Tatra Mt. Math. Publ. ?? (2026), 1–20
DOI: 10.2478/tmmp-2026-0014

TATRA MOUNTAINS
Mathematical Publications

OSCILLATORY BEHAVIOUR OF SOLUTIONS OF FOURTH-ORDER NONLINEAR DYNAMIC EQUATIONS

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ABSTRACT. This paper presents novel oscillation criteria for certain fourth-order dynamic equations on time scales. The proposed results are derived by comparing these equations with lower-order dynamic equations whose oscillatory behaviour is well understood. We establish new and enhanced oscillation conditions that differ from existing findings in the literature. Additionally, an example is provided to illustrate the significance of our results.

1. Introduction

Since Sturm's groundbreaking work on second-order linear differential equations, the oscillatory behaviour of differential and difference equations has been widely explored using various methods, including the Riccati transformation, integral averaging, and comparison techniques. Over the past three decades, significant advancements have been made in the oscillation theory of second-order and third-order delay dynamic equations on time scales; see, for instance, [1, 3, 4, 7]. However, compared to second- and third-order equations, relatively less attention has been given to investigating the oscillatory and asymptotic properties of higher-order dynamic equations [2, 8, 10, 11, 12, 13, 24, 25, 26].

© 2026 Mathematical Institute, Slovak Academy of Sciences.
2020 Mathematics Subject Classification: 34C10, 34K11, 34N05, 39A10.
Keywords: Oscillation, comparison method, Lebesgue's theorem, dynamic equation, fourth order.
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A notable contribution in this area is the work of Grace et al. [12], where the authors investigated fourth-order dynamic equations of the form

$$\mathcal{X}^{\Delta^4}(\ell) + P(\ell)\mathcal{X}^{\lambda}(\ell) = 0, \tag{1.1}$$

where λ is the ratio of positive odd integers. In their study, the authors developed oscillation criteria using an integral averaging technique, Taylor's formula on time scales, and a comparative approach involving two second-order dynamic equations.

Such higher-order dynamic equations have broad applications across multiple disciplines, including beam and plate vibrations in structural mechanics [16], stability analysis in fluid dynamics, population models in biological systems [18], optimal control and signal processing in engineering [20], and thermo-elasticity in material science [9]. Despite these applications, further exploration is needed to develop a comprehensive understanding of their oscillatory properties, particularly in the framework of time-scale calculus.

With this motivation, we consider a class of fourth-order nonlinear dynamic equations of the type

$$\left[\psi(\ell)\left(\mathcal{X}^{\Delta\Delta}(\ell)\right)^{\alpha_1}\right]^{\Delta\Delta} + P(\ell)H(\mathcal{X}(g(\ell))) = 0 \tag{1.2}$$

on a time scale $\mathbb{T}$, where α_1 is the ratio of odd positive integers. Here, $\sup\mathbb{T} = \infty$ and $\ell \in [\ell_0, \infty)_{\mathbb{T}} = [\ell_0, \infty) \cap \mathbb{T}$ with $\ell_0 \in \mathbb{T}$. Throughout the paper, we assume that

(i) $P(\ell) \in \mathrm{C}_{rd}([\ell_0, \infty)_{\mathbb{T}}, \mathbb{R}_+)$ with $P(\ell) \not\equiv 0$, $\psi(\ell) \in \mathrm{C}_{rd}([\ell_0, \infty)_{\mathbb{T}}, (0, \infty)_{\mathbb{R}})$ with $\psi^{\Delta}(\ell) \geq 0$ and

$$\int_{\ell_0}^{\infty} \psi^{\frac{-1}{\alpha_1}}(s)\Delta s = \infty; \tag{1.3}$$

(ii) $g(\ell) \in \mathrm{C}_{rd}([\ell_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $g(\ell) \leq \ell$, $g(\ell)$ is non-decreasing and $\lim_{\ell\to\infty} g(\ell) = \infty$;

(iii) $H \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ is a non-decreasing function satisfying $\mathcal{X}H(\mathcal{X}) > 0$ for $\mathcal{X} \neq 0$ and

$$-H(-uv) \geq H(uv) \geq H(u)H(v) \quad \text{for } uv > 0. \tag{1.4}$$

The remainder of the paper is organized as follows. Section 2 reviews the basic concepts and results of time scale calculus and establishes two auxiliary theorems for lower-order nonlinear dynamic equations. In Section 3, we establish new oscillation criteria for (1.2) by comparing it with lower-order equations, particularly second-order and first-order equations. In Section 4, these criteria are applied to specific time scales, namely $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, yielding corresponding continuous and discrete oscillation results. Finally, Section 5 summarizes the main findings of the paper and discusses some directions for future research.

2. Preliminaries

The theory of dynamic equations on time scales provides a unified framework for the study of differential and difference equations. Comprehensive treatments of this theory and its applications can be found in the monographs of Bohner and Peterson [3, 4]. For the reader's convenience, we recall several fundamental concepts and preliminary results that will be used throughout the paper. Unless otherwise stated, the following definitions are taken from [3, 4].

Definition 2.1. Let $\mathbb{T}$ be a time scale. The *forward jump operator* $\sigma\colon \mathbb{T} \to \mathbb{T}$ and the *backward jump operator* $\rho\colon \mathbb{T} \to \mathbb{T}$ are defined, respectively, by

$$\sigma(\ell) = \inf\{s \in \mathbb{T} : s > \ell\}, \qquad \rho(\ell) = \sup\{s \in \mathbb{T} : s < \ell\}.$$

For completeness, we adopt the conventions $\sigma(\ell) = \ell$ whenever $\ell = \sup \mathbb{T}$ and $\rho(\ell) = \ell$ whenever $\ell = \inf \mathbb{T}$.

Definition 2.2. A point $\ell \in \mathbb{T}$ is classified as follows:

(1) right-scattered if $\sigma(\ell) > \ell$;
(2) left-scattered if $\rho(\ell) < \ell$;
(3) right-dense if $\ell < \sup \mathbb{T}$ and $\sigma(\ell) = \ell$;
(4) left-dense if $\ell > \inf \mathbb{T}$ and $\rho(\ell) = \ell$;
(5) isolated if $\rho(\ell) < \ell < \sigma(\ell)$;
(6) dense if $\rho(\ell) = \ell = \sigma(\ell)$.

