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A NOTE ON ASYMPTOTIC ESTIMATE FOR DIFFERENCE EQUATION WITH SEVERAL PROPORTIONAL DELAYS

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ABSTRACT. The main goal of this article is to derive asymptotic estimate for solutions of difference equation with several proportional delays. Studied difference equation arises from generalized pantograph equation (with several proportional delays) via the Euler method with a constant stepsize.

1. Introduction

The main goal of this article is to derive asymptotic estimate for delay difference equation with several proportional delays

$$\Delta x_n = -ahx_n + h\sum_{i=1}^{r} b_i x_{\lfloor \lambda_i n \rfloor}, \qquad n = 0, 1, 2, \dots, \tag{1}$$

where $a > 0$, $b_i \neq 0$, $i = 1, \dots, r$, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_r < 1$ are reals, symbol $\lfloor\ \rfloor$ means the floor function and $h > 0$ is the stepsize. Difference equation (1) can be considered as a discrete form of multi-pantograph equation

$$\dot{z}(t) = -az(t) + \sum_{i=1}^{r} b_i z(\lambda_i t), \qquad t \in I = [0, \infty). \tag{2}$$

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Then $x_n \approx z(t_n)$, $t_n = nh$, $n = 0, 1, 2, \dots$ This discretization is based on the Euler method for delay differential equations, where a piecewise constant interpolation of delayed terms is utilized. It has been shown in [5] that this numerical method is convergent (see also [1]). To recall some papers dealing with the asymptotic properties of solutions of the pantograph equation and its discretizations, we can mention, e.g., [2, 3, 7, 11, 13] and [6, 8, 10, 12], respectively.

This work arises from previous author's research. It started by [9], where the asymptotic properties of solutions of the pantograph difference equation with one delayed argument (considering not only proportional delay) were discussed. Now, our aim is to formulate some asymptotic estimate of solutions of difference equation (1) (under some assumptions on coefficients a, b_i) and to compare it with the known asymptotic estimate of its continuous counterpart (2).

2. Preliminaries

In this section we introduce an auxiliary inequality, which will be utilized in the proof of the main result. Its solution plays the key role in the formulation of asymptotic estimate of solutions of (1).

The discussion of the asymptotic behaviour of solutions of the pantograph equation in the continuous case often utilizes some auxiliary functional equations. Dealing with the discrete case we consider the inequality

$$a\rho_n \geq \sum_{i=1}^{r} |b_i| \rho_{\lfloor \lambda_i n \rfloor}, \qquad n = 1, 2, \dots \tag{3}$$

Although we can use any positive and nondecreasing solution $\rho_n, n = 1, 2, \dots$ of (3) in the formulation of the main result (provided additional assumptions on $a, b_i, i = 1, 2, \dots$), we introduce a concrete suitable solution in the following remark.

Remark 1. Denote $B := \sum_{i=1}^{r} |b_i|$. Let us show that if $B \geq a$, then $\rho_n = (nh)^\alpha$, $\alpha = \frac{\log \frac{B}{a}}{\log \lambda_r^{-1}}$, $n = 1, 2, \dots$ is a positive and nondecreasing solution of (3). Indeed,

$$\alpha = \frac{\log \frac{B}{a}}{\log \lambda_r^{-1}} \geq 0 \quad \Rightarrow \quad \left(\frac{nh}{\lambda_r nh} \right)^\alpha = \frac{B}{a},$$

hence

$$a(nh)^\alpha = B(\lambda_r nh)^\alpha \geq B\big(\lfloor \lambda_r n \rfloor h\big)^\alpha.$$

Then, because of $\lambda_r > \lambda_i, i = 1, 2, \dots, r-1$, we get

$$a(nh)^\alpha \geq \sum_{i=1}^{r} |b_i| \big(\lfloor \lambda_i n \rfloor h \big)^\alpha,$$

which is the inequality (3) with $\rho_n = (nh)^\alpha$.

3. Main result

Theorem 1. *Let x_n be a solution of equation* (1)*, where $a, h > 0$, $b_i \neq 0$ and $\lambda_i, i = 1, 2, \dots, r$ are real scalars such that $0 < ah < 1$, $B \geq a$, $0 < \lambda_1 < \lambda_2 < \dots < \lambda_r < 1$. Then $x_n = O(n^\alpha)$ as $n \to \infty$, where $\alpha = \frac{\log \frac{B}{a}}{\log \lambda_r^{-1}}$.*

