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EXISTENCE OF ASYMPTOTICALLY PERIODIC SOLUTIONS OF SCALAR VOLTERRA DIFFERENCE EQUATIONS

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ABSTRACT. There is used a version of Schauder's fixed point theorem to prove the existence of asymptotically periodic solutions of a scalar Volterra difference equation. Along with the existence of asymptotically periodic solutions, sufficient conditions for the nonexistence of such solutions are derived. Results are illustrated on examples.

1. Introduction

We consider a Volterra difference equation

$$x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^{n} K(n,i)x(i), \tag{1}$$

where

$$n \in \mathbb{N} := \{0,1,2,\dots\}, \; a, b, x \colon \mathbb{N} \to \mathbb{R},\; K \colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$$

and $\mathbb{R}$ denotes the set of all real numbers. By a solution of equation (1) we mean a sequence $x \colon \mathbb{N} \to \mathbb{R}$ whose terms satisfy (1) for every $n \in \mathbb{N}$. Throughout this paper we will assume that sequences a and K are not identically equal to zero.

We will also adopt the customary notations

$$\sum_{i=k+s}^{k} \mathcal{O}(i) = 0, \qquad \prod_{i=k+s}^{k} \mathcal{O}(i) = 1,$$

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where k is an integer, s is a positive integer and "$\mathcal{O}$" denotes the function considered independently of whether it is defined for the arguments indicated or not.

Definition 1. Let ω be a positive integer. The sequence $y\colon \mathbb{N} \to \mathbb{R}$ is called ω-periodic if $y(n+\omega) = y(n)$ for all $n \in \mathbb{N}$. The sequence y is called asymptotically ω-periodic if there exist two sequences $u, v\colon \mathbb{N} \to \mathbb{R}$ such that u is ω-periodic, $\lim_{n\to\infty} v(n) = 0$ and $y(n) = u(n) + v(n)$ for all $n \in \mathbb{N}$.

The background for discrete Volterra equations can be found in the well known monograph [1] by Agarwal, as well as in Elaydi [3] and Kocić and Ladas [7].

Uniform asymptotic stability in linear Volterra difference equations was studied by Elaydi and Murakami in [4]. Periodic and asymptotically periodic solutions of linear difference equations were investigated, e.g., by Agarwal and Popenda in [2], and by Popenda and Schmeidel in [9, 10].

In [5] and [6], Furumochi considered the behavior of solutions of the following classes of Volterra difference equations

$$x(n+1) = a(n) - \sum_{i=0}^{n} D\big(n, i, x(i)\big)$$

and

$$x(n+1) = p(n) - \sum_{i=-\infty}^{n} P\big(n, i, x(i)\big),$$

and their inter-relations. Boundedness, attractivity, and convergence of solutions were investigated.

2. Asymptotically periodic solutions

In this section, sufficient conditions for the existence of an asymptotically ω-periodic solution of equation (1) are given. The following version of Schauder's fixed point theorem, which can be found in [8], will be used to prove the main result of this paper.

Lemma 1. *Let Ω be a Banach space and S its nonempty, closed and convex subset, and let T be a continuous mapping such that $T(S)$ is contained in S, and the closure $\overline{T(S)}$ is compact. Then T has a fixed point in S.*

Let ω be a positive integer and $b\colon \mathbb{N} \to \mathbb{R} \setminus \{0\}$ be ω-periodic. Then we define an ω-periodic function $\beta\colon \mathbb{N} \to \mathbb{R}$ as

$$\beta(n) = \begin{cases} \displaystyle\prod_{j=0}^{n-1} \frac{1}{b(j)} & \text{if} \quad n \geq 1, \\ \beta(\omega) & \text{if} \quad n = 0. \end{cases} \tag{2}$$

Further we define

$$m := \min\bigl\{|\beta(1)|, |\beta(2)|, \dots, |\beta(\omega)|\bigr\}$$

and

$$M := \max\bigl\{|\beta(1)|, |\beta(2)|, \dots, |\beta(\omega)|\bigr\}.$$

Theorem 1 (Main result). *Let ω be a positive integer and $b\colon \mathbb{N} \to \mathbb{R} \setminus \{0\}$ be ω-periodic. Assume that*

$$\prod_{i=0}^{\omega-1} b(i) = 1, \tag{3}$$

$$\sum_{i=0}^{\infty} |a(i)| < \infty \tag{4}$$

and

$$\sum_{j=0}^{\infty} \sum_{i=0}^{j} |K(j,i)| < \frac{m}{M}\,. \tag{5}$$

Then, for any nonzero constant c, there exists an asymptotically ω-periodic solution x of (1) such that

$$x(n) = u(n) + v(n), \qquad n \in \mathbb{N} \tag{6}$$

with

$$u(n) := c \prod_{k=0}^{n^*} b(k) \quad \text{and} \quad \lim_{n\to\infty} v(n) = 0 \tag{7}$$

where n^* is the remainder of dividing $n-1$ by ω.