Definition 2.3. A function $f\colon \mathbb{T} \to \mathbb{R}$ is called *regulated* if it admits finite right-hand limits at every right-dense point and finite left-hand limits at every left-dense point of $\mathbb{T}$. The collection of all regulated functions is denoted by $\mathrm{C}_r(\mathbb{T}, \mathbb{R})$.

Definition 2.4. A function $f\colon \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* if it is continuous at every right-dense point of $\mathbb{T}$ and possesses finite left-hand limits at every left-dense point. The set of all rd-continuous functions is denoted by $\mathrm{C}_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.5. The *graininess function* $\mu\colon \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(\ell) = \sigma(\ell) - \ell.$$

If $\mathbb{T}$ possesses a left-scattered maximum point m, then we set

$$\mathbb{T}^{\kappa} = \mathbb{T} \setminus m,$$

otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

Definition 2.6. Let $f\colon \mathbb{T} \to \mathbb{R}$ and let $\ell \in \mathbb{T}^{\kappa}$. The *delta derivative* (or *Hilger derivative*) of f at ℓ is the number $f^{\Delta}(\ell)$, provided it exists, such that for every $\varepsilon > 0$ there is a neighborhood

$$U = (\ell - \delta, \ell + \delta) \cap \mathbb{T}, \quad \delta > 0,$$

satisfying

$$\big|f(\sigma(\ell)) - f(r) - f^{\Delta}(\ell)(\sigma(\ell) - r)\big| \leq \varepsilon |\sigma(\ell) - r|, \quad r \in U.$$

Definition 2.7. A function $F\colon \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f\colon \mathbb{T} \to \mathbb{R}$ if

$$F^{\Delta}(\ell) = f(\ell), \quad \ell \in \mathbb{T}^{\kappa}.$$

In this case, the delta integral of f from a to b is defined by

$$\int_a^b f(\ell)\Delta\ell = F(b) - F(a).$$

Motivated by [17, Theorem 3.1.1] and [17, Theorem 4.3.1], we extend the corresponding oscillation criterion to the framework of time scales. Since results of this type for first-order and second-order sublinear and superlinear dynamic equations on time scales appear to be unavailable in the literature, we include a complete proof here for future reference.

For the next two results, in addition to hypotheses (ii) and (iii), we assume that

$$\lim_{x \to 0^+} \frac{x}{H(x)} = M < \infty. \tag{2.5}$$

Theorem 2.8. *Let $q \in \mathrm{C}_{rd}([\ell_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$. If*

$$\limsup_{\ell \to \infty} \int_{g(\ell)}^{\ell} q(s)\, \Delta s > M, \tag{2.6}$$

then the first-order dynamic equation

$$Y^{\Delta}(\ell) + q(\ell)\, H(Y(g(\ell))) = 0 \tag{2.7}$$

is oscillatory.

P r o o f. Assume, for the sake of contradiction, that Y is a non-oscillatory solution of (2.7). Without loss of generality, suppose that $Y(\ell) > 0$ and $Y(g(\ell)) > 0$ for all $\ell \in [\ell_1, \infty)_{\mathbb{T}} \subset [\ell_0, \infty)_{\mathbb{T}}$. From (2.7), we obtain

$$Y^{\Delta}(\ell) = -q(\ell)\, H(Y(g(\ell))) < 0, \quad \ell \geq \ell_1.$$

Hence, $Y(\ell)$ is non-increasing on $[\ell_1, \infty)_{\mathbb{T}}$, and since $Y(\ell) > 0$, the limit $\lim\limits_{\ell \to \infty} Y(\ell) = L$ exists and is finite.

We first claim that $L = 0$. Suppose that $L > 0$. Since Y is non-increasing and $\lim_{\ell\to\infty} g(\ell) = \infty$, there exists a constant $\ell_2 > \ell_1$ such that

$$Y(g(\ell)) \geq L, \quad \ell \in [\ell_2, \infty)_{\mathbb{T}}.$$

Consequently, since H is non-decreasing and $L > 0$, we have

$$H(Y(g(\ell))) \geq H(L) \quad \text{for all } \ell \in [\ell_2, \infty)_{\mathbb{T}}$$

and, by the condition $xH(x) > 0$ for $x \neq 0$, $H(L) > 0$.

From condition (2.6), there exist a sequence $\{\nu_k\}_{k=1}^{\infty}$ and a constant $\delta > M$ such that $\lim_{k\to\infty} \nu_k = \infty$ and

$$\int_{g(\nu_k)}^{\nu_k} q(s)\,\Delta s \geq \delta \quad \text{for all } k. \tag{2.8}$$

Since $\lim_{k\to\infty} g(\nu_k) = \infty$, we may construct a subsequence of $\{\nu_k\}$ (which, for simplicity, is denoted again by $\{\nu_k\}$) recursively as follows. Choose ν_1 such that $g(\nu_1) > \nu_0$, where $\nu_0 \geq \ell_2$. Having chosen ν_m, select ν_{m+1} sufficiently large so that $g(\nu_{m+1}) > \nu_m$. Since $g(\nu_m) \leq \nu_m$, it follows that

$$\nu_0 < g(\nu_1) < \nu_1 < g(\nu_2) < \nu_2 < \cdots < g(\nu_k) < \nu_k.$$

Consequently, the intervals $[g(\nu_i), \nu_i]_{\mathbb{T}}$ are pairwise disjoint.

