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MULTIPLICATIVE INVERSE CUBIC FUNCTIONAL EQUATIONS IN FUZZY BANACH SPACES

Abasalt Bodaghi[1] — Beri Venkatachalapathy Senthil Kumar[2]

[1]Department of Mathematics, WT.C., Islamic Azad University, Tehran, Iran

[2]Department of Mathematics, C. Abdul Hakeem College of Engineering and Technology, Melvisharam, Tamil Nadu, India

ABSTRACT. In this article, two novel multiplicative inverse cubic functional equations are proposed. The stability results in the fuzzy version of these functional equations are demonstrated. In other words, some approximations for mappings from a restricted domain of real numbers into complete fuzzy normed spaces are established. The real-life interpretation of the equations dealt in this article is connected with the lubrication theory to model fluid film forces through inverse cube law.

1. Introduction

Zadeh [20] proposed the idea of fuzzy sets in 1965, expanding on the traditional concept of sets by permitting elements to have degrees of membership instead of a rigid binary classification. Fuzzy sets assign a membership value in the interval $[0, 1]$, which indicates the degree to which an element meets a particular property, in contrast to crisp sets, where an element either belongs or does not belong to a set. Many real-world systems and human thought processes are characterized by uncertainty, ambiguity, and imprecision, all of which can be naturally modeled using this paradigm. Fuzzy set theory has become a foundational tool in areas such as fuzzy logic, approximate reasoning, decision-making, control systems, and pattern recognition. Its capacity to comprehend linguistic concepts like "high", "low", "moderate", or "near" makes it especially useful for creating intelligent systems that communicate with imprecise, complex settings.

The problem that is concerning to finding or existence an exact solution close to an approximate solution of a functional equation is said to be the Ulam's stability of the functional equations [19]. Ulam's problem was answered positively by Hyers [6], assuming that the groups are Banach spaces and that the

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homomorphisms are additive mappings; see also [15] for the stability of linear mappings on normed spaces. Following the functional equations whose solutions are monomials, Ravi and Senthil Kumar [16] defined the reciprocal functional equation for the first time as follows:

$$R(r_1 + r_2) = \frac{R(r_1)R(r_2)}{R(r_1) + R(r_2)}. \tag{1.1}$$

They obtained some stability results of a mapping on the space of nonzero real numbers. Obviously, $R(r) = \frac{c}{r}$ is a solution of the functional equation (1.1). Next, Jung [7] studied the stability of (1.1) by using a fixed point technique.

Bodaghi and Kim [3] introduced the quadratic reciprocal functional equation

$$\mathcal{R}_q(2x + y) + \mathcal{R}_q(2x - y) = \frac{2\mathcal{R}_q(x)\mathcal{R}_q(y)[4\mathcal{R}_q(y) + \mathcal{R}_q(x)]}{(4\mathcal{R}_q(y) - \mathcal{R}_q(x))^2} \tag{1.2}$$

for the first time. They also studied the Găvruta [5] and Rassias [15] stabilities for (1.2) on non-zero real numbers; for generalized form of (1.2), we refer to [2].

The first reciprocal-cubic functional equation was introduced by Kim et al. [9]. In [17], Senthil Kumar et al. proposed the multiplicative inverse quadratic functional equation which involves the arguments in rational form as follows:

$$Q\left(\frac{rs}{2r + s}\right) + Q\left(\frac{rs}{2r - s}\right) = 2Q(r) + 8Q(s),$$

where $Q\colon S \longrightarrow \mathbb{R}$ is a function with $S = \{r, s \in \mathbb{R} : r, s \neq 0, 2r + s \neq 0, 2r - s \neq 0\}$. We remind that the solution and various stabilities of a rational functional equation involving two variables by means of fixed point technique in the vicinity of Felbin's type fuzzy normed spaces were established in [18]. Many mathematicians dealt with a number of intriguing and motivating stability results of different functional equations in fuzzy version [8, 10, 12, 13, 14]. Latterly, various forms of rational or multiplicative inverse functional equations pertinent to multi-variable functions were dealt with in [1, 4], and their stability results were studied in different spaces.

