An Experiment in Numerical Prediction with two Non-Geostrophic Barotropic Models

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Abstract

The result of a test with two simple non-geostrophic barotropic models are discussed. One of these models is based on ideas presented in a previous article (CHARASH 1957).

The forecasts obtained are found to differ significantly from forecasts obtained with the quasi-geostrophic model. Both non-geostrophic models appear to be free from one major weakness of the non-geostrophic model, i.e. the hypertrophy of anticyclones.

It is suggested that the disappointing forecasting performance of baroclinic models hitherto to a considerable degree springs from a similar weakness as exposed here in the quasi-geostrophic barotropic model.

Introduction

In practically all atmospheric models constructed for the purpose of numerical prediction the height field of an isobaric surface is treated as a pseudo-streamfield. That procedure implies the neglect—partially at least—of the variation of the Coriolis parameter. Justification for doing so has been found in the fact that that variation is comparatively small, and it has been concluded that the so-called geostrophic vorticity constituted a very good first approximation.

One of the authors in a previous paper (CHARASCH 1957) argues that failure to take into account fully the variation of the Coriolis parameter in evaluating the geostrophic vorticity is responsible for a systematic aberration in the exchange of kinetic energy between the basic current and the disturbances. That energy conversion constitutes an effective mechanism for the growth of disturbances and it is the only mechanism of wave growth available in the barotropic model.

Systematic appreciable differences may be expected therefore, in numerical predictions with a barotropic model which—in contrast to the quasi-geostrophic model—does account for the full variation of the Coriolis parameter.

The results of an experiment along those lines are presented in this paper.

1. The equations

The prognostic equation for the barotropic model is

$$\frac{\partial \zeta}{\partial t} = -\mathbf{v} \cdot \nabla (\zeta + f) \tag{1}$$

where ζ denotes the relative vorticity, $\mathbf{v}$ the wind-velocity, and f the Coriolis parameter. If we introduce the *quasi*-geostrophic vorticity

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$$\zeta_q = \frac{1}{f} \nabla^2 \varphi \tag{2}$$

eq. (1) becomes

$$\frac{1}{f} \frac{\partial}{\partial t} \nabla^2 \varphi = -\mathbf{v} \cdot \nabla \left(\frac{1}{f} \nabla^2 \varphi + f \right) \tag{3}$$

If we use the geostrophic wind we get

$$\frac{1}{f} \frac{\partial}{\partial t} \nabla^2 \varphi = \left(\frac{1}{f} \nabla \varphi \times \mathbf{k} \right) \cdot \nabla \left(\frac{1}{f} \nabla^2 \varphi + f \right) \tag{4}$$

or

$$\nabla^2 \frac{\partial \varphi}{\partial t} = J \left(\frac{1}{f} \nabla^2 \varphi + f, \varphi \right) \tag{4a}$$

If we instead introduce the complete geostrophic vorticity

$$\zeta_g = \frac{1}{f} \nabla^2 \varphi - \frac{\nabla f}{f^2} \nabla \varphi \tag{5}$$

in eq. (1), we obtain

$$\frac{\partial}{\partial t} \left(\frac{1}{f} \nabla^2 \varphi - \frac{\nabla f}{f^2} \nabla \varphi \right) = -\mathbf{v} \cdot \nabla \left(\frac{1}{f} \nabla^2 \varphi - \frac{\nabla f}{f^2} \nabla \varphi + f \right) \tag{6}$$

It is convenient to introduce here a streamfunction ψ_a defined by

$$\nabla^2 \psi_a = \zeta_g = \frac{1}{f} \nabla^2 \varphi - \frac{\nabla f}{f^2} \nabla \varphi \tag{7}$$

The wind field in which the advection of the vorticity takes place—to be consistent—is then defined by

$$\mathbf{v} = \mathbf{k} \times \nabla \psi_a \tag{8}$$

i.e. the streamfunction which is defined by the geostrophic vorticity in turn defines a (divergence-free) wind field (BOLIN 1955).

With these definitions in mind, it is now possible to substitute in equation (6) and to reformulate the barotropic prognostic equation

$$\frac{\partial}{\partial t} \nabla^2 \psi_a = J (\nabla^2 \psi_a + f, \psi_a) \tag{9}$$

In equation (9) the vorticity is derived initially from the geostrophic equation, however, (9) is the prognostic equation for a non-geostrophic barotropic model.