Now, integrating (2.7) from ν_0 to ν_k, we obtain

$$Y(\nu_k) = Y(\nu_0) - \int_{\nu_0}^{\nu_k} q(s)\,H(Y(g(s)))\Delta s.$$

Substituting our lower bound $H(Y(g(s))) \geq H(L)$ into the integral yields

$$Y(\nu_k) \leq Y(\nu_0) - H(L)\int_{\nu_0}^{\nu_k} q(s)\,\Delta s.$$

Because $q(\ell) \in \mathbb{R}^+$, the integral over the large domain $[\nu_0, \nu_k]_{\mathbb{T}}$ is bounded below by the sum of the integrals over the smaller, non-overlapping intervals $[g(\nu_i), \nu_i]_{\mathbb{T}}$ contained inside it

$$\int_{\nu_0}^{\nu_k} q(s)\,\Delta s \geq \sum_{i=1}^{k} \int_{g(\nu_i)}^{\nu_i} q(s)\,\Delta s \geq \sum_{i=1}^{k} \delta = k\delta.$$

Substituting this back into our inequality for $Y(\nu_k)$ produces

$$Y(\nu_k) \leq Y(\nu_0) - H(L)\delta k.$$

Taking the limit as $k \to \infty$, the right-hand side tends to $-\infty$, which is impossible since $Y(\ell) > 0$ for all $\ell \geq \ell_1$. Thus, our assumption was false, and we must have $L = 0$.

Since $Y(\ell)$ is non-increasing and $\lim_{\ell\to\infty} Y(\ell) = 0$, it follows that $\lim_{\ell\to\infty} Y(g(\ell)) = 0$. Integrating (2.7) from $g(\ell)$ to ℓ, we obtain

$$Y(\ell) - Y(g(\ell)) + \int_{g(\ell)}^{\ell} q(s)\, H(Y(g(s)))\Delta s = 0,$$

which implies

$$\int_{g(\ell)}^{\ell} q(s)\, H(Y(g(s)))\Delta s \le Y(g(\ell)).$$

Since g is strictly increasing, for any $s \in [g(\ell), \ell]_{\mathbb{T}}$, we have $g(s) \le g(\ell)$. Utilizing the non-increasing property of Y, it follows that $Y(g(s)) \ge Y(g(\ell))$, and consequently, using the monotonicity of H, $H(Y(g(s))) \ge H(Y(g(\ell)))$. Using this in the integral inequality yields

$$H(Y(g(\ell))) \int_{g(\ell)}^{\ell} q(s)\, \Delta s \le Y(g(\ell)),$$

so that

$$\int_{g(\ell)}^{\ell} q(s)\, \Delta s \le \frac{Y(g(\ell))}{H(Y(g(\ell)))}. \tag{2.9}$$

Evaluating the inequality (2.9) explicitly along our chosen subsequence $\ell = \nu_k$, and using (2.8), we find

$$\delta \le \int_{g(\nu_k)}^{\nu_k} q(s)\, \Delta s \le \frac{Y(g(\nu_k))}{H(Y(g(\nu_k)))}.$$

Passing the limit $k \to \infty$ and using $Y(g(\nu_k)) \to 0$ together with (2.5), we obtain

$$\delta \le \lim_{k\to\infty} \frac{Y(g(\nu_k))}{H(Y(g(\nu_k)))} = M,$$

which contradict $\delta > M$. This completes the proof. □

Remark 2.9. Consider the particular case $H(x) = x^{\beta}$ with $0 < \beta < 1$. Then

$$\lim_{x\to 0^+} \frac{x}{H(x)} = \lim_{x\to 0^+} x^{1-\beta} = 0.$$

Hence $M = 0$, and condition (2.6) reduces to $\limsup_{\ell\to\infty} \int_{g(\ell)}^{\ell} q(s)\, \Delta s > 0$. Therefore, Theorem 2.8 recovers the classical oscillation criterion for the sublinear dynamic equation

$$Y^{\Delta}(\ell) + q(\ell) Y^{\beta}(g(\ell)) = 0.$$

Theorem 2.10. *Let $q \in \mathrm{C}_{rd}([\ell_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$. If*

$$\limsup_{\ell\to\infty}\int_{g(\ell)}^{\ell} q(s)\,H\big(g(\ell)-g(s)\big)\,\Delta s > M, \tag{2.10}$$

then every bounded solutions of the second-order dynamic equation

$$Z^{\Delta\Delta}(\ell) - q(\ell)H(Z(g(\ell))) = 0 \tag{2.11}$$

is oscillatory.

P r o o f. Assume, to the contrary, that Z is a non-oscillatory solution of (2.11). Without loss of generality, suppose that $Z(\ell) > 0$ and $Z(g(\ell)) > 0$ for all $\ell \in [\ell_1,\infty)_{\mathbb{T}} \subset [\ell_0,\infty)_{\mathbb{T}}$. It follows from (2.11) that

$$Z^{\Delta\Delta}(\ell) = q(\ell)H(Z(g(\ell))) > 0, \quad \ell \in [\ell_1,\infty)_{\mathbb{T}}.$$

Hence, $Z^{\Delta}(\ell)$ is non-decreasing on $[\ell_1,\infty)_{\mathbb{T}}$. Consequently, one of the following cases must occur:

(I) $Z^{\Delta\Delta}(\ell) > 0$, and $Z^{\Delta}(\ell) > 0$ for $\ell \geq \ell_1$;

(II) $Z^{\Delta\Delta}(\ell) > 0$, and $Z^{\Delta}(\ell) < 0$ for $\ell \geq \ell_1$.

Case (I): Since Z^{Δ} is positive and non-decreasing, there exist $\ell_2 > \ell_1$ and a constant $b > 0$ such that

$$Z^{\Delta}(\ell) \geq b, \quad \ell \in [\ell_2,\infty)_{\mathbb{T}}.$$

Integrating from ℓ_2 to ℓ and letting $\ell \to \infty$ gives

$$Z(\ell) \to \infty,$$

which contradicts the boundedness of Z.

