may be used to relate $\frac{W}{H}$ and $\frac{V}{S}$, by use of the vorticity equation, viz.

$$\frac{d\zeta}{dt} + \beta v + (f+\zeta)\left(D + \frac{2w}{r}\right) + w_x\left(v_z + \frac{v}{r}\right) - \\ - w_y\left(u_z + \frac{u}{r} + j\right) = -p_y\left(\frac{1}{\varrho}\right)_x + p_x\left(\frac{1}{\varrho}\right)_y \tag{19}$$

in which

$$\left.\begin{aligned} \beta &= f_y \\ \zeta &= v_x - u_y + \frac{u}{r}\tan\phi \end{aligned}\right\}. \tag{20}$$

By (4) and (18), none of the terms in (19) containing w or D are of higher order than the term fD. The remaining terms on the left-hand side, viz. $\frac{d_h\zeta}{dt}$ and βv, are of the orders

$\frac{V^2}{S^2}$ and $\frac{fV}{a}$ respectively, while the solenoidal terms on the right may be written as

$$\frac{p}{\varrho}\left(\frac{p_y}{p}\cdot\frac{\varrho_x}{\varrho}-\frac{p_x}{p}\cdot\frac{\varrho_y}{\varrho}\right)$$

which, by (9) and (10), is of the order $\frac{f^2V^2}{gH}$. Therefore we have

$$f\frac{W}{H}\sim\frac{V^2}{S^2}+\frac{fV}{a}+\frac{f^2V^2}{gH}. \qquad (21)$$

However, by (5), the solenoidal term never exceeds the β-term in order of magnitude, so that (21) becomes

$$\frac{W}{H}\sim\frac{V}{S}\left(\frac{V}{fS}+\frac{S}{a}\right), \qquad (22)$$

which is the key-relation on which our subsequent argumentation will be based. We may note at this stage that, from (3) and (4)

$$\frac{W}{H}\lesssim\frac{V}{S}. \qquad (23)$$

This implies that the vertical advection terms are not more important than the horizontal advection terms in the horizontal equations of motion, and therefore smallness of the Rossby parameter (representing the ratio of horizontal advection terms and coriolis terms) is sufficient for validating the geostrophic approximation.

4. Application to moderately long waves

Turning now to the interpretation of the key-relation (22), we note that, within the range of its validity, it determines the order of W uniquely in terms of V and S, so that also σ is determined in terms of V and S by (13). Evidently we should distinguish in (22) between the two cases where $\frac{V}{fS}\gtrsim\frac{S}{a}$ and $\frac{V}{fS}\lesssim\frac{S}{a}$, and our aim is to stress the significance of the latter case. In this section, however, we consider the former case. Thus, when

$$\frac{S}{a}\lesssim\frac{V}{fS} \qquad (24)$$

the relation (22) becomes

$$\frac{W}{H}\sim\frac{1}{f}\cdot\frac{V^2}{S^2} \qquad (25)$$

showing that W increases with increasing V and decreasing S. Substitution of this into (13) yields

$$\frac{S}{H}\sim\frac{\sqrt{g\sigma}}{f}, \qquad (26)$$

and this is a fundamental relation for the ratio of the horizontal and vertical scales, which is already implicit in the work of PRANDTL (1936). The condition (24), however, represents definite limits to its applicability. The relation (26) may be written as

$$\sigma H\sim\frac{f^2S^2}{gH} \qquad (27)$$

which shows a fast decrease of σ with decreasing S. On substituting from (24), we get

$$\sigma H\lesssim\frac{fVa}{gH} \qquad (28)$$

so that, since $V\lesssim 100$ m/sec,

$$\sigma H\lesssim 1$$

in consistence with (14).