In this paper, we consider two multiplicative inverse cubic functional equations as follows:

$$C\left(\frac{pq}{2p + q}\right) + C\left(\frac{pq}{p + 2q}\right) = 3C(p) + 3C(q) + 48C\left(\frac{2pq}{p + q}\right) \tag{1.3}$$

defined on $\mathcal{X}_1 = \{p, q \in \mathbb{R} : p, q \neq 0, 2p + q \neq 0, p + 2q \neq 0, p + q \neq 0\}$, and

$$2C\left(\frac{pq}{2p + q}\right) + 2C\left(\frac{pq}{2p - q}\right) = C\left(\frac{pq}{p + 2q}\right) + C\left(\frac{pq}{p - 2q}\right) + 30C(q) \tag{1.4}$$

defined on $\mathcal{X}_2 = \{p, q \in \mathbb{R} : p, q \neq 0, 2p+q \neq 0, p+2q \neq 0, 2p-q \neq 0, p-2q \neq 0\}$. It is easy to verify that the reciprocal-cubic mapping $C(p) = \frac{c}{p^3}$ is a solution of the functional equations (1.3) and (1.4). The fuzzy version of the stability

results of the functional equations (1.3) and (1.4) is demonstrated in Section 2. We also present the real-life interpretation of the functional equation (1.3) in lubrication theory in modeling fluid film forces via inverse cube law. At the end of this study, we discuss the concluding remarks on the results obtained.

2. Fuzzy stability results of the functional equation (1.3)

The main findings of this study are derived from the following basic notion of fuzzy Banach space.

Definition 2.1 ([11]). Let $\mathcal{P}$ be a vector space over the field of reals. For each $p, q \in \mathcal{P}$, and each $\alpha, \beta \in \mathbb{R}$, a fuzzy norm on $\mathcal{P}$ is a function $\mathsf{F}\colon \mathcal{P} \times \mathbb{R} \longrightarrow [0,1]$ satisfying the following:

(N_1) For any $\beta > 0$, $\mathsf{F}(p, \beta) = 0$.

(N_2) $p = 0$ if and only if $\mathsf{F}(p, \beta) = 1$ for every $\beta > 0$.

(N_3) If $\ell \neq 0$, then $\mathsf{F}(\ell p, \beta) = \mathsf{F}\left(p, \frac{\beta}{|\ell|}\right)$.

(N_4) $\mathsf{F}(p + q, \alpha + \beta) \geq \min\{\mathsf{F}(p, \alpha), \mathsf{F}(q, \beta)\}$.

(N_5) Non-decreasing functions of $\mathbb{R}$ include $\mathsf{F}(p, \cdot)$ and $\lim\limits_{\beta \to \infty} \mathsf{F}(p, \beta) = 1$.

(N_6) For $p \neq 0$, $\mathsf{F}(p, \cdot)$ is continuous on $\mathbb{R}$.

It is claimed that the pair $(\mathcal{P}, \mathsf{F})$ is a *fuzzy normed vector space*.

Definition 2.2 ([11]). Let $(\mathcal{P}, \mathsf{F})$ be a vector space defined with fuzzy norm. If there is an $p \in \mathcal{P}$ such that $\lim\limits_{n \to \infty} \mathsf{F}(p_n - p, \beta) = 1$ for all $\beta > 0$, then the sequence $\{p_n\}$ in $\mathcal{P}$ is *convergent* or *converges*. The limit of the sequence $\{p_n\}$ in this case is p, which we represent by $\mathsf{F}\text{-}\lim\limits_{n \to \infty} p_n = p$.

Definition 2.3 ([11]). A fuzzy normed vector space $(\mathcal{P}, \mathsf{F})$ is considered. If there exists an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and all $k > 0$, we have $\mathsf{F}(p_{n+k} - p_n, \beta) > 1 - \epsilon$, then a sequence $\{p_n\}$ in $\mathcal{P}$ is Cauchy.

Every convergent sequence in a fuzzy normed vector space is known to be Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete*, and the fuzzy normed vector space is called a *fuzzy Banach space*.

If the sequence $\{f(p_n)\}$ converges to $f(p_0)$ for every sequence $\{p_n\}$ convergent to p_0 in $\mathcal{P}_1$, then we say that a mapping $g\colon \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ between fuzzy normed vector spaces $\mathcal{P}_1$ and $\mathcal{P}_2$ is continuous at a point $p_0 \in \mathcal{P}_1$. In addition, g is continuous if it is continuous at every point of $\mathcal{P}_1$.

For every $p \in \mathcal{P}_1$, if $g\colon \mathcal{P}_1 \longrightarrow \mathcal{P}_2$, then $h\colon \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is *continuous* on $\mathcal{P}_1$.