Equations (4a) and (9) are both derived from Eq. (1), but differ from each other in two respects, namely

1. The neglect of the β-term in computing the geostrophic vorticity in (4a).
2. The introduction of the geostrophic divergence in (4a).

In order to investigate the importance of these differences it was decided to compare a number of forecasts using (4a) and (9) respectively. This does not make it possible to separate the two effects, but in the light of the theory presented by CHARASCH (1957) it is felt that the β-term is more important than the geostrophic divergence.

The significance of the streamfunction ψ_a is illuminated by the following consideration.

The balance equation may be written

$$\nabla^2 \psi = \frac{1}{f} \nabla^2 \varphi - \frac{\nabla f}{f} \nabla \psi - \frac{2}{f} J \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \right) \tag{10}$$

If the geostrophic approximation is made in the second term on the right hand side, we obtain

$$\nabla^2 \psi_b = \frac{1}{f} \nabla^2 \varphi - \frac{\nabla f}{f^2} \nabla \varphi - \frac{2}{f} J \left(\frac{\partial \psi_b}{\partial x}, \frac{\partial \psi_b}{\partial y} \right) \tag{11}$$

It should be stressed that (11) is a very good approximation to (10) as:

the approximated term is small compared to $\frac{1}{f} \nabla^2 \varphi$ and the error in the approximated term is a fraction of its true value.

If the non-linear term is neglected we get back to equation (7). ψ_a may thus be regarded as a first approximation to the streamfunction ψ_b, which in turn is an approximation to the exact streamfunction ψ defined by (10). ψ_a and ψ_b differ only with respect to the non-linear term and it was decided to investigate the importance of this term by making parallel forecasts with ψ_a and ψ_b. The prognostic equation using ψ_b is exactly similar to the one using ψ_a.

$$\frac{\partial}{\partial t} \nabla^2 \psi_b = J (\nabla^2 \psi_b + f, \psi_b) \tag{12}$$

II. Set-up of the experiment

A. The forecasts were computed with finite difference formulae corresponding to eqs. (4a), (9) and (12). The timestep was in all cases one hour and the meshwidth was 300 km at 50° N in a polar stereographic projection. The boundary conditions were

$$\frac{\partial \varphi}{\partial t} = \frac{\partial \zeta}{\partial t} = 0 \tag{13}$$

and analogous for ψ_a and ψ_b. Smoothing was applied at 0^h, 24^h and 48^h using the formulae (BEST, 1956)

$$\varphi^{(1)} = \varphi^{(0)} + k \nabla^2 \varphi^{(0)} \tag{14}$$

and

$$(\nabla^2 \varphi)^{(1)} = (\nabla^2 \varphi)^{(0)} + k \nabla^2 (\nabla^2 \varphi)^{(0)} \tag{15}$$

and the corresponding formulae for ψ_a and ψ_b (∇ denotes the finite Laplace operator).

B. The ψ_a-field was obtained from the φ-field by solving the Laplace-equation (7). To obtain suitable boundary values the geostrophic approximation was made use of in the following way:

$$\frac{\partial \psi}{\partial s} = \frac{\dfrac{\partial \varphi}{\partial s}}{f} - \gamma \tag{16}$$

where

$$\gamma = \frac{1}{L} \oint \frac{\dfrac{\partial \varphi}{\partial s}}{f} ds \tag{17}$$

L is the length of the boundary and $\dfrac{\partial}{\partial s}$ denotes differentiation along the boundary.

In order to obtain a good first approximation to the solution in the interior the geostrophic approximation was integrated in a similar way from one side of the rectangular area across the region to the opposite boundary.

The φ-fields corresponding to the 24^h and 48^h ψ_a-forecasts were obtained in an exactly analogous way with equation (11).

C. The ψ_b-field was computed from the ψ_a-field by a second relaxation procedure.

If equation (7) is used equation (11) can be written

$$f \nabla^2 \psi_b = f \nabla^2 \psi_a - 2J\left(\frac{\partial \psi_b}{\partial x}, \frac{\partial \psi_b}{\partial y}\right) \tag{18}$$

It was decided to solve ψ_b from this equation utilizing the boundary values already computed for ψ_a.