In the case treated by Charney, i.e. where $S\sim 1{,}000$ km, the value of σH yielded by (27) is 10^{-1}, in conformity with the value used by Charney. We write (27) as

$$\frac{f^2S^2}{gH}<<1 \qquad (29)$$

and therefore

$$\frac{f^2V^2}{gH}<<\frac{V^2}{S^2} \qquad (30)$$

which means that the solenoidal terms are negligible against the horizontal advection of vorticity. Further, (13) yields

$$\frac{fV^2}{gH}\sim\frac{W}{H}(\sigma H)<<\frac{W}{H} \qquad (31)$$

which shows that the horizontal time derivative of density is negligible in the continuity equation. The fact that here

$$\frac{S}{a} << 1 \qquad (32)$$

allows metrical terms containing $\frac{1}{r}$ to be omitted from the equations, and D and ζ may be simplified to

$$\left.\begin{aligned} D &= u_x + v_y \\ \zeta &= v_x - u_y \end{aligned}\right\} \qquad (33)$$

Charney's further assumption $V \sim 10$ m/sec now implies

$$\frac{V}{fS} << 1 \qquad (34)$$

so that the geostrophic approximation holds and, from (25)

$$\frac{W}{H} << \frac{V}{S} \qquad (35)$$

This justifies the omission of terms containing w in the vorticity equation, and also, by (18), incidentally implies that

$$D << \zeta \qquad (36)$$

This also means that D is of lower order than its component terms; in fact it is of the same order as the errors in its components, and is therefore very difficult to represent accurately.

At this stage the equations, as simplified under Charney's assumptions, may be directly written down. The equations of motion (1) become the geostrophic and hydrostatic approximations, the adiabatic equation remains unchanged, while the continuity equation becomes

$$\left.\begin{aligned} &D + w(\ln \varrho)_z + w_z = 0 \\ &\text{i.e. } D + \frac{(\varrho w)_z}{\varrho} = 0 \end{aligned}\right\} \qquad (37)$$

which supplies an alternative way of representing D. The disappearance of time-derivatives from the equations reflects the quasi-stationary nature of the flow, and, as is well known, constitutes a major difficulty in the application of the primitive equations for prognostic purposes. The value of the vorticity equation (19) lies in the fact that the time-derivative does not drop out; in fact (taking into account $C \sim V$) it becomes

$$\frac{d_h \zeta}{dt} + \beta v + fD = 0 \qquad (38)$$

or, if D is eliminated from (37),

$$\frac{d_h \zeta}{dt} + \beta v = \frac{(\varrho w)_z}{\varrho}. \qquad (39)$$

In this equation the left-hand side may now be geostrophically approximated. As is known, this is not allowed for D in (38). We may remark in passing that the reason for this is not, as is occasionally stated, "that the geostrophic divergence is nearly zero", but simply that, being of lower order than its individual terms, it is masked by errors in these terms, whether they are geostrophically approximated or not. This may be illustrated by writing down the geostrophic divergence:

$$\left.\begin{aligned} D_g &= \frac{1}{f}\left[-\left(\frac{1}{\varrho}\right)_x p_y + \left(\frac{1}{\varrho}\right)_y p_x\right] - \frac{\beta}{f^2}\cdot\frac{p_x}{\varrho} \\ &= \frac{1}{f}\left[-\left(\frac{1}{\varrho}\right)_x p_y + \left(\frac{1}{\varrho}\right)_y p_x - \beta v_g\right] \\ &\sim \frac{fV^2}{gH} + \frac{V}{a} \end{aligned}\right\} \qquad (40)$$

and therefore, because of (5),

$$D_g \sim \frac{V}{a}.$$

From (22) we see that

$$\frac{W}{H} \gtrsim \frac{V}{a}$$

which, by (18), means

$$D \gtrsim D_g.$$

However, it should be noted that in the present case, where $S \sim 1{,}000$ km and $V \sim 10$ m/sec,

$$D \sim D_g.$$

In fact (22) shows that only for larger $\frac{V}{fS}$ $\left(\text{though not exceeding } \frac{S}{a}\right)$, i.e. for small S

and large V where the geostrophic approximation is generally bad, does D_g become small compared to D. However, the results of Landers (1955), who obtains relatively large values of D, are not quite clear in the light of these considerations.