Let us assume that $(\mathcal{F}, \mathsf{F})$ is a complete fuzzy normed space (or fuzzy Banach space). In the ensuing results, we demonstrate various stabilities of the functional

equation (1.3) related to Ulam stability theory in the context of complete fuzzy normed spaces. In order to facilitate comprehension of the results, let us assign operators for differences by $\Delta_1, \Delta_2 \colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{F}$ via

$$\Delta_1 C(p,q) := C\left(\frac{pq}{2p+q}\right) + C\left(\frac{pq}{p+2q}\right) - 3C(p) - 3C(q) - 48C\left(\frac{2pq}{p+q}\right)$$

for all $p, q \in \mathcal{X}_1$, and

$$\Delta_2 C(p,q) := 2C\left(\frac{pq}{2p+q}\right) + 2C\left(\frac{pq}{2p-q}\right) - C\left(\frac{pq}{p+2q}\right) - C\left(\frac{pq}{p-2q}\right) - 30C(q)$$

for all $p, q \in \mathcal{X}_2$. Since the arguments are in rational form in the functional equations (1.3) and (1.4), we assume that the vector space $\mathbb{R}$ over the field of reals is the domain of a mapping C. Also, we assume in the following theorems and corollaries that $C \colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{F}$ is a mapping.

The next theorem is the main result of the present paper which is the Găvruţa stability of reciprocal-cubic functional equation (1.3).

Theorem 2.4. *Suppose $\zeta \colon \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$ is a general control function satisfying the condition*

$$\tilde{\zeta}(p,q) = \frac{1}{54} \sum_{r=1}^{\infty} 3^{3r} \zeta\left(3^r p, 3^r q\right) < \infty \tag{2.1}$$

for all $p, q \in \mathbb{R}$. If $C \colon \mathbb{R} \longrightarrow \mathcal{F}$ satisfies the following condition:

$$\lim_{\beta \to \infty} \mathsf{F}(\Delta_1 C(p,q), \beta\zeta(p,q)) = 1 \tag{2.2}$$

uniformly on $\mathbb{R} \times \mathbb{R}$, then, for each $p \in \mathbb{R}$, $R_c(p) = \mathsf{F}\text{-}\lim_{r \to \infty} 3^{3r} C\left(3^r p\right)$ exists, and the mapping $R_c \colon \mathbb{R} \longrightarrow \mathcal{F}$ is reciprocal-cubic such that if, for some $\alpha > 0$,

$$\mathsf{F}(\Delta_1 C(p,q), \zeta(p,q)) \geq \alpha \tag{2.3}$$

for all $p, q \in \mathbb{R}$, then

$$\mathsf{F}\left(C(p) - R_c(p), \widetilde{\zeta}(p,p)\right) \geq \alpha \tag{2.4}$$

for all $p \in \mathbb{R}$. Moreover, the reciprocal-cubic mapping $R_c \colon \mathbb{R} \longrightarrow \mathcal{F}$ is unique such that

$$\lim_{\beta \to \infty} \mathsf{F}(C(p) - R_c(p), \beta\tilde{\zeta}(p,p)) = 1 \tag{2.5}$$

uniformly on $\mathbb{R}$.

P r o o f. Using (2.2), we may determine a $\beta_0 > 0$ such that, for a given $\epsilon > 0$,

$$\mathsf{F}(\Delta_1 C(p,q), \beta\zeta(p,q)) \geq 1 - \epsilon \tag{2.6}$$

for all $\beta \geq \beta_0$. Putting $p = q$ in (2.2), we obtain

$$\mathsf{F}\Big(C(p) - 3^3 C\left(3p\right), \frac{1}{2}\beta\zeta\left(3p, 3p\right)\Big) \geq 1 - \epsilon \tag{2.7}$$

for all $p \in \mathbb{R}$. Using a positive integer m and the mathematical induction process, we demonstrate that

$$\mathsf{F}\Big(C(p) - 3^{3m}C\left(3^m p\right), \frac{1}{54}\beta\sum_{s=1}^{m} 3^{3s}\zeta\left(3^s p, 3^s p\right)\Big) \geq 1-\epsilon \tag{2.8}$$

for all $\beta \geq \beta_0$, all $p \in \mathbb{R}$, and all $m \in \mathbb{N}$. By (2.7), the result is valid for $m = 1$. In addition, (2.7) implies that

$$\mathsf{F}\Big(3^{3m}C\left(3^m p\right) - 3^{3(m+1)}C\left(3^{m+1}p\right), \frac{1}{2}\beta 3^{3m}\zeta\left(3^{m+1}p, 3^{m+1}p\right)\Big) \geq 1-\epsilon. \tag{2.9}$$