Case (II): Because $Z^{\Delta}(\ell)$ is negative and increasing on $[\ell_1,\infty)_{\mathbb{T}}$, there exists a finite limit

$$L := \lim_{\ell\to\infty} Z^{\Delta}(\ell) \leq 0.$$

We claim that $L = 0$. Suppose, on the contrary, that $L < 0$. Then there exist $\varepsilon > 0$ and $\ell_2 \geq \ell_1$ such that

$$Z^{\Delta}(\ell) \leq -\varepsilon, \quad \ell \in [\ell_2,\infty)_{\mathbb{T}}.$$

Integrating from ℓ_2 to ℓ, we obtain

$$Z(\ell) = Z(\ell_2) + \int_{\ell_2}^{\ell} Z^{\Delta}(s)\,\Delta s \leq Z(\ell_2) - \varepsilon(\ell-\ell_2).$$

Letting $\ell \to \infty$, we arrive at

$$Z(\ell) \to -\infty,$$

a contradiction to $Z(\ell) > 0$ eventually. Therefore, $\lim\limits_{\ell\to\infty} Z^{\Delta}(\ell) = 0$, and consequently

$$-Z^{\Delta}(\ell) \to 0^+ \quad \text{as} \quad \ell \to \infty. \tag{2.12}$$

Now, integrating (2.11) from $g(\ell)$ to ℓ, we obtain

$$Z^{\Delta}(\ell) - Z^{\Delta}(g(\ell)) = \int_{g(\ell)}^{\ell} q(s)H(Z(g(s)))\,\Delta s.$$

Since $Z^{\Delta}(\ell) < 0$, it follows that

$$-Z^{\Delta}(g(\ell)) \geq \int_{g(\ell)}^{\ell} q(s)H(Z(g(s)))\,\Delta s. \tag{2.13}$$

For $s \in (g(\ell), \ell)_{\mathbb{T}}$, we have

$$Z(g(s)) - Z(g(\ell)) = \int_{g(s)}^{g(\ell)} -Z^{\Delta}(u)\,\Delta u.$$

Since $-Z^{\Delta}(\ell)$ is positive and decreasing,

$$-Z^{\Delta}(u) \geq -Z^{\Delta}(g(\ell)) \quad \text{for } u \in [g(s), g(\ell)]_{\mathbb{T}}.$$

Hence

$$Z(g(s)) - Z(g(\ell)) \geq -Z^{\Delta}(g(\ell))\big(g(\ell) - g(s)\big).$$

Since $Z(g(\ell)) > 0$, we obtain

$$Z(g(s)) \geq -Z^{\Delta}(g(\ell))\big(g(\ell) - g(s)\big),$$

which in view of hypothesis (iii) implies that

$$\begin{aligned} H(Z(g(s))) &\geq H\left(-Z^{\Delta}(g(\ell))(g(\ell) - g(s))\right) \\ &\geq H(-Z^{\Delta}(g(\ell)))H(g(\ell) - g(s)). \end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.13), we get

$$-Z^{\Delta}(g(\ell)) \geq H(-Z^{\Delta}(g(\ell))) \int_{g(\ell)}^{\ell} q(s)H(g(\ell) - g(s))\,\Delta s.$$

Therefore,

$$\int_{g(\ell)}^{\ell} q(s)H(g(\ell) - g(s))\,\Delta s \leq \frac{-Z^{\Delta}(g(\ell))}{H(-Z^{\Delta}(g(\ell)))}. \tag{2.15}$$

Taking the limit superior on both sides of (2.15) and using (2.12) together with (2.5), we obtain

$$\limsup_{\ell\to\infty} \int_{g(\ell)}^{\ell} q(s)H(g(\ell) - g(s))\,\Delta s \leq \lim_{\ell\to\infty} \frac{-Z^{\Delta}(g(\ell))}{H(-Z^{\Delta}(g(\ell)))} = M.$$

This contradicts (2.10). The proof is complete. □

3. Main results

Let us denote

$$\Psi[\ell,\ell_0]=\int_{\ell_0}^{\ell}\int_{\ell_0}^{s}\left(\frac{x}{\psi(x)}\right)^{\frac{1}{\alpha_1}}\Delta x\Delta s.$$

Theorem 3.1. *Assume that* (1.3) *and* (1.4) *are satisfied. Furthermore, let there exist a function* $\omega(\ell)\in \mathrm{C}_{rd}([\ell_0,\infty)_{\mathbb{T}},\mathbb{R})$ *satisfying*

$$g(\ell)<\omega(\ell)<\ell \quad \text{and} \quad \omega^{\Delta}(\ell)\geq 0 \quad \text{for } \ell\geq\ell_0.$$

If there exist constants $e_1,\bar{e}\in(0,1)_{\mathbb{R}}$ *such that the first-order dynamic equation*

$$Y^{\Delta}(\ell)+e_1P(\ell)H(\Psi[g(\ell),\ell_0])H(Y^{\frac{1}{\alpha_1}}(g(\ell)))=0 \tag{3.16}$$

is oscillatory and every bounded solution of the second-order dynamic equation

$$Z^{\Delta\Delta}(\ell)-\bar{e}P(\ell)H(g(\ell))H\left[\frac{\omega(\ell)-g(\ell)}{\psi^{\frac{1}{\alpha_1}}(\omega(\ell)}\right]H(Z^{\frac{1}{\alpha_1}}(\omega(\ell)))=0 \tag{3.17}$$

is also oscillatory, then (1.2) *oscillates.*

P r o o f. Assume, for the sake of contradiction, that $\mathcal{X}$ is a non-oscillatory solution of (1.2) satisfying $\mathcal{X}(\ell)>0$ and $\mathcal{X}(g(\ell))>0$ for all $\ell\in[\ell_1,\infty)_{\mathbb{T}}\subset[\ell_0,\infty)_{\mathbb{T}}$. Due to (1.3), for $\ell_2>\ell_1$, there are two possible cases:

(I) $\left(\psi(\ell)\left(\mathcal{X}^{\Delta\Delta}(\ell)\right)^{\alpha_1}\right)^{\Delta}>0$, $\psi(\ell)\left(\mathcal{X}^{\Delta\Delta}(\ell)\right)^{\alpha_1}>0$, and $\mathcal{X}^{\Delta}(\ell)>0$ for $\ell\geq\ell_2$;

(II) $\left(\psi(\ell)\left(\mathcal{X}^{\Delta\Delta}(\ell)\right)^{\alpha_1}\right)^{\Delta}>0$, $\psi(\ell)\left(\mathcal{X}^{\Delta\Delta}(\ell)\right)^{\alpha_1}<0$, and $\mathcal{X}^{\Delta}(\ell)>0$ for $\ell\geq\ell_2$.