Turing from the vorticity equation to the other popular differentiated form of the equations of motion, viz. the divergence equation, we find that it is again quasi-stationary, being simply equivalent to the geostrophic approximation. Written out in full, the divergence equation is

$$\left.\begin{aligned}&\frac{dD}{dt}+D\left(D+\frac{2w}{r}\right)+2\left(v_x u_y-u_x v_y\right)+\\&\quad+w_x\left(u_z+\frac{u}{r}+j\right)+w_y\left(v_z+\frac{v}{r}\right)+\\&+2\tan\phi\left(uu_y+vv_y\right)+\frac{u^2+v^2}{r^2}-f\zeta+\\&+\beta u=-\left(\frac{p_x}{\varrho}\right)_x-\left(\frac{p_y}{\varrho}\right)_y+\frac{p_y}{\varrho}\frac{\tan\phi}{r}\end{aligned}\right\}\quad(41)$$

An alternative form of the right-hand side is

$$-f\zeta_g+\beta u_g \qquad (42)$$

where the subscript g again denotes the geostrophic value. These two terms are of orders $\frac{fV}{S}$ and $\frac{fV}{a}$ respectively, so that the whole expression is of order $\frac{fV}{S}$. Thus knowing the order of the right-hand side of (41), it may be simplified by being written in the further alternative form:

$$\left.\begin{aligned}&-\frac{1}{\varrho}\left(p_{xx}+p_{yy}\right)+\frac{p_y}{\varrho}\frac{\tan\phi}{r}-\left(\frac{1}{\varrho}\right)_x p_x-\\&\qquad-\left(\frac{1}{\varrho}\right)_y p_y.\end{aligned}\right\}\quad(43)$$

The middle one of these terms is of order $\frac{fV}{a}$, and therefore negligible in the present case, while the last two, like the solenoidal terms in the vorticity equation, are of the order $\frac{f^2V^2}{gH}$, and therefore, by (30) and (34), also negligible. It is now easily seen that (41) simplifies to

$$f\zeta=f\zeta_g=-\frac{1}{\varrho}\left(p_{xx}+p_{yy}\right), \qquad (44)$$

which simply expresses the geostrophic relation.

For the sake of interest we conclude this section by relating the Rossby parameter to the Richardson number. Using (27), we namely have

$$Ri\equiv\frac{g\sigma}{v_z^2}\sim\frac{f^2S^2}{H^2}\Big/\frac{V^2}{H^2}=\left(\frac{fS}{V}\right)^2 \qquad (45)$$

Therefore (34) implies

$$Ri>>1 \qquad (46)$$

which is the condition used by Eady (1949) instead of the geostrophic approximation. We note that, in the present context, (46) is a less restrictive condition than (34), since the Rossby number occurs squared in (45).

5. Application to planetary waves

Turning now to the case where

$$\frac{V}{fS}\lesssim\frac{S}{a} \qquad (47)$$

we note that (22) becomes

$$\frac{W}{H}\sim\frac{V}{a} \qquad (48)$$

which, substituted in (13), yields

$$\sigma H\sim\frac{fVa}{gH} \qquad (49)$$

instead of the relations (25) and (27) of the complementary case. Thus the Prandtl relation (26) is no longer valid, and both σ and W increase linearly with V but are independent of S. Again (49) is consistent with (14) for $V\lesssim 100$ m/sec.

Focussing attention on the planetary waves, for which

$$\frac{S}{a}\sim 1 \qquad (50)$$

we note that (since $V\lesssim 100$ m/sec) the Rossby parameter is small:

$$\frac{V}{fS}<<1 \qquad (51)$$

so that the geostrophic approximation is well satisfied. This also means

$$\frac{V}{fS} << \frac{S}{a} \tag{52}$$

so that

$$\frac{V^2}{S^2} << \frac{fV}{a} \tag{53}$$

which implies that the horizontal time-derivative of vorticity is negligible against the β-term. This leads to the important conclusion that, *on the planetary scale, even the vorticity equation displays a quasi-stationary character.*