Assume that for $m \in \mathbb{N}$, (2.8) holds. By (2.8) and (2.9), we have

$$\begin{aligned}
&\mathsf{F}\Big(C(p) - 3^{3(m+1)}C\left(3^{m+1}p\right), \frac{1}{54}\beta\sum_{s=1}^{m+1} 3^{3s}\zeta\left(3^s p, 3^s p\right)\Big)\\
&\quad\geq \min\Big\{\mathsf{F}\Big(C(p) - 3^{3m}C\left(3^m p\right), \frac{1}{54}\beta\sum_{s=1}^{m} 3^{3s}\zeta\left(3^s p, 3^s p\right)\Big),\\
&\qquad\qquad \mathsf{F}\Big(3^{3m}C\Big(3^m p\Big) - 3^{3(m+1)}C\left(3^{m+1}p\right), \frac{1}{2}\beta 3^{3m}\zeta\left(3^{m+1}p, 3^{m+1}p\right)\Big)\Big\}\\
&\quad\geq \min\{1-\epsilon, 1-\epsilon\} = 1-\epsilon.
\end{aligned}$$

The induction argument is now complete. Letting $\beta = \beta_0$ and substituting p and $3^k p$ in (2.8), we get

$$\mathsf{F}\Big(3^{3k}C\left(3^k p\right) - 3^{3(m+k)}C\left(3^{m+k}p\right), \frac{1}{2}\beta_0\sum_{s=1}^{m} 3^{3(s+k-1)}\zeta\left(3^{k+s}p, 3^{k+s}p\right)\Big) \geq 1-\epsilon \tag{2.10}$$

for all positive integers m and k. By the equality

$$\sum_{s=1}^{m} 3^{3(s+k-1)}\zeta\left(3^{k+s}p, 3^{k+s}p\right) = \sum_{s=k+1}^{m+k} 3^{3(s-1)}\zeta\left(3^s p, 3^s p\right), \tag{2.11}$$

and (2.1), it follows that, for a given positive δ, there is a positive integer m_0 such that, for all $m \geq m_0$ and $k > 0$,

$$\frac{1}{2}\beta_0\sum_{s=k+1}^{m+k} 3^{3(s-1)}\zeta\left(3^s p, 3^s p\right) < \delta. \tag{2.12}$$

We now deduce from (2.10), (2.11) and (2.12) that, for all $m \geq m_0$ and all $k > 0$,

$$\begin{aligned}
&\mathsf{F}\left(3^{3k}C\left(3^k p\right) - 3^{3(m+k)}C\left(3^{m+k}p\right), \delta\right)\\
&\geq \mathsf{F}\Big(3^{3k}C\left(3^k p\right) - 3^{3(m+k)}C\left(3^{m+k}p\right), \frac{1}{2}\beta_0\sum_{s=1}^{m} 3^{3(s+k-1)}\zeta\left(3^{k+s}p, 3^{k+s}p\right)\Big) \geq 1-\epsilon.
\end{aligned} \tag{2.13}$$

Thus, in $\mathcal{F}$, the sequence $\left\{3^{3m}C\left(3^m p\right)\right\}$ transforms into Cauchy. Since $\mathcal{F}$ is a fuzzy Banach space, the mentioned sequence converges to some $R_c(p) \in \mathcal{F}$. Thus, we may use $R_c(p) = \mathsf{F}\text{-}\lim_{n\to\infty} 3^{3m}C\left(3^m p\right)$ to establish a mapping $R_c\colon \mathbb{R} \longrightarrow \mathcal{F}$. For any $\beta > 0$ and $p \in \mathbb{R}$, $\lim_{m\to\infty} \mathsf{F}\left(3^{3m}C\left(3^m p\right) - R_c(p), \beta\right) = 1$.

Let $p, q \in \mathbb{R}$. Fix $0 < \epsilon < 1$ and $\beta > 0$. Given that $\lim_{m\to\infty} 3^{3m}\zeta\left(3^m p, 3^m p\right) = 0$, there is an $m_1 > m_0$ such that $\beta_0\zeta\left(3^m p, 3^m p\right) < \frac{3^{-3m}\beta}{3}$ for all $m \geq m_1$. Consequently, for any $m \geq m_1$, we have