Some problems occurred in connection with the transformation of (18) to difference form and regarding the relaxation scheme. SCHUMAN (1955) describes a method for solving the balance equation that has several advantages. However, in his presentation the non-linear term $(\psi_{xx}\,\psi_{yy} - \psi_{xy}^2)$ is approximated by

$$\frac{m^4}{(\Delta s)^4} \Big\{ (\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j})\,(\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}) - \frac{1}{16}(\psi_{i+1,j+1} - \psi_{i+1,j-1} - \psi_{i-1,j+1} + \psi_{i-1,j-1})^2 \Big\} \tag{19}$$

where m is a mapping factor.

As pointed out by BOLIN (1956) this introduces a systematic error as the two terms are evaluated with different gridsizes. A proper approximation is instead

$$\frac{m^4}{4(\Delta s)^4} \Big\{ (\psi_{i+1,j+1} + \psi_{i-i,j-1} - 2\psi_{i,j})(\psi_{i+1,j-1} + \psi_{i-1,j+1} - 2\,\psi_{i,j}) - (\psi_{i+1,j} - \psi_{i,j+1} - \psi_{i,j-1} + \psi_{i-1,j})^2 \Big\} \tag{20}$$

If this approximation is introduced in Schuman's method the latter loses its advantages unless the term $\nabla^2 \psi_b$ is evaluated in a similar way, i.e.

$$\frac{m^2}{2(\Delta s)^2} (\psi_{i+1,i+1} + \psi_{i+1,j-1} + \psi_{i-1,j+1} + \psi_{i-1,j-1} - 4\psi_{i,j}) \tag{21}$$

However, a ψ-field determined in this way consists of two almost independent sets only coupled to each other by the non-linear term. A field computed in this way showed, as could be expected, short wave deviations from the true field, the deviations being distributed in the following way

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \end{matrix}$$

The possibility to remedy this defect by further manipulations with the difference formulae was not investigated, but instead a line of attack somewhat different from Schuman's was taken.

In (18) we approximate $\nabla^2\psi$ in the usual way by

$$\frac{m^2}{(\Delta s)^2}\left\{\psi_{i+1,j} + \psi_{i,j+1} + \psi_{i,j-1} + \psi_{i-1,j} - 4\,\psi_{i,j}\right) \tag{23}$$

and the non-linear term by (20). Let us denote the difference equation obtained in this way

$$L(\psi_{i,j}) = 0 \tag{24}$$

Suppose now that we have an approximate solution

$$L(\psi'_{i,j}) = \varepsilon_{i,j} \tag{25}$$

Next we want to change the value of ψ' in the point (i, j) by an amount $\delta\psi'_{i,j}$ so that $\varepsilon_{i,j}$ is diminished. Due to the non-linear character of L we get a second order equation

$$a_{i,j}(\delta\psi'_{i,j})^2 + b_{i,j}\ \delta\psi'_{i,j} = \delta\varepsilon_{i,j} \tag{26}$$

We neglect the quadratic term and put $\delta\varepsilon_{i,j} = -\varepsilon_{i,j}$ and obtain

$$\delta\psi'_{i,j} = \varepsilon_{i,j} \Bigg/ \left(4f + \frac{m^2}{(\Delta s)^2}(\psi_{i+1,j+1} + \psi_{i+1,j-1} + \psi_{i-1,j+1} + \psi_{i-1,j-1} - 4\psi_{i,j})\right) \tag{27}$$

In this way we determine a new value

$$\psi''_{i,j} = \psi'_{i,j} + \delta\psi'_{i,j} \tag{28}$$

Then we proceed to the next point and make now use of the new value ψ'' in the points already treated.

Using this method the residuals in a field of 1,200 points were found to decrease by a factor ten in on the average slightly less than twenty iterations. The total time needed to compute ψ_b from ψ_a using the Swedish computer BESK was around fifteen minutes when the residuals were decreased to about one per cent of their initial value.

The method described above is very similar to a method used by BOLIN (1955 and 1956). The only difference lies in fact in the denominator in the quotient determining $\delta\psi'$. Bolin had here

$$f + \frac{m^2}{\Delta s^2}(\psi_{i+1,j} + \psi_{i,j+1} + \psi_{i,j-1} + \psi_{i-1,j} - 4\psi_{i,j}) \tag{29}$$

In this context we will touch shortly on the ellipticity criterion.