We proceed to establish the approximate form of the vorticity equation in this case. Combination of (48) with (50) yields

$$\frac{W}{H} \sim \frac{V}{S} \tag{54}$$

in contrast to (35) which applies in Charney's case. From (18) and (48) we see that

$$\beta v \sim fD \tag{55}$$

By (5), (51), (53), and (54), all other terms on the left-hand side of (19) vanish against the two terms occurring in (55). As for the right-hand side, the solenoidal terms $\left(\sim \frac{f^2V^2}{gH}\right)$ predominate over the vorticity advection terms:

$$\frac{f^2V^2}{gH} >> \frac{V^2}{a^2} \sim \frac{V^2}{S^2}, \tag{56}$$

while (5) shows that they are never more important than the terms of (55):

$$\frac{f^2V^2}{gH} \lesssim \frac{fV}{a} \tag{57}$$

If we assume, as in Charney's case, that $V \sim 10$ m/sec (i.e. $V << 100$ m/sec), the symbol $\lesssim$ becomes $<<$, but (56) is independent of the value of V. This leads to the result that, for the planetary scale and $V \sim 10$ m/sec, the vorticity equation becomes

$$\beta v + fD = 0 \tag{58}$$

By (40) this equation is evidently simply equivalent to the geostrophic approximation, and it indicates that the very fast retrogression which would be expected from the theory of Rossby waves, does not in fact occur.

For larger V the solenoidal terms become increasingly important, and should therefore eventually be inserted on the right-hand side of (58). However, in the "p-system", in which the pressure p is used as vertical coordinate, the solenoidal terms do not occur at all. We therefore refer, at this point, to the connection between our coordinate system and the p-system. This connection is characterized by noting that the derivative u_x transforms as follows:

$$\left.\begin{aligned} (u_x)_{p\text{-system}} &= u_x + u_z(z_x)_{p\text{-system}} = \\ &= u_x + u_z\frac{fv_g}{g} \\ &\sim \frac{V}{S} + \frac{fV^2}{gH} \end{aligned}\right\} \tag{59}$$

Thus u_x has approximately the same value in the two systems if the second term of (59) is negligible compared to the first, i.e. if

$$\frac{fV}{g} << \frac{H}{S} \tag{60}$$

We know that this second term of (59) is never predominant, since, by (5),

$$\frac{fV}{g} \lesssim \frac{H}{a} \lesssim \frac{H}{S}$$

and it is negligible if

$$VS << \frac{gH}{f} \sim 10^9 \text{ m}^2 \text{ sec}^{-1}$$

In the case where $S \sim 1{,}000$ km this is satisfied for all permissible values of V, whereas in the present case where $S \sim 10{,}000$ km, we have to require V to be ~ 10 m/sec. If both V and S approach their upper limits, the two terms of (59) are of the same order, and the value of v_x is no longer even approximately the same in the two systems (though the order of magnitude will generally be unaltered). This, as is to be expected, is exactly the case in which the solenoidal terms in the z-system become of the same order as the terms of (58). It should be noted that here the relation between the vertical speed w and the corresponding quantity in the p-system, $\omega \equiv \frac{dp}{dt}$, also becomes more complicated. We namely have

$$\left.\begin{aligned} \frac{\omega}{\varrho g} &= -w + \left(\frac{d_h z}{dt}\right)_{p\text{-system}} \\ &\sim W + \frac{fV^2}{g} \end{aligned}\right\} \qquad (61)$$

which, by (22) may be written as

$$\begin{aligned} \frac{\omega}{\varrho g} &\sim \frac{HV}{S}\left(\frac{V}{fS} + \frac{S}{a}\right) + \frac{fV^2}{g} \\ &\sim \frac{HV}{S}\left(\frac{V}{fS} + \frac{S}{a} + \frac{fVS}{gH}\right) \end{aligned}$$

and we note that, for $S \sim a$, the condition that the second term on the right-hand side of (61) is negligible against the first, is identical to (60). In summary, we may say that our results, obtained in the z-system, are easily transferable to the p-system, excepting that some care is needed in the cases when (60) is violated.