$$\begin{aligned}
\mathsf{F}(\Delta_1 C(p,q), \beta) \geq \min\Bigg\{ & \mathsf{F}\bigg(R_c\Big(\frac{pq}{2p+q}\Big) - 3^{3m}C\Big(3^m\frac{pq}{2p+q}\Big), \frac{\beta}{6}\bigg), \\
& \mathsf{F}\bigg(\Big(R_c\Big(\frac{pq}{p+2q}\Big) - 3^{3m}C\Big(3^m\frac{pq}{p+2q}\Big)\Big), \frac{\beta}{6}\bigg), \\
& \mathsf{F}\bigg(3R_c(p) - 3^{3m+1}C\Big(3^m p\Big), \frac{\beta}{6}\bigg), \mathsf{F}\bigg(3R_c(q) - 3^{3m+1}C\Big(3^m q\Big), \frac{\beta}{6}\bigg), \\
& \mathsf{F}\bigg(48R_c\Big(\frac{2pq}{p+q}\Big) - 3^{3m}\cdot 48C\Big(\frac{2\cdot 3^m pq}{p+q}\Big), \frac{\beta}{6}\bigg), \\
& \mathsf{F}\bigg(3^{3m}\Big(C\Big(\frac{3^m pq}{2p+q}\Big) + C\Big(\frac{3^m pq}{p+2q}\Big) \\
& \quad - 3C(3^m p) - 3C(3^m q) - 48C\Big(\frac{2\cdot 3^m pq}{p+q}\Big)\Big), \frac{\beta}{6}\bigg)\Bigg\}.
\end{aligned}$$

The first five terms on the right side of the above inequality approach 1 as $m \to \infty$, but the 6th term is larger than or equal to

$$\mathsf{F}\left(C\left(\frac{3^m pq}{2p+q}\right) + C\left(\frac{3^m pq}{p+2q}\right) - 3C\left(3^m p\right) - 3C\left(3^m q\right) - 48C\left(\frac{2\cdot 3^m pq}{p+q}\right), \frac{3^{-3m}\beta}{6}\right),$$

whose value is $1-\epsilon$ or more. Hence,

$$\mathsf{F}(\Delta_1 R_c(p,q), \beta) \geq 1 - \epsilon$$

for all $\beta > 0$, and so, $\mathsf{F}(\Delta_1 R_c(p,q), \beta) = 1$ for all $\beta > 0$ and all $p, q \in \mathbb{R}$. The mapping $R_c\colon \mathbb{R} \longrightarrow \mathcal{F}$ is therefore reciprocal-cubic, meaning that, for any $p, q \in \mathbb{R}$, $\Delta_1 R_c(p,q) = 0$. Let

$$\zeta_m(p,p) = \frac{1}{54}\sum_{s=1}^{m} 3^{3s}\zeta\left(3^s p, 3^s p\right)$$

for all $p \in \mathbb{R}$. Let $p \in \mathbb{R}$. From (2.3), one can infer by using the same logic as in (2.13) that

$$\mathsf{F}\left(C(p) - 3^{3m}C\left(3^m p\right), \zeta_m(p,p)\right) \geq \alpha \tag{2.14}$$

for all integers m which are positive. Let $\beta > 0$. We now have

$$\begin{aligned}
& \mathsf{F}(C(p) - R_c(p), \zeta_m(p,p) + \beta) \\
& \quad \geq \min\left\{\mathsf{F}\left(C(p) - 3^{3m}C\left(3^m p\right), \zeta_m(p,p)\right), \mathsf{F}\left(3^{3m}C\left(3^m p\right) - R_c(p), \beta\right)\right\}.
\end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15), and using the fact that

$$\lim_{m\to\infty} \mathsf{F}\left(3^{3m}C\left(3^m p\right) - R_C(p), \beta\right) = 1,$$

we note that

$$\mathsf{F}(C(p) - R_c(p), \zeta_m(p,p) + \beta) \geq \alpha$$

for sufficiently large $m \in \mathbb{N}$. We notice that $\mathsf{F}\big(C(p) - R_c(p), \tilde{\zeta}(p,p) + \beta\big) \geq \alpha$ because the function $\mathsf{F}(C(p) - R_c(p), \cdot)$ is continuous. Taking $\beta \to 0$ in the last inequality, we find (2.4). We now need to show that the mapping is unique in order to wrap up the evidence. Assume that there is another reciprocal-cubic mapping R_c' that also meets the specified criterion (2.5). Make sure $c > 0$. By (2.5), for R_c and R_c', there exists a $\beta_0 > 0$ such that, for each $\epsilon > 0$,

$$\mathsf{F}\left(C(p) - R_c(p), \beta\tilde{\zeta}(p,p)\right) \geq 1 - \epsilon,$$
$$\mathsf{F}\left(C(p) - R_c'(p), \beta\tilde{\zeta}(p,p)\right) \geq 1 - \epsilon$$

for all $\beta \geq 2\beta_0$ and all $p \in \mathbb{R}$. Determine some integer m_0 and fix some $p \in \mathbb{R}$ so that