Proof. Let $\tilde{a}$ be such that $\tilde{a}^h = 1 - ah$. Then we can rewrite the difference equation (1) as

$$x_{n+1} = \tilde{a}^h x_n + h \sum_{i=1}^{r} b_i x_{\lfloor \lambda_i n \rfloor} \qquad n = 1, 2, 3, \dots \tag{4}$$

We introduce the substitution $y_n = \frac{x_n}{\rho_n}$ in (4) to obtain

$$\rho_{n+1} y_{n+1} = \tilde{a}^h \rho_n y_n + h \sum_{i=1}^{r} b_i \rho_{\lfloor \lambda_i n \rfloor} y_{\lfloor \lambda_i n \rfloor}, \qquad n = 1, 2, 3, \dots, \tag{5}$$

where ρ_n, $n = 1, 2, 3, \dots$ satisfies the inequality (3) with the properties mentioned in Remark 1. Our aim is to show that every solution y_n of (5) is bounded as $n \to \infty$.

Multiplying (5) by $1/\tilde{a}^{(n+1)h}$ we get

$$\frac{\rho_{n+1} y_{n+1}}{\tilde{a}^{(n+1)h}} = \frac{\rho_n y_n}{\tilde{a}^{nh}} + h \sum_{i=1}^{r} \frac{b_i}{\tilde{a}^{(n+1)h}} \rho_{\lfloor \lambda_i n \rfloor} y_{\lfloor \lambda_i n \rfloor},$$

i.e.,

$$\Delta \left(\frac{\rho_n y_n}{\tilde{a}^{nh}} \right) = h \sum_{i=1}^{r} \frac{b_i}{\tilde{a}^{(n+1)h}} \rho_{\lfloor \lambda_i n \rfloor} y_{\lfloor \lambda_i n \rfloor}. \tag{6}$$

Next, we split the interval $I = [0, \infty)$ into a sequence of intervals using the maximum proportional coefficient λ_r. We put $T_{-1} := 0$ and $T_m := \lambda_r^{-m}$, $m = 0, 1, 2, \dots$ Then we set $I_m := [T_{m-1}, T_m]$, hence the functions $\lambda_i t$, $i = 1, \dots, r$ are mapping I_{m+1} into $\bigcup_{j=0}^{m} I_j$ for all $m = 0, 1, 2, \dots$

Now we take any positive integer v such that $v \in I_{m+1}$, $m = 1, 2, \dots$ Then we denote $u := v - 1 - \lfloor v - T_m \rfloor$. Let us emphasize that $u \in \bigcup_{j=0}^{m} I_j$. Summing the equation (6) from u to $v-1$, we get

$$y_v = \frac{\rho_u \tilde{a}^{vh}}{\rho_v \tilde{a}^{uh}} y_u + \frac{\tilde{a}^{vh}}{\rho_v} \sum_{s=u}^{v-1} \frac{h}{\tilde{a}^{(s+1)h}} \left(\sum_{i=1}^{r} b_i \rho_{\lfloor \lambda_i s \rfloor} y_{\lfloor \lambda_i s \rfloor} \right).$$

Let us denote $M_m := \sup\left\{ |y_i|,\ i \in \bigcup_{j=0}^{m} I_j \right\}$. In accordance with (3) we obtain

$$\begin{aligned} |y_v| &\leq \frac{\rho_u \tilde{a}^{vh}}{\rho_v \tilde{a}^{uh}} |y_u| + \frac{\tilde{a}^{vh}}{\rho_v} \sum_{s=u}^{v-1} \frac{h}{\tilde{a}^{(s+1)h}} \left(\sum_{i=1}^{r} |b_i| \rho_{\lfloor \lambda_i s \rfloor} |y_{\lfloor \lambda_i s \rfloor}| \right) \\ &\leq \frac{\rho_u \tilde{a}^{vh}}{\rho_v \tilde{a}^{uh}} M_m + \frac{\tilde{a}^{vh}}{\rho_v} \sum_{s=u}^{v-1} \frac{h M_m}{\tilde{a}^{(s+1)h}} \left(\sum_{i=1}^{r} |b_i| \rho_{\lfloor \lambda_i s \rfloor} \right) \\ &\leq \frac{\rho_u \tilde{a}^{vh}}{\rho_v \tilde{a}^{uh}} M_m + \frac{\tilde{a}^{vh} M_m}{\rho_v} \sum_{s=u}^{v-1} \frac{1 - \tilde{a}^h}{\tilde{a}^{(s+1)h}} \rho_s . \end{aligned}$$