Returning, now, to the vorticity equation, it can be argued that, since the observational errors are reasonably small on the planetary scale, due to the goodness of the geostrophic approximation, such errors need not mask the time-derivative of vorticity. However, the very relation (51) which validates the geostrophic approximation, also implies (53), so that the more geostrophic the flow is, the more quasi-stationary is the character of the vorticity equation. If in spite of this, the time-derivative is retained, all other terms of comparable magnitude should of course also be retained in order to yield sensible prognostic information. We note that (54) implies that the horizontal advection of vorticity is equalled in order of magnitude by the vertical advection and twisting terms, and therefore we obtain in this case:

$$\left.\begin{aligned} &\frac{d\zeta}{dt} + \beta v + fD + w_x v_z - w_y u_z = \\ &= -\left(\frac{1}{\varrho}\right)_x p_y + \left(\frac{1}{\varrho}\right)_y p_x \end{aligned}\right\} \qquad (62)$$

which is not essentially simpler than the full vorticity equation. This points to the fact that it is very difficult to get specific information on the planetary scale beyond expressing its quasi-stationary character.

Furthermore, it should be remembered that, due to (50) the metrical terms do not drop out from the expressions (17) and (20) for D and ζ, though, of course, r may be replaced by a. As for the geostrophic divergence, this contains no metrical terms, as is evident from its vectorial form:

$$D_g = \boldsymbol{k} \cdot \nabla p \times \nabla \frac{1}{f\varrho}$$

where $\boldsymbol{k}$ is the vertical unit vector, and ∇ the gradient operator. The geostrophic vorticity is, in vector notation,

$$\zeta_g = \frac{1}{f\varrho} \nabla_h^2 p + \nabla_h p \cdot \nabla_h \frac{1}{f\varrho} \qquad (63)$$

where ∇_h denotes the gradient operator in a horizontal surface. Here the Laplacian generally does contain metrical terms. However, this is not the case with cartesian coordinates on a conformal plane projection of the earth's surface, as is customarily used in numerical prediction. Here, if X and Y denote the cartesian coordinates on the map, and m is the mapping factor (ratio of an arc increment on the map to that on the earth's surface), we namely have

$$\nabla_h^2 = m^2 \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)$$

Thus two circumstances, viz. firstly, the use of geostrophic values, and, secondly, the use of cartesian coordinates on a conformal map, combine rather fortuitously to render a discussion of the relative importance of metrical terms in ζ and D irrelevant.

Proceeding, now, from the vorticity equation to the other equations for the planetary scale, not much is changed compared to the previous section. The equations of motion are evidently very well represented by the geostrophic and hydrostatic equations, the adiabatic equation remains unaltered, and, (if we again assume $V \sim 10$ m/sec), the continuity equation still has the form (37). If V increases, i.e. if (60) ceases to hold, (54) and (10) show that the horizontal time-derivaitve of density becomes less negligible (for large S) in the continuity equation. In the divergence equation (41), evidently βu on the left-hand side should now be included, being of the same order of magnitude as $f\zeta$. On the right-hand side, the last two terms of (43) may be omitted for $V \sim 10$ m/sec; for larger V they should be retained, as with the solenoidal terms belonging with (58). Therefore (for $V \sim 10$ m/sec) we have

$$\left.\begin{aligned} -f\zeta + \beta u &= -f\zeta_g + \beta u_g = \\ &= -\frac{1}{\varrho}\left(p_{xx} + p_{yy} - \frac{\tan\phi}{a}p_y\right) \end{aligned}\right\}, \qquad (64)$$

again simply expressing the geostrophic approximation. In fact, we could more conveniently replace this equation by directly writing down the relevant geostrophic formula for any quantity as required. Thus, for instance, the spherical coordinate form of (63) is

$$\zeta_g = \frac{1}{f\varrho}\left(p_{xx} + p_{yy} - \frac{\tan\phi}{a}p_y\right) - $$
$$- \frac{\beta p_y}{f^2\varrho} + \frac{1}{f}p_y\left(\frac{1}{\varrho}\right)_y + \frac{1}{f}p_x\left(\frac{1}{\varrho}\right)_x$$

and, if the last two terms are omitted, as they were in (64), this becomes

$$\zeta_g = \frac{1}{f\varrho}\left(p_{xx} + p_{yy} - \frac{\tan\phi}{a}p_y\right) - \frac{\beta p_y}{f^2\varrho} \qquad (65)$$

which is exactly the value of ζ obtained from (64) by replacing βu on the left-hand side by its geostrophic value.