Proof. We note that $n^* = n - 1 - \omega\left[\frac{n-1}{\omega}\right]$ where the function $[\,\cdot\,]$ is the greatest integer function.

From the ω-periodicity of sequence b, the definition of β by (2), and the property (3) we have

$$\beta(n) \in \bigl\{\beta(1),\, \beta(2), \dots, \beta(\omega)\bigr\}$$

for any $n \in \mathbb{N}$. Thus,

$$m \leq |\beta(n)| \leq M \tag{8}$$

for any $n \in \mathbb{N}$. Let $c > 0$. We set

$$\alpha(0) := \frac{M \sum_{i=0}^{\infty} |a(i)| + \frac{cM}{m} \sum_{j=0}^{\infty} \sum_{i=0}^{j} |K(j,i)|}{1 - \frac{M}{m} \sum_{j=0}^{\infty} \sum_{i=0}^{j} |K(j,i)|}$$

and

$$\alpha(n) := M \sum_{i=n}^{\infty} |a(i)| + \frac{\big(c + \alpha(0)\big)M}{m} \sum_{j=n}^{\infty} \sum_{i=0}^{j} |K(j,i)|\,,$$

for $n \geq 1$. It is easy to see that

$$\lim_{n \to \infty} \alpha(n) \;=\; 0. \tag{9}$$

We show, moreover, that

$$\alpha(n) \;\leq\; \alpha(0) \tag{10}$$

for any $n \in \mathbb{N}$. Let us first remark that

$$\alpha(0) \;= M \sum_{i=0}^{\infty} |a(i)| + \frac{\big(c + \alpha(0)\big)M}{m} \sum_{j=0}^{\infty} \sum_{i=0}^{j} |K(j,i)|\,.$$

Then, due to the convergence of series (4), (5), the inequality

$$\begin{aligned} \alpha(0) \;&= M \sum_{i=0}^{\infty} |a(i)| + \frac{\big(c + \alpha(0)\big)M}{m} \sum_{j=0}^{\infty} \sum_{i=0}^{j} |K(j,i)| \\ &\geq M \sum_{i=n}^{\infty} |a(i)| + \frac{\big(c + \alpha(0)\big)M}{m} \sum_{j=n}^{\infty} \sum_{i=0}^{j} |K(j,i)| = \alpha(n) \end{aligned}$$

obviously holds for every $n \in \mathbb{N}$ and (10) is proved.

Let B be the Banach space of all real bounded sequences $z \colon \mathbb{N} \to \mathbb{R}$ equipped with the usual supremum norm. We define a subset $S \subset B$ as

$$S := \big\{ z(n) \in B \colon c - \alpha(0) \leq z(n) \leq c + \alpha(0), \quad n \in \mathbb{N} \big\}.$$

It is not difficult to prove that S is a nonempty, bounded, convex, and closed subset of B.

Let us define a mapping $T \colon S \to B$ as follows

$$(Tz)(n) = c - \sum_{i=n}^{\infty} a(i)\beta(i+1) - \sum_{j=n}^{\infty} \sum_{i=0}^{j} \frac{\beta(j+1)}{\beta(i)} K(j,i) z(i) \tag{11}$$

for any $n \in \mathbb{N}$. We will prove that the mapping T has a fixed point in B. We first show that $T(S) \subset S$. Indeed, if $z \in S$, then

$$|z(n) - c| \le \alpha(0) \qquad \text{for} \quad n \in \mathbb{N},$$

and, by (8) and (11), we have

$$\begin{aligned} |(Tz)(n) - c| &\le M \sum_{i=n}^{\infty} |a(i)| + \frac{\big(c + \alpha(0)\big)M}{m} \sum_{j=n}^{\infty} \sum_{i=0}^{j} |K(j,i)| \\ &= \alpha(n) \le \alpha(0). \end{aligned} \tag{12}$$