Case (I): Setting $\Phi(\ell)=\psi(\ell)\left(\mathcal{X}^{\Delta\Delta}(\ell)\right)^{\alpha_1}$, we observe that $\Phi(\ell)>0$ and $\Phi^{\Delta}(\ell)>0$ for $\ell\in[\ell_2,\infty)_{\mathbb{T}}$. Furthermore, from equation (1.2), it follows that

$$\Phi^{\Delta\Delta}(\ell)=\left[\psi(\ell)\left(\mathcal{X}^{\Delta\Delta}(\ell)\right)^{\alpha_1}\right]^{\Delta\Delta}=-P(\ell)H(\mathcal{X}(g(\ell)))<0.$$

Thus, $\Phi^{\Delta}(\ell)$ is non-increasing on $[\ell_2,\infty)_{\mathbb{T}}$, which implies that $\Phi^{\Delta}(s)\geq\Phi^{\Delta}(\ell)$ for all $s\in[\ell_2,\ell]_{\mathbb{T}}$. Integrating $\Phi^{\Delta}(s)$ from ℓ_2 to ℓ yields

$$\Phi(\ell)=\Phi(\ell_2)+\int_{\ell_2}^{\ell}\Phi^{\Delta}(s)\Delta s\geq\Phi^{\Delta}(\ell)\int_{\ell_2}^{\ell}\Delta s=(\ell-\ell_2)\Phi^{\Delta}(\ell).$$

Therefore, for any constant $d_1\in(0,1)_{\mathbb{R}}$, there exist $\ell_3>\ell_2$ such that $(\ell-\ell_2)>d_1\ell$. Consequently,

$$\Phi(\ell)\geq d_1\ell\Phi^{\Delta}(\ell) \quad \text{for } \ell\in[\ell_3,\infty)_{\mathbb{T}}.$$

Therefore, for $\ell\in[\ell_3,\infty)_{\mathbb{T}}$

$$\mathcal{X}^{\Delta\Delta}(\ell)\geq(d_1)^{\frac{1}{\alpha_1}}\left(\frac{\ell}{\psi(\ell)}\right)^{\frac{1}{\alpha_1}}\left(\Phi^{\Delta}(\ell)\right)^{\frac{1}{\alpha_1}}:=d\left(\frac{\ell}{\psi(\ell)}\right)^{\frac{1}{\alpha_1}}\left(\Phi^{\Delta}(\ell)\right)^{\frac{1}{\alpha_1}}, \tag{3.18}$$

where $d = (d_1)^{\frac{1}{\alpha_1}}$. By integrating the inequality (3.18) twice from ℓ_3 to ℓ, we get

$$\mathcal{X}(\ell) \geq d \int_{\ell_3}^{\ell} \int_{\ell_3}^{s} \left(\frac{x}{\psi(x)}\right)^{\frac{1}{\alpha_1}} \left(\Phi^{\Delta}(x)\right)^{\frac{1}{\alpha_1}} \Delta x \Delta s.$$

Since Φ^{Δ} is non-increasing, then $\Phi^{\Delta}(x) \geq \Phi^{\Delta}(\ell)$ for $x \in [\ell_3, \ell]_{\mathbb{T}}$, which implies that $\left(\Phi^{\Delta}(x)\right)^{\frac{1}{\alpha_1}} \geq \left(\Phi^{\Delta}(\ell)\right)^{\frac{1}{\alpha_1}}$. As a result,

$$\mathcal{X}(\ell) \geq d \left(\Phi^{\Delta}(\ell)\right)^{\frac{1}{\alpha_1}} \int_{\ell_3}^{\ell} \int_{\ell_3}^{s} \left(\frac{x}{\psi(x)}\right)^{\frac{1}{\alpha_1}} \Delta x \Delta s := d\Psi[\ell, \ell_3] \left(\Phi^{\Delta}(\ell)\right)^{\frac{1}{\alpha_1}}.$$

Since $g(\ell) \to \infty$ as $\ell \to \infty$, we can choose $\ell_4 \geq \ell_3$ such that $g(\ell) \geq \ell_3$ and

$$\mathcal{X}(g(\ell)) \geq d\Psi[g(\ell), \ell_3] \left(\Phi^{\Delta}(g(\ell))\right)^{\frac{1}{\alpha_1}} \quad \text{for } \ell \in [\ell_4, \infty)_{\mathbb{T}}. \tag{3.19}$$

Substituting equation (3.19) into equation (1.2) and then utilizing the monotonicity of H and (1.4), we get

$$Y^{\Delta}(\ell) + H(d)P(\ell)H\left(\Psi[g(\ell), \ell_3]\right) H\left(Y^{\frac{1}{\alpha_1}}(g(\ell))\right) \leq 0 \quad \text{for } \ell \in [\ell_4, \infty)_{\mathbb{T}}, \tag{3.20}$$

where $Y(\ell) = \Phi^{\Delta}(\ell)$.

Now, by [14, Lemma 3], there exists a positive solution $Y = \Phi^{\Delta}$ of equation (3.16) with $\lim_{\ell \to \infty} Y(\ell) = 0$, which is a contradiction.

Case (II): Since $\psi(\ell) > 0$ and α_1 is a ratio of odd integers, then $\psi(\ell) \left(\mathcal{X}^{\Delta\Delta}(\ell)\right)^{\alpha_1} < 0$ implies that

$$\mathcal{X}^{\Delta\Delta}(\ell) < 0 \quad \text{for } \ell \in [\ell_2, \infty)_{\mathbb{T}}.$$

Therefore, $\mathcal{X}^{\Delta}(\ell)$ is positive and decreasing on $[\ell_2, \infty)_{\mathbb{T}}$.