For the differential equation (18) it can easily be shown that the ellipticity criterion

$$f\nabla\psi_a + \frac{f^2}{2} > 0 \tag{30}$$

implies

$$\nabla^2\psi_b + f > 0 \tag{31}$$

When we transform the differential equation to a difference equation we would expect to find analogous conditions. That is we would expect to find a condition corresponding to (31) to be an immediate consequence of a condition corresponding to (30). If in the difference equation we combine either expressions (19) and (23) or expressions (20) and (21) this can easily be shown to be true. Combining (20) and (23) as we have done here, makes things more difficult and so far we have not been able to find any difference relations for this case that correspond to (30) and (31) and are coupled in the same way as these.

In the relaxation scheme used by Bolin where the same expressions (20) and (23) were utilized, it is necessary that expression (29) is always larger than zero. Bolin accordingly made certain that a condition corresponding to (30) was fulfilled and this proved to be sufficient in the cases ha ran. The denominator in (27) corresponds to

$$\nabla^2\psi + 2f \tag{32}$$

and could for a 500-mb surface be expected to be positive everywhere even if (30) is not fulfilled everywhere. Experiments showed also that a ψ_b-field could be determined by the method described above even when condition (30) was not fulfilled. However, the ψ_b-fields used for the forecasts presented here were computed from ψ_a-fields where (30) was fulfilled. (The ψ_a-forecasts were computed from the unchanged ψ_a-fields.)

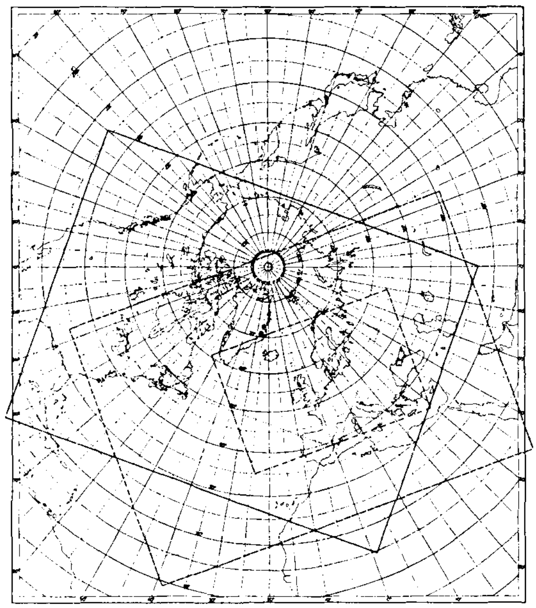

Fig. 1.

Altogether five synoptic cases have been chosen for the experiment. They have been picked arbitrarily from five different months. The analyses of two of the cases have been done by the conventional subjective method and of the remaining three cases the analyses have been obtained by the numerical procedure developed by Bergthorsson and Döös. The arbitrariness of choice of the latter three cases has been restricted merely by the requirement that their analyses should agree generally with a reasonable subjective analysis of the respective situations. Only analyses obtained by the subjective method were used for the purpose of verification of forecasts. The forecast area is shown by the solid line in fig. 1. In one case, the 13.12.51, the forecast area was oriented differently. The area for this case is shown by the dashed line.

For each of the cases 24^h- and 48^h-forecasts have been computed using the prognostic equations (4 a), (9) and (12).

The time required to obtain a 48^h-forecast with the geostrophic prognostic equation is about 50 minutes using BESK. The ψ_a-forecasts require about 65 minutes and the ψ_b-forecasts about 80 minutes.

III. Results

The results of the forecasts are summarized in table 1. Here r denotes the correlation coefficient between observed and computed height changes, ε denotes the mean error, and σ the observed mean change. $\bar{y}$ is the average computed change and x the average observed change. The subscripts g, a and b refer to the geostrophic forecast, the ψ_a-forecast and the ψ_b-forecast respectively.

Table I.