Next we prove that T is continuous. Let $z^{(p)}$ be a sequence in S such that $z^{(p)} \to z$ as $p \to \infty$. Because S is closed, $z \in S$. Now, by (5), (8) and (11), we get

$$\begin{aligned} |(Tz^{(p)})(n) - (Tz)(n)| &= \left| \sum_{j=n}^{\infty} \sum_{i=0}^{j} \frac{\beta(j+1)}{\beta(i)} K(j,i) \big(z^{(p)}(n) - z(n)\big) \right| \\ &\le \frac{M}{m} \cdot \frac{m}{M} \cdot \sup_{i \ge 0} |z^{(p)}(i) - z(i)|, \qquad n \in \mathbb{N}. \end{aligned}$$

Therefore

$$\left| \Big(Tz^{(p)}\Big)(n) - (Tz)(n) \right| \le \sup_{i \ge 0} \left| z^{(p)}(i) - z(i) \right|, \qquad n \in \mathbb{N}$$

and

$$\lim_{p \to \infty} \left| Tz^{(p)} - Tz \right| = 0.$$

This means that T is continuous.

To prove that $\overline{T(S)}$ is compact, we take $\varepsilon > 0$. Then, from (9), we conclude that there exists $n_\varepsilon \in \mathbb{N}$ such that $\alpha(n) < \varepsilon$, for $n \ge n_\varepsilon$. We cover the segment $\big[c - \alpha(0), c + \alpha(0)\big]$ with a finite number k_ε of intervals each having a length of ε. Let the points c_{k_ε} be the centres of the ε-length intervals. We conclude that, for an arbitrarily small $\varepsilon > 0$, we can collect a finite set of intervals with centres at c_{k_ε} and with radii $\varepsilon/2$ which covers $\overline{T(S)}$. Hence $\overline{T(S)}$ is compact.

By Schauder's fixed point theorem (see Lemma 1), there exists a $z \in S$ such that

$$z(n) = (Tz)(n) \qquad \text{for } n \in \mathbb{N}.$$

Thus

$$\begin{aligned} z(n) = c &- \sum_{i=n}^{\infty} a(i)\beta(i+1) \\ &- \sum_{j=n}^{\infty} \sum_{i=0}^{j} \frac{\beta(j+1)}{\beta(i)} K(j,i) z(i) \qquad \text{for any } n \in \mathbb{N}. \end{aligned} \tag{13}$$

Due to (9) and (12), for fixed point $z \in S$ of T, we have

$$\lim_{n\to\infty} |z(n) - c| = \lim_{n\to\infty} |(Tz)(n) - c| \le \lim_{n\to\infty} \alpha(n) = 0$$

or, equivalently,

$$\lim_{n\to\infty} z(n) = c. \tag{14}$$

Finally, we will show that there exists a connection of the fixed point $z \in S$ with the existence of asymptotically ω-periodic solution of (1). Considering (13) for $z(n+1)$ and $z(n)$, we get

$$\Delta z(n) = a(n)\beta(n+1) + \sum_{i=0}^{n} \frac{\beta(n+1)}{\beta(i)} K(n,i)z(i), \qquad n \in \mathbb{N}.$$

Hence, by (2) (taking into account that $\beta(0) = \beta(\omega) = 1$ in view of (3)), we have

$$\begin{aligned} z(n+1) - z(n) &= a(n)\beta(n+1) + \frac{\beta(n+1)}{\beta(0)} K(n,0)z(0) \\ &\quad + \sum_{i=1}^{n} \frac{\beta(n+1)}{\beta(i)} K(n,i)z(i) \\ &= a(n) \prod_{k=0}^{n} \frac{1}{b(k)} + \left(\prod_{k=0}^{n} \frac{1}{b(k)} \right) K(n,0)z(0) \\ &\quad + \sum_{i=1}^{n} \left(\prod_{k=i}^{n} \frac{1}{b(k)} \right) K(n,i)z(i), \qquad n \in \mathbb{N}. \end{aligned} \tag{15}$$