Using the integration-by-parts formula on time scale [3, Theorem 1. 77 (vi)], we obtain

$$\mathcal{X}(u) - \mathcal{X}(\ell_2) = (u - \ell_2)\mathcal{X}^{\Delta}(u) - \int_{\ell_2}^{u} (\sigma(s) - \ell_2)\mathcal{X}^{\Delta\Delta}(s)\Delta s.$$

Since $(\sigma(s) - \ell_2) > 0$ and $\mathcal{X}^{\Delta\Delta}(s) < 0$, then the integrand is negative. As a result,

$$\mathcal{X}(u) - \mathcal{X}(\ell_2) > (u - \ell_2)\mathcal{X}^{\Delta}(u).$$

Consequently,

$$\mathcal{X}(u) > \mathcal{X}(\ell_2) + (u - \ell_2)\mathcal{X}^{\Delta}(u) > (u - \ell_2)\mathcal{X}^{\Delta}(u).$$

Dividing by the positive quantity $u\mathcal{X}^{\Delta}(u)$, we obtain

$$\frac{\mathcal{X}(u)}{u\mathcal{X}^{\Delta}(u)} > 1 - \frac{\ell_2}{u}.$$

Taking the limit inferior as $u \to \infty$, we find that

$$\liminf_{u\to\infty} \frac{\mathcal{X}(u)}{u\mathcal{X}^{\Delta}(u)} \geq 1.$$

Therefore, for any fixed constant $e \in (0,1)$, there exists $\ell_3 \geq \ell_2$ such that

$$\mathcal{X}(u) \geq e\, u\mathcal{X}^{\Delta}(u) \quad \text{for all } u \in [\ell_3,\infty)_{\mathbb{T}}.$$

Since $g(\ell) \to \infty$ as $\ell \to \infty$, we can choose $\ell_4 \geq \ell_3$ such that $g(\ell) \geq \ell_3$ and

$$\mathcal{X}(g(\ell)) \geq eg(\ell)\mathcal{X}^{\Delta}(g(\ell)) \quad \text{for } \ell \in [\ell_4,\infty)_{\mathbb{T}}. \tag{3.21}$$

Substituting (3.21) into (1.2), the using (1.4), we have

$$\left[\psi(\ell)\left(\mathcal{X}^{\Delta\Delta}(\ell)\right)^{\alpha_1}\right]^{\Delta\Delta} + H(e)P(\ell)H(g(\ell))H\left(\mathcal{X}^{\Delta}(g(\ell))\right) \leq 0 \quad \text{for } \ell \in [\ell_4,\infty)_{\mathbb{T}}.$$

Letting $\varphi(\ell) = \mathcal{X}^{\Delta}(\ell)$ and $\bar{e} = H(e)$ in the above inequality, we obtain

$$\left[\psi(\ell)\left(\varphi^{\Delta}(\ell)\right)^{\alpha_1}\right]^{\Delta\Delta} + \bar{e}P(\ell)H(g(\ell))H(\varphi(g(\ell))) \leq 0. \tag{3.22}$$

Next, we utilize the fact that $\varphi(\ell) > 0$ is a non-increasing function, while $\varphi^{\Delta}(\ell)$ is non-decreasing, which leads to

$$\varphi(s) \geq (\ell - s)(-\varphi^{\Delta}(\ell)) \quad \text{for } \ell \geq s \geq \ell_4.$$

Setting $s = g(\ell)$ and $\ell = \omega(\ell)$ in the preceding inequality, we get

$$\begin{aligned}\varphi(g(\ell)) &\geq (\omega(\ell) - g(\ell))(-\varphi^{\Delta}(\omega(\ell)) \\ &= \frac{\omega(\ell) - g(\ell)}{\psi^{\frac{1}{\alpha_1}}(\omega(\ell))}\left(-\psi(\omega(\ell))\left(\varphi^{\Delta}(\omega(\ell))\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}}.\end{aligned} \tag{3.23}$$

Substituting (3.23) into (3.22) and applying (1.4), we derive

$$Z^{\Delta\Delta}(\ell) \geq \bar{e}P(\ell)H(g(\ell))H\left(\frac{\omega(\ell) - g(\ell)}{\psi^{\frac{1}{\alpha_1}}(\omega(\ell))}\right)H(Z^{\frac{1}{\alpha_1}}(\omega(\ell))), \tag{3.24}$$

where $Z(\ell) = -\psi(\ell)(\varphi^{\Delta}(\ell))^{\alpha_1}$.

Hence, for all $\ell \geq \ell_2$, $Z(\ell)$ is strictly positive and strictly decreasing. Since $Z^{\Delta}(\ell) < 0$, it follows that $-Z^{\Delta}(\ell) > 0$. So, integrating (3.24) from ℓ to ∞, we obtain

$$-Z^{\Delta}(\ell) \geq \bar{e}\int_{\ell}^{\infty} P(s)\,H(g(s))\,H\left(\frac{\omega(s) - g(s)}{\psi^{\frac{1}{\alpha_1}}(\omega(s))}\right)\,H\left(Z^{\frac{1}{\alpha_1}}(\omega(s))\right)\Delta s. \tag{3.25}$$

Since $Z(\ell) \geq \int_{\ell}^{\infty} \big(- Z^{\Delta}(v)\big)\, \Delta v$, integrating once again the above inequality from ℓ to ∞ yields

$$Z(\ell) \geq \bar{e} \int_{\ell}^{\infty} \int_{v}^{\infty} P(s)\, H(g(s))\, H\left(\frac{\omega(s) - g(s)}{\psi^{\frac{1}{\alpha_1}}(\omega(s))} \right) H\left(Z^{\frac{1}{\alpha_1}}(\omega(s)) \right) \Delta s\, \Delta v. \tag{3.26}$$

Consider the Banach space

$$X = \{U : U \in C_{rd}\big([\ell_4, \infty)_{\mathbb{T}}, \mathbb{R}\big), \sup_{\ell \in [\ell_4, \infty)_{\mathbb{T}}} |U(\ell)| < \infty\}.$$