24^h	r_g	r_a	r_b	ε_g	ε_a	ε_b	σ	$\bar{y}_g$	$\bar{y}_a$	$\bar{y}_b$	$\bar{x}$
30.1.56	0.72	0.71	0.75	86	96	93	95	18	— 43	— 53	— 2
20.2.56	0.65	0.66	0.67	89	107	90	79	47	69	50	0
19.3.56	0.49	0.57	0.59	98	77	76	80	53	25	24	— 12
13.4.56	0.65	0.69	0.74	56	48	43	65	32	11	11	11
13.12.51	0.81	0.81	0.74	77	61	75	101	60	17	— 5	31
mean	0.66	0.69	0.70	81	78	75	84	42	16	5	6

48^h	r_g	r_a	r_g	ε_g	ε_a	ε_b	σ	$\bar{y}_g$	$\bar{y}_a$	$\bar{y}_b$	$\bar{x}$
30.1.56	0.58	0.51	0.60	154	167	188	121	— 18	— 92	—126	7
20.2.56	0.64	0.63	0.64	163	194	168	105	92	120	96	— 18
19.3.56	0.49	0.52	0.60	136	113	105	93	102	73	66	8
13.4.56	0.68	0.75	0.80	88	70	59	91	46	35	30	23
13.12.51	0.61	0.81	0.77	173	91	114	137	76	14	— 23	42
mean	0.60	0.64	0.68	143	127	127	109	60	30	9	12

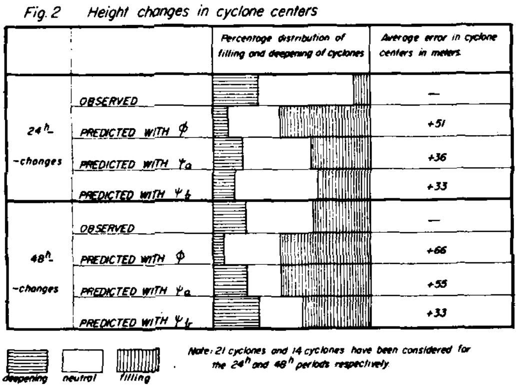

Fig. 2.

The verification was made over the area indicated by the dotted line in fig. 1. The verification area for the 13.12.51 is shown by the dash-dotted line.

1. The table shows a noticeable improvement from the quasi-geostrophic model to the non-geostrophic ones and also a slight improvement due to the non-linear term. The average changes in the non-geostrophic forecasts are not predominantly too high as in the quasi-geostrophic forecasts, but show on the other hand a much larger dispersion.

2. A subjective inspection of the maps reveals that systematic and in some cases great differences result from the inclusion of the β-term. These changes are generally of a rather large scale and the most striking feature is that the exaggeration of anticyclones, which is a rather common phenomenon in quasi-geostrophic forecasts, is substantially suppressed by including the β-term. In one case, however, namely the one based on the 20. 2. 56 an anticyclone intensifies inordinately in the non-geostrophic models rather than in the quasi-geostrophic model; but in this case the dimension of the anticyclone does not even approach that of a "super-anticyclone", the central height being predicted at 48^h about 120 m and 100 m too high using ψ_a and ψ_b respectively.

3. Another fact revealed by the test is that the tendency to fill out cyclones, which is strongly present in the quasi-geostrophic model, is much weakened in the non-geostrophic models. This is illustrated by figure 2 which shows the percentage distribution of filling and deepening of cyclones in the different forecasts. The average error in the cyclone centers is also shown.

4. The observed differences with respect to prediction of the intensity of disturbances are in agreement with the theory formulated by CHARASCH (1957).

5. As regards prediction of positions of cyclones and anticyclones no systematic differences have been observed in this very small material.

IV. Summary

Significant differences have been observed between predictions with the quasi-geostrophic and the non-geostrophic models in all of the cases tested. These differences appear to be due mainly to the neglect of the β-term in computing the "geostrophic" vorticity of the quasi-geostrophic model; an investigation is required to establish clearly the effect of the geostrophic divergence. The above differences are great enough to be comparable with the neglect of baroclinic factors and divergence; there is also evidence that the non-geostrophic models are devoid of a certain bias with regard to the average height changes that is present in the quasi-geostrophic model.

The non-linear term appears also to have a significant influence; the differences observed due to this term are on a much smaller scale than those due to the β-term.

Although the sample of forecasts investigated is statistically inadequate it is felt, in the light of the simple theory underlying the investigation, that the β-term must not be neglected in any numerical prediction model, barotropic or baroclinic, and that non-linear term should be included whenever it is possible for practical reasons.

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