Putting

$$z(n) = \left(\prod_{k=0}^{n-1} \frac{1}{b(k)} \right) x(n), \qquad n \in \mathbb{N} \tag{16}$$

in (15), we get equation (1) since

$$\begin{aligned} \frac{x(n+1)}{\prod\limits_{k=0}^{n} \frac{1}{b(k)}} - \frac{x(n)}{\prod\limits_{k=0}^{n-1} \frac{1}{b(k)}} &= a(n) \prod_{k=0}^{n} \frac{1}{b(k)} + \left(\prod_{k=0}^{n} \frac{1}{b(k)} \right) K(n,0) \left(\prod_{k=0}^{-1} \frac{1}{b(k)} \right) x(0) \\ &\quad + \sum_{i=1}^{n} \left(\prod_{k=i}^{n} \frac{1}{b(k)} \right) K(n,i) \left(\prod_{k=0}^{i-1} \frac{1}{b(k)} \right) x(i), \qquad n \in \mathbb{N} \end{aligned}$$

yields

$$x(n+1) = a(n) + b(n)x(n) + K(n,0)x(0) + \sum_{i=1}^{n} K(n,i)x(i), \qquad n \in \mathbb{N}.$$

Consequently, x defined by (16) is a solution of (1). From (14) and (16), we obtain

$$\left(\prod_{k=0}^{n-1}\frac{1}{b(k)}\right)x(n)=z(n)=c+o(1)$$

for $n\to\infty$ (where $o(1)$ is the Landau order symbol). Hence

$$x(n)=c\prod_{k=0}^{n-1}b(k)+\left(\prod_{k=0}^{n-1}b(k)\right)o(1),\qquad n\to\infty.$$

From (3) we get

$$\prod_{k=0}^{n-1}b(k)=\prod_{k=0}^{n^*}b(k).$$

The proof is complete since the sequence $\left\{\prod_{k=0}^{n^*}b(k)\right\}$ is ω-periodic and due to properties of Landau order symbols we have

$$\left(\prod_{k=0}^{n^*}b(k)\right)o(1)=o(1),\qquad n\to\infty.$$

If $c<0$, the proof, which we omit, can be carried out in a manner similar to the one used above if x is changed to $-x$. □

Remark 1. Tracing the proof of Theorem 1 we see that it remains valid even in the case of $c=0$. Then there exists an asymptotically "ω"-periodic solution x of (1) as well. The formula (6) reduces to

$$x(n)=v(n)=o(1),\qquad n\in\mathbb{N}.$$

From the point of view of Definition 1, we can consider this case as follows. We set (as a singular case) $u\equiv 0$ with an arbitrary (possibly other than "ω") period and with $v=o(1)$ for $n\to\infty$.

In the following example, a sequence b is 1-periodic. Then it is 2-periodic, too. By virtue of Theorem 1, there exists a 2-periodic solution of the equation in question.

Example 1. Put

$$a(n)=(-1)^{n+1}\cdot\frac{1}{3\cdot 2^{n+3}}+\frac{53}{48\cdot 2^{2n}}+\frac{1}{2^{3n+4}},$$

$$b(n)\equiv -1\quad\text{ and }\quad K(n,i)=\frac{2^i}{4^{n+2}}$$

in (1). We consider the sequence b as a 2-periodic sequence and put $\omega=2$. Obviously, $m=M=1$ and

$$\sum_{j=0}^{\infty}\sum_{i=0}^{j}K(j,i)=\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{2^i}{4^{j+2}}=\sum_{j=0}^{\infty}\frac{2^{j+1}-1}{4^{j+2}}=\frac{1}{6}<1.$$

Then all the assumptions of Theorem 1 are satisfied and (by (6), (7)) there exists an asymptotically 2-periodic solution

$$x(n) = u(n) + v(n), \qquad n \in \mathbb{N}$$

of the equation (1) where

$$u(n) = c \prod_{k=0}^{n^*} b(k) = c \prod_{k=0}^{n^*} (-1) = c(-1)^n, \qquad \lim_{n\to\infty} v(n) = 0.$$

Indeed, a sequence

$$x(n) = c(-1)^n + \frac{1}{4^n}$$

with $c = 1$ is such a solution.

3. Nonexistence of asymptotically periodic solutions

Finally, we present sufficient conditions for the nonexistence of asymptotically periodic solution of (1) satisfying some auxiliary conditions.

Let $x(n) = u(n) + v(n)$ be an asymptotically periodic solution of (1) such that the sequence u is ω-periodic and $\lim_{n\to\infty} v(n) = 0$.