$$\beta_0 \sum_{s=m}^{\infty} 3^{3s}\zeta\left(3^s p, 3^s p\right) < \frac{c}{2} \tag{2.16}$$

for all $m \geq m_0$. Since

$$\begin{aligned}\sum_{s=m}^{\infty} 3^{3s}\zeta\left(3^s p, 3^s p\right) &= 3^{3m}\sum_{s=m}^{\infty} 3^{3(s-m)}\zeta\left(3^{s-m}\left(3^m p\right), 3^{s-m}\left(3^m p\right)\right) \\ &= 3^{3m}\sum_{s=0}^{\infty} 3^{3s}\zeta\left(3^s\left(3^m p\right), 3^s\left(3^m p\right)\right) = 3^{3m}\tilde{\zeta}\left(3^m p, 3^m p\right),\end{aligned} \tag{2.17}$$

we have

$$\begin{aligned}&\mathsf{F}(R_c(p) - R_c'(p), c) \\ &\quad\geq \min\left\{\mathsf{F}\left(3^{3m}C\left(3^m p\right) - R_c(p), \frac{c}{2}\right), \mathsf{F}\left(R_c'(p) - 3^{3m}C\left(3^m p\right), \frac{c}{2}\right)\right\} \\ &\quad= \min\left\{\mathsf{F}\left(C\left(3^m p\right) - R_c\left(3^m p\right), 3^{-3m}\frac{c}{2}\right), \mathsf{F}\left(R_C'\left(3^m p\right) - C\left(3^m p\right), 3^{-3m}\frac{c}{2}\right)\right\} \\ &\quad\geq \min\left\{\mathsf{F}\Big(C\left(3^m p\right) - R_c\left(3^m p\right), 3^{-3m}\beta_0\sum_{s=m}^{\infty} 3^{3s}\zeta\left(3^s p, 3^s p\right)\Big),\right. \\ &\qquad\left.\mathsf{F}\Big(R_c'\left(3^m p\right) - C\left(3^m p\right), 3^{-3m}\beta_0\sum_{s=m}^{\infty} 3^{3s}\zeta\left(3^s p, 3^s p\right)\Big)\right\} \geq 1 - \epsilon.\end{aligned} \tag{2.18}$$

Consequently, by (2.16), (2.17) and (2.18), we have $\mathsf{F}(R_c(p) - R_c'(p), c) = 1$ for all $c > 0$. Therefore, $R_c(p) = R_c'(p)$ for all $p \in \mathbb{R}$, which finishes the proof. □

In the following corollary, we prove the stability of the reciprocal-cubic functional equation (1.3) in sense of Rassias.

Corollary 2.5. *Let $\lambda \geq 0$ and ℓ be a real number such that $\ell < -3$. Suppose that $C\colon \mathbb{R} \longrightarrow \mathcal{F}$ satisfies the equality*

$$\lim_{\beta\to\infty} \mathsf{F}\Big(\Delta_1 C(p,q), \beta\lambda\left(\|p\|^{\ell} + \|q\|^{\ell}\right)\Big) = 1$$

uniformly on $\mathbb{R}\times\mathbb{R}$. Then, for each $p \in \mathbb{R}$, $R_c(p) = \mathsf{F}\text{-}\lim_{r\to\infty} 3^{3r} C\left(3^r p\right)$ exists and the mapping $R_c\colon \mathbb{R} \longrightarrow \mathcal{F}$ is reciprocal-cubic such that if, for some $\alpha > 0$,

$$\mathsf{F}\Big(\Delta_1 C(p,q), \lambda\left(\|p\|^{\ell} + \|q\|^{\ell}\right)\Big) \geq \alpha$$

for all $p, q \in \mathbb{R}$, then

$$\mathsf{F}\Big(C(p) - R_c(p), \frac{\lambda}{3^{-\ell} - 27}\|p\|^{\ell}\Big) \geq \alpha$$

for all $p \in \mathbb{R}$. Furthermore, the uniqueness of the reciprocal-cubic mapping $R_c\colon \mathbb{R} \longrightarrow \mathcal{F}$ means that

$$\lim_{\beta\to\infty} \mathsf{F}\Big(C(p) - R_c(p), \frac{\beta\lambda}{3^{-\ell} - 27}\|p\|^{\ell}\Big) = 1$$

uniformly on $\mathbb{R}$.

P r o o f. Taking into consideration $\zeta(p,q) = \lambda\left(\|p\|^{\ell} + \|q\|^{\ell}\right)$, and proceeding with comparable arguments as in Theorem 2.4, we obtain the desired result. □

This theorem follows a similar line of reasoning to Theorem 2.4 and is a dual outcome of that theorem. As a result, we just outline the key steps in the evidence.