Using the relation $\frac{1-\tilde{a}^h}{\tilde{a}^{(s+1)h}} = \Delta \left(\frac{1}{\tilde{a}} \right)^{sh}$ we get

$$|y_v| \leq \frac{\rho_u \tilde{a}^{vh}}{\rho_v \tilde{a}^{uh}} M_m + \frac{\tilde{a}^{vh} M_m}{\rho_v} \sum_{s=u}^{v-1} \Delta \left(\frac{1}{\tilde{a}} \right)^{sh} \rho_s ,$$

and summing by parts we obtain

$$\begin{aligned} |y_v| &\leq \frac{\rho_u \tilde{a}^{vh}}{\rho_v \tilde{a}^{uh}} M_m + M_m \frac{\tilde{a}^{vh}}{\rho_v} \left(\frac{\rho_v}{\tilde{a}^{vh}} - \frac{\rho_u}{\tilde{a}^{uh}} - \sum_{s=u}^{v-1} \left(\frac{1}{\tilde{a}} \right)^{(s+1)h} \Delta \rho_s \right) \\ &= M_m \left(1 - \frac{\tilde{a}^{vh}}{\rho_v} \sum_{s=u}^{v-1} \left(\frac{1}{\tilde{a}} \right)^{(s+1)h} \Delta \rho_s \right). \end{aligned}$$

Because we can take $\rho_n = (nh)^\alpha$ as a positive and nondecreasing solution of (3), where $\alpha = \frac{\log \frac{B}{a}}{\log \lambda_r^{-1}}$, $B = \sum_{i=1}^{r} |b_i|$, then $\Delta \rho_n$ is nonnegative for $n = 1, 2, 3, \dots$, hence $|y_v| \leq M_m$. Since v was arbitrary such that $v \in I_{m+1} = [\lambda^{-m}, \lambda^{-m-1}]$, we have

$$M_{m+1} \leq M_m \leq M_1 .$$

Hence, the solution y_n is bounded. $\square$

4. Final remarks

Remark 2. To compare our result with a continuous case we utilize asymptotic estimate derived in [4, Theorem 4.1]. Although this estimate is formulated for a more general differential equation (including power coefficients) than (2), it is suitable also in our case. Under the above assumptions on coefficients a, b_i and λ_i, $i = 1, \dots, r$, the asymptotic estimate for solutions of equation (2) can be formulated as

$$z(t) = O(t^{\alpha}) \qquad \text{for} \quad t \to \infty, \tag{7}$$

where $\alpha = \frac{\log \frac{B}{a}}{\log \lambda_r^{-1}}$. One can see that the asymptotic estimates coincide in the continuous and discrete case.

Remark 3. The assumption on the stepsize h ($h < \frac{1}{a}$) enables us to preserve the correlation of asymptotic estimates in the discrete and continuous case. In other words, the estimates of solutions in the discrete case and the continuous case are expressed via the same function, resp. sequence (provided the stepsize h is sufficiently small). Note that the equation (2) is obtained from (1) by letting $h \to 0$ and in this case the inequality $ah < 1$ becomes trivial.

The following example just illustrates the application of the asymptotic estimate.

Example 1. Consider the initial value problem

$$\dot{z}(t) = -2z(t) - z\left(\frac{t}{5}\right) + 3z\left(\frac{t}{4}\right) + 2z\left(\frac{t}{3}\right), \qquad z(0) = 1, \quad t \in [0, \infty). \tag{8}$$

In accordance with (7) we have the asymptotic estimate

$$z(t) = O(t) \qquad \text{as} \quad t \to \infty.$$

In the discrete case we consider formula (1) in the form

$$x_{n+1} = (1 - 2h)x_n - hx_{\lfloor \frac{n}{5} \rfloor} + 3hx_{\lfloor \frac{n}{4} \rfloor} + 2hx_{\lfloor \frac{n}{3} \rfloor}, \qquad x_0 = 1, \quad n = 0, 1, 2, \dots$$

Then according to Theorem 1 we get asymptotic estimate $x_n = O(n)$ as $n \to \infty$ provided $h < \frac{1}{2}$. This corresponds to asymptotic estimate for solution of (8).

Open problems: The paper deals with the asymptotic estimate of solutions of difference equation (1) considering the coefficient a is positive. The application of Theorem 1 is mainly limited by assumptions $0 < ah < 1$ and $B \geq a$. The first one comes from the introduction of $\tilde{a}$ in the proof, and the second one arises from the utilization of solution ρ_n of (3) introduced in Remark 1. There is not discussed the case $B \leq a$ in this paper, hence the problem of asymptotic stability remains opened. Otherwise, the investigation of asymptotic estimates of solutions of equation (1) can be expanded in the sense of assumptions on coefficients a and b_i (e.g., a negative, considering non-constant coefficients), delays (considering not only proportional delays) or involving some forcing term.

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