Define the closed, bounded, and convex subset $S \subset X$ by

$$S = \{U \in X : 0 \leq U(\ell) \leq Z(\ell),\ \ell \in [\ell_4, \infty)_{\mathbb{T}}\}.$$

We define the operator $T \colon S \to X$ by

$$(TU)(\ell) = \begin{cases} Z(\ell), & \ell \in [\ell_4, \ell_5]_{\mathbb{T}}, \\ \bar{e} \displaystyle\int_{\ell}^{\infty} \int_{v}^{\infty} P(s) H(g(s)) H\left(\frac{\omega(s) - g(s)}{\psi^{\frac{1}{\alpha_1}}(\omega(s))} \right) H\left(U^{\frac{1}{\alpha_1}}(\omega(s)) \right) \Delta s\, \Delta v, & \\ \qquad\qquad \ell \in [\ell_5, \infty)_{\mathbb{T}}. \end{cases}$$

The remainder of the proof follows the same argument as that employed in the proof of [15, Theorem 1] after equation (38), with the necessary modifications to accommodate the present function. Therefore, the details are omitted. This completes the proof. □

The following example is an illustration of Theorem 3.1.

Example 3.2. Consider

$$\left[\ell(\mathcal{X}^{\Delta\Delta}(\ell))^3\right]^{\Delta\Delta} + \ell^2 \mathcal{X}\left(\frac{\ell}{2}\right) = 0, \quad \ell \geq \ell_0 = 0. \tag{3.27}$$

Here, $\alpha_1 = 3$, $\psi(\ell) = \ell$, $P(\ell) = \ell^2$, $g(\ell) = \frac{\ell}{2}$, $H(\mathcal{X}) = \mathcal{X}$, $\psi^{\Delta}(\ell) = 1$. From (1.3), we have

$$\int_{\ell_0}^{\infty} \frac{1}{\psi^{\frac{1}{\alpha_1}}(s)} \Delta s = \int_{\ell_0}^{\infty} \frac{1}{s^{\frac{1}{3}}} \Delta s = \infty.$$

Moreover,

$$\begin{aligned} H\left(\Psi[g(\ell), \ell_0]\right) = \Psi[g(\ell), \ell_0] &= \Psi\left(\frac{\ell}{2}, 0\right) \\ &= \int_0^{\ell/2} \int_0^{s} \left(\frac{x}{\psi(x)}\right)^{1/\alpha_1} \Delta x \Delta s \\ &= \int_0^{\ell/2} \int_0^{s} 1 \Delta x \Delta s = h_2\left(\frac{\ell}{2}, 0\right), \end{aligned}$$

where h_2 is the second-order time-scale monomial.

From (3.16), for $e_1 \in (0,1)_{\mathbb{R}}$, we obtain

$$Y^{\Delta}(\ell) + e_1 \ell^2 h_2\left(\frac{\ell}{2}, 0\right) Y^{\frac{1}{3}}\left(\frac{\ell}{2}\right) = 0. \tag{3.28}$$

Indeed by Theorem 2.8, equation (3.28) is oscillatory.

Next, choose $\omega(\ell) = \frac{3\ell}{4}$. Indeed, $g(\ell) < \omega(\ell) < \ell$ for all $\ell \geq \ell_0$ and $\omega^{\Delta}(\ell) = \frac{3}{4} > 0$. From (3.17), for $\bar{e} \in (0,1)_{\mathbb{R}}$, we obtain

$$Z^{\Delta\Delta}(\ell) - \frac{\bar{e}}{8}\left(\frac{4}{3}\right)^{\frac{1}{3}} \ell^{\frac{11}{3}} Z^{\frac{1}{3}}\left(\frac{3\ell}{4}\right) = 0. \tag{3.29}$$

Furthermore,

$$\int_{g(\ell)}^{\ell} q(s)\, H\big(g(\ell) - g(s)\big)\, \Delta s = \frac{\bar{e}}{8}\left(\frac{4}{3}\right)^{\frac{1}{3}} \int_{\frac{\ell}{2}}^{\ell} s^{\frac{11}{3}} \left(\frac{\ell - s}{2}\right) \Delta s \to \infty \quad \text{as } \ell \to \infty,$$

Therefore, by Theorem 2.10, every bounded solution of (3.29) is oscillatory.

Hence, all the assumptions of Theorem 3.1 are satisfied. Consequently, equation (3.27) is oscillatory.

Remark 3.3. Before stating the next theorem, we clarify that the comparison framework in Theorem 3.1 is based on a nonlinear second-order delay dynamic equation of unstable type, where bounded solutions are necessarily oscillatory. In contrast, the next result removes this restriction and establishes a comparison with a standard second-order dynamic equation without imposing any boundedness condition on its solutions.

For the next results, assume that

$$\int_{\ell}^{\infty} \int_{s}^{\infty} P(x) \Delta x\, \Delta s < \infty, \quad \ell \in [\ell_0, \infty)_{\mathbb{T}}. \tag{3.30}$$

Then, we define the function

$$\widehat{P}(\ell) = \left(\frac{1}{\psi(\ell)} \int_{\ell}^{\infty} \int_{s}^{\infty} P(x) \Delta x \Delta s\right)^{\frac{1}{\alpha_1}} \quad \text{for } \ell \in [\ell_0, \infty)_{\mathbb{T}}. \tag{3.31}$$

Theorem 3.4. *Let* (1.3) *and* (1.4) *hold. If there exists number* $e_1 \in (0,1)_{\mathbb{R}}$ *such that equation* (3.16) *and the delay equation*

$$\Xi^{\Delta\Delta}(\ell) + \widehat{P}(\ell) H^{\frac{1}{\alpha_1}}(\Xi(g(\ell))) = 0 \tag{3.32}$$

are oscillatory, then the conclusion of Theorem 3.1 *holds true.*

P r o o f. Following the proof of Theorem 3.1, we distinguish two cases, Case (I) and Case (II), for $\ell \geq \ell_1$. Since the proof of Case (I) is identical to that of Case (I) of Theorem 3.1, it is omitted.