Theorem 2.6. *Suppose $\zeta\colon \mathbb{R}\times\mathbb{R} \longrightarrow [0,\infty)$ is a general control with the condition*

$$\tilde{\zeta}(p,q) = \frac{1}{54}\sum_{r=0}^{\infty} 3^{-3r}\zeta\left(3^{-r}p, 3^{-r}q\right) < \infty$$

for all $p, q \in \mathbb{R}$. Let $C\colon \mathbb{R} \longrightarrow \mathcal{F}$ satisfy (2.2). Then, for every $p \in \mathbb{R}$, $R_c(p) = \mathsf{F}\text{-}\lim_{r\to\infty} 3^{-3r} C\left(3^{-r}p\right)$ exists and the mapping $R_C\colon \mathbb{R} \longrightarrow \mathcal{F}$ is reciprocal-cubic such that if, for some $\alpha > 0$,

$$\mathsf{F}\Big(\Delta_1 C(p,q), \zeta(p,q)\Big) \geq \alpha \tag{2.19}$$

for all $p, q \in \mathbb{R}$, then

$$\mathsf{F}\left(C(p) - R_c(p), \tilde{\zeta}\left(p,p\right)\right) \geq \alpha$$

for all $p \in \mathbb{R}$. Additionally, the uniqueness of the reciprocal-cubic mapping $R_c\colon \mathbb{R} \longrightarrow \mathcal{F}$ is that

$$\lim_{\beta\to\infty} \mathsf{F}\left(C(p) - R_c(p), \beta\tilde{\zeta}\left(p,p\right)\right) = 1$$

uniformly on $\mathbb{R}$.

P r o o f. Given $\epsilon > 0$, there exists a $\beta_0 > 0$ such that (2.6) holds for any $\beta \geq \beta_0$ by (2.2). Similar to the proof of Theorem 2.4, by induction, one can show that, for any integer $m > 0$,

$$\mathsf{F}\Big(C(p) - 3^{-3m}C\left(3^{-m}p\right), \frac{1}{54}\beta\sum_{s=0}^{m-1} 3^{-3s}\zeta\left(3^{-s}p, 3^{-s}p\right)\Big) \geq 1 - \epsilon$$

for all $\beta \geq \beta_0$, all $p \in \mathbb{R}$, and all $n \in \mathbb{N}$. The remaining steps in the proof are left out since they are comparable to those in Theorem 2.4. □

A reciprocal-cubic functional equation possesses the Hyers stability as follows.

Corollary 2.7. *Let $\lambda > 0$. If $C\colon \mathbb{R} \longrightarrow \mathcal{F}$ satisfies*

$$\lim_{\beta\to\infty} \mathsf{F}(\Delta_1 C(p, q), \beta\epsilon) = 1$$

uniformly on $\mathbb{R} \times \mathbb{R}$, then, for each $p \in \mathbb{R}$, $R_c(p) = \mathsf{F}\text{-}\lim_{r\to\infty} 3^{-3r}C\left(3^{-r}p\right)$ exists and the mapping $R_c\colon \mathbb{R} \longrightarrow \mathcal{F}$ is reciprocal-cubic such that if, for some $\alpha > 0$,

$$\mathsf{F}(\Delta_1 C(p, q), \lambda) \geq \alpha$$

for all $p, q \in \mathbb{R}$, then

$$\mathsf{F}\left(C(p) - R_c(p), \frac{\lambda}{52}\right) \geq \alpha$$

for all $p \in \mathbb{R}$. Moreover, the uniqueness of the reciprocal-cubic mapping $R_c\colon \mathbb{R} \longrightarrow \mathcal{F}$ implies that

$$\lim_{\beta\to\infty} \mathsf{F}\left(C(p) - R_c(p), \frac{\beta\lambda}{52}\right) = 1$$

uniformly on $\mathbb{R}$.

P r o o f. Letting $\zeta(p, q) = \lambda$ in Theorem 2.4 yields the proof. □

In the next result, we present another Rassias stability of the reciprocal-cubic functional equations.