Theorem 2. *If sequences $a\colon \mathbb{N} \to \mathbb{R}$ and $b\colon \mathbb{N} \to \mathbb{R}$ are bounded and there exists a positive integer ω such that*

$$K(n,i) = K(n+\omega, i+\omega) \tag{17}$$

for all $n, i \in \mathbb{N}$, then the equation (1) does not have any asymptotically ω-periodic solution $x(n) = u(n) + v(n)$ such that

$$\sum_{i=0}^{\omega-1} K(\omega-1, i)u(i) \neq 0 \tag{18}$$

and

$$\sum_{i=0}^{\infty} |v(i)| < \infty. \tag{19}$$

P r o o f. Suppose, on the contrary, that assumptions of Theorem 2 are satisfied and there exists an asymptotically ω-periodic solution x of equation (1) which satisfies conditions (18) and (19). Without loss of generality we may assume that

$$\sum_{i=0}^{\omega-1} K(\omega-1, i)u(i) > 0. \tag{20}$$

From (17) we have

$$K(n,i) = K\left(n - \omega\left[\frac{n}{\omega}\right], i - \omega\left[\frac{i}{\omega}\right]\right) = K\big((n+1)^*, (i+1)^*\big).$$

Because $x(n) = u(n) + v(n)$, from (17) and the ω-periodicity of the sequence u, we have

$$\begin{aligned}\sum_{i=0}^{n} K(n,i)x(i) &= \sum_{i=0}^{n} K\big((n+1)^*, (i+1)^*\big)u\big((i+1)^*\big) \\ &\quad + \sum_{i=0}^{n} K\big((n+1)^*, (i+1)^*\big)v(i) \\ &= \left[\frac{n}{\omega}\right] \cdot \sum_{i=0}^{\omega-1} K(\omega-1, i)u(i) \\ &\quad + \sum_{i=0}^{(n+1)^*} K\big((n+1)^*, i\big)u(i) \\ &\quad + \sum_{i=0}^{n} K\big((n+1)^*, (i+1)^*\big)v(i).\end{aligned}$$

By (20)

$$\limsup_{n\to\infty} \left[\frac{n}{\omega}\right] \cdot \sum_{i=0}^{\omega-1} K(\omega-1, i)u(i) = \infty.$$

We remark that the sum

$$\sum_{i=0}^{(n+1)^*} K\big((n+1)^*, i\big)u(i)$$

is bounded (for $n \to \infty$) and there exists a positive constant K^{**} such that

$$\left|K\big((n+1)^*, (i+1)^*\big)\right| \le K^{**}$$

for all $n, i \in \mathbb{N}$. Then

$$\left|K\big((n+1)^*, (i+1)^*\big)\right| |v(i)| \le K^{**} |v(i)|$$

and, by (19), the series

$$\sum_{i=0}^{\infty} K\big((n+1)^*, (i+1)^*\big)v(i)$$

is absolutely convergent. Thus

$$\limsup_{n\to\infty} \sum_{i=0}^{n} K(n,i)x(i) = \infty.$$

Rewriting (1), we get

$$x(n+1) - a(n) - b(n)x(n) = \sum_{i=0}^{n} K(n,i)x(i),$$

where the left-hand side of the above equation is bounded while the right-hand side is unbounded. This contradiction completes the proof. □

Remark 2. We will emphasize the necessity of (18) in Theorem 2. If

$$\sum_{i=0}^{\omega-1} K(\omega-1,i)u(i) = 0$$

then (1) can have an asymptotically ω-periodic solution.

Let, e.g., $K(j,i) = \big(1+(-1)^i\big)/2$. Then, taking sequences a and b in (1) in a proper manner, the equation (1) will have an asymptotically 4-periodic solution $x(n) = u(n) + v(n)$ with 4-periodic function $u(n) := (0,1,0,2,\dots)$. In this case

$$\sum_{i=0}^{\omega-1} K(\omega-1,i)u(i) = \sum_{i=0}^{\omega-1} \frac{1+(-1)^i}{2} u(i) = 1\cdot 0 + 0\cdot 1 + 1\cdot 0 + 0\cdot 2 = 0$$

and (18) does not hold. Then

$$\limsup_{n\to\infty} \left[\frac{n}{\omega}\right] \cdot \sum_{i=0}^{\omega-1} K(\omega-1,i)u(i) = 0$$

and we do not get the final contradiction in the proof of Theorem 2.

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