If we attempt to retain some of the non-linear terms of the divergence equation in (64), the position is similar to that in (62), where also terms in w as well as the density derivatives should be retained.

Finally, we note that in the present case the relation between the Richardson and Rossby numbers differs from that of the previous section. Here we have

$$Ri = \frac{g\sigma}{v_z^2} \sim \frac{fVa}{H^2}\bigg/\frac{V^2}{H^2} = \frac{fa}{V}$$

and if attention is restricted to the case (50), this becomes

$$Ri \sim \frac{fS}{V} \qquad (66)$$

Evidently, therefore, $Ri >> 1$ is on the whole a less restrictive condition than $\frac{V}{fS} << 1$, over the range of values considered.

6. Conclusion

Based on the above results, the position may be summarized as follows:

(a) The simplified form (39) of the vorticity equation, geostrophically approximated, is a useful tool in forecasting on the moderately large scale of motion.

(b) The same equation applied to waves of planetary size, should fail in the details of its prognosis of their motion.

(c) If both the scales of a and b are present, the planetary waves will contribute only a small part to the forecast change. This means that the major part of the change is well handled, so that the overall performance should still be good.

(d) The fact that the changes are of lower order of magnitude on the planetary scale, is caused by a balance between the divergence and β-terms in the vorticity equation. Therefore, a divergence term has to be present in order to express this balanced state. This confirms the proposed addition of a divergence term to the barotropic model by Bolin (1955, 1956). Further since the geostrophic approximation is here well satisfied, while also on this scale the separate terms in the divergence do not tend to cancel, the divergence term may, as far as the requirements of the planetary scale are concerned, be geostrophically approximated.

(e) Any attempt beyond expressing the state of quasibalance on the planetary scale, is hampered, not only by the masking effect of errors on different scales, but also by the fact that a number of extra terms, normally considered negligible, seem not to be negligible for this purpose. However, merely adding such extra terms will increase the number of degrees of freedom of the system, also on the scales where these terms are in fact negligible, and the overall effect will probably not be satisfactory. If, for the planetary scale, an adequate representation of the quasi-stationary character only is sought, a possible amended form of the barotropic vorticity equation for use over different scales, is therefore (38), with a geostrophically approximated divergence term. However, the moderate scales will probably be adversely affected by the geostrophic approximation in the divergence term.

(f) As with the divergence term in the vorticity equation, the β-term in the divergence equation is of prime importance on the planetary scale. Using the divergence equation (or balance equation) in the form (64), or, equivalently, using the value (65) for ζ, should therefore considerably improve the representation on this scale. Here the presence of metrical terms $\left(\text{containing } \frac{1}{a}\right)$ is important in spherical coordinates; however, as has been shown above, the use of a square grid on a conformal plane map of the earth's surface, which is customary in numerical prediction, automatically allows for the metrical terms. It should be noted that (64) is still entirely equivalent to the geostrophic approximation, and that inclusion of the β-term in the balance equation is not a step towards nongeostrophic representation.

(g) As far as the moderate scales are concerned, the condition of smallness of the Rossby number, which justifies the geostrophic approximation, is not too well satisfied for the higher wind-speeds. In considerations aimed at proceeding towards nongeostrophic effects the condition of largeness of the Richardson number may be regarded as a next approximation, being less stringent and better satisfied than smallness of the Rossby number.

(h) The Prandtl relation (26), expressing the ratio of horizontal and vertical scales, ceases to be valid on the planetary scale.

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