Corollary 2.8. *Assume that $\lambda \geq 0$, and suppose a real number ℓ is such that $\ell > -3$. Let $C\colon \mathbb{R} \longrightarrow \mathcal{F}$ satisfy (2.19). Then, for every $p \in \mathbb{R}$, $R_c(p) = \mathsf{F}\text{-}\lim_{r\to\infty} 3^{-3r}C\left(3^{-r}p\right)$ exists, and the mapping $R_C\colon \mathbb{R} \longrightarrow \mathcal{F}$ is reciprocal-cubic such that if, for some $\alpha > 0$,*

$$\mathsf{F}\Big(\Delta_1 C(p, q), \lambda\left(\|p\|^{\ell} + \|q\|^{\ell}\right)\Big) \geq \alpha$$

for all $p, q \in \mathbb{R}$, then

$$\mathsf{F}\Big(C(p) - R_c(p), \frac{\lambda}{27 - 3^{-\ell}}\|p\|^{\ell}\Big) \geq \alpha$$

for all $p \in \mathbb{R}$. Furthermore, a unique reciprocal-cubic mapping $R_C \colon \mathbb{R} \longrightarrow \mathcal{F}$ exists such that

$$\lim_{\beta \to \infty} \mathsf{F}\Big(C(p) - R_c(p), \frac{\beta\lambda}{27 - 3^{-\ell}} \|p\|^{\ell}\Big) = 1$$

uniformly on $\mathbb{R}$.

P r o o f. By considering $\zeta(p,q) = \lambda\left(\|p\|^{\ell} + \|q\|^{\ell}\right)$ in Theorem 2.6 with similar arguments, the conclusion follows immediately. □

Remark 2.9. Regarding to the stability of the functional equation (1.4), by adding the oddness of a mapping C in all results in the current work, one can obtain the fuzzy stability results pertinent to the functional equation (1.4) similarly.

3. Real-life interpretation of the functional equation (1.3)

The applications of reciprocal-cubic mapping $C(p) = \frac{1}{p^3}$ are found in various fields where inverse cube relationships are observed. This mapping is directly related to phenomena governed by an inverse cube law, where a quantity is inversely proportional to the cube of the distance. In certain fluid flow scenarios, particularly those involving viscous forces or boundary layers, relationships involving inverse powers of distance, including the inverse cube, can arise in the mathematical models.

We discuss the role of the functional equation (1.3) connected with lubrication theory (Thin film flow). Consider two parallel plates with a thin film of viscous fluid between them as shown in Figure 1.

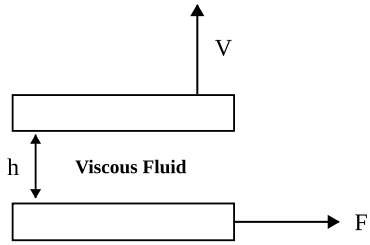


FIGURE 1. Parallel plates with a think film between them

Let us assume that plate 1 is fixed and plate 2 is pushed vertically with velocity V. Suppose the thickness of the gap between the plates 1 and 2 is h. Let μ be

the fluid viscosity with plate area A. According to inverse cube law, the squeezing force F is inversely proportional to the cube of h. Then from lubrication theory, the required squeezing force is given by

$$F = \frac{3\mu A^2}{2\pi h^3} V.$$

Suppose p and q are thicknesses in different directions of the fluid film. The arguments $\frac{pq}{2p+q}$, $\frac{pq}{p+2q}$ and $\frac{pq}{p+q}$ represent effective film thicknesses when forces act in series or parallel directions. Then, the functional equation (1.3) represents the fact that the sum of forces from one configuration equals that from another configuration. Thus, the functional equation (1.3) directly models fluid film forces where the governing inverse cube law is $F \propto \frac{1}{h^3}$.

4. Concluding remarks

In this section, we close our study with concluding remarks on the results obtained. In this study, we have used fuzzy complete normed spaces to investigate the stability of the functional equations (1.3) and (1.4). The results obtained in this study imply that the stability results hold in the setting of fuzzy complete normed spaces. A graded, membership-based concept of closeness is combined with the traditional robustness question of Hyers–Ulam stability to investigate Ulam stability of functional equations in fuzzy normed spaces. Because fuzzy norms represent "closeness" in degrees rather than binary terms, this paradigm is especially well-suited for real-world models that are affected by scale-dependent noise, language uncertainty, or measurement imprecision. In order to explicitly account for uncertainty, stability results in fuzzy settings offer strict guarantees that approximate solutions are close to exact solutions. The fuzzy method provides tools that are helpful for both theoretical advancements and real-world applications in fuzzy control, signal processing, and decision systems.

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Abasalt Bodaghi
Department of Mathematics, WT.C.
Islamic Azad University
Tehran
IRAN
E-mail: abasalt.bodaghi@gmail.com

Department of Mathematics
C. Abdul Hakeem College of Engineering and Technology
Melvisharam 632 509
Tamil Nadu
INDIA
E-mail: bvskumarmaths@gmail.com