Case (II): Integrating (1.2) twice from $\ell(\geq \ell_1)$ to w and letting $w \to \infty$, we obtain

$$\begin{aligned} -\mathcal{X}^{\Delta\Delta}(\ell) &\geq \left[\frac{1}{\psi(\ell)} \int_{\ell}^{\infty} \int_{s}^{\infty} P(x) H(\mathcal{X}(g(\ell))) \Delta x \Delta s\right]^{\frac{1}{\alpha_1}} \\ &\geq \widehat{P}(\ell) H^{\frac{1}{\alpha_1}}(\mathcal{X}(g(\ell))). \end{aligned} \tag{3.33}$$

Integrating (3.33) again from $\ell(\geq \ell_1)$ to w and letting $w \to \infty$ yields

$$\mathcal{X}^{\Delta}(\ell) \geq \int_{\ell}^{\infty} \widehat{P}(y) H^{\frac{1}{\alpha_1}}(\mathcal{X}(g(y))) \Delta y, \tag{3.34}$$

which on integration from ℓ_1 to ℓ yields

$$\mathcal{X}(\ell) \geq \mathcal{X}(\ell_1) + \int_{\ell_1}^{\ell} \int_{s}^{\infty} \widehat{P}(y) H^{\frac{1}{\alpha_1}}(\mathcal{X}(g(y))) \Delta y \Delta s. \tag{3.35}$$

Thus, consider the equation

$$\Phi(\ell) = \mathcal{X}(\ell_1) + \int_{\ell_1}^{\ell} \int_{s}^{\infty} \widehat{P}(y) H^{\frac{1}{\alpha_1}}(\Phi(g(y))) \Delta y \Delta s. \tag{3.36}$$

We observe that if (3.36) admits a solution $\Phi(\ell) > 0$ for $\ell \geq \ell_1$, then Φ serves as a positive solution of (3.35). To demonstrate the existence of such a function Φ, we define the sequence $\{\Phi_m\}$ as follows:

$$\Phi_1(\ell) = \mathcal{X}(\ell), \quad \ell \in [\ell_1, \infty)_{\mathbb{T}},$$

and for $m > 1$,

$$\Phi_{m+1}(\ell) = \begin{cases} \mathcal{X}(\ell), & \ell \in [\ell_0, \ell_1]_{\mathbb{T}} \\ \mathcal{X}(\ell_1) + \displaystyle\int_{\ell_1}^{\ell} \int_{s}^{\infty} \widehat{P}(y) H^{\frac{1}{\alpha_1}}\left(\Phi_m(g(y))\right) \Delta y \Delta s, & \ell \in [\ell_1, \infty)_{\mathbb{T}}. \end{cases}$$

It is not difficult to verify that $\{\Phi_m\}$ is well-defined, and it satisfies

$$\mathcal{X}(\ell_1) \leq \Phi_m(\ell) \leq \mathcal{X}(\ell) \quad \text{for } \ell \in [\ell_1, \infty)_{\mathbb{T}}.$$

We will prove this by induction. For $m = 1$, using $\Phi_1(\ell) = \mathcal{X}(\ell)$ and the fact that $\mathcal{X}$ is increasing, we have

$$\mathcal{X}(\ell_1) \leq \mathcal{X}(\ell) = \Phi_1(\ell), \quad \ell \in [\ell_1, \infty)_{\mathbb{T}}.$$

For some $m \geq 1$, assume that

$$\mathcal{X}(\ell_1) \leq \Phi_m(\ell) \leq \mathcal{X}(\ell), \quad \ell \in [\ell_1, \infty)_{\mathbb{T}}.$$

Since H is non-decreasing, it follows that

$$H^{\frac{1}{\alpha_1}}\big(\Phi_m(g(y))\big) \leq H^{\frac{1}{\alpha_1}}\big(\mathcal{X}(g(y))\big).$$

Using the definition of Φ_{m+1} for $\ell \geq \ell_1$, we obtain

$$\begin{aligned}\Phi_{m+1}(\ell) &= \mathcal{X}(\ell_1) + \int_{\ell_1}^{\ell}\int_{s}^{\infty} \widehat{P}(y) H^{\frac{1}{\alpha_1}}\left(\Phi_m(g(y))\right) \Delta y\, \Delta s \\ &\leq \mathcal{X}(\ell_1) + \int_{\ell_1}^{\ell}\int_{s}^{\infty} \widehat{P}(y) H^{\frac{1}{\alpha_1}}\left(\mathcal{X}(g(y))\right) \Delta y\, \Delta s \\ &= \mathcal{X}(\ell)\end{aligned}$$

because of (3.35). Moreover, since $\widehat{P} \geq 0$ and $H^{\frac{1}{\alpha_1}}(\Phi_m(g(y))) \geq 0$, we have

$$\Phi_{m+1}(\ell) = \mathcal{X}(\ell_1) + \int_{\ell_1}^{\ell}\int_{s}^{\infty} \widehat{P}(y) H^{\frac{1}{\alpha_1}}\left(\Phi_m(g(y))\right) \Delta y\, \Delta s \geq \mathcal{X}(\ell_1).$$

Therefore, by mathematical induction,

$$\mathcal{X}(\ell_1) \leq \Phi_m(\ell) \leq \mathcal{X}(\ell), \quad \ell \in [\ell_1, \infty)_{\mathbb{T}}, \quad m \in \mathbb{N}.$$

It is not difficult to verify that $\{\Phi_m\}$ is monotone decreasing with respect to m. Indeed, for $m = 1$ and in view of (3.35),

$$\Phi_2(\ell) = \mathcal{X}(\ell_1) + \int_{\ell_1}^{\ell}\int_{s}^{\infty} \widehat{P}(y) H^{\frac{1}{\alpha_1}}\left(\mathcal{X}(g(y))\right) \Delta y\, \Delta s \leq \mathcal{X}(\ell) = \Phi_1(\ell).$$

Assume that $\Phi_m(\ell) \leq \Phi_{m-1}(\ell)$ for $\ell \in [\ell_1, \infty)_{\mathbb{T}}$. Since H is non-decreasing, we have

$$H^{\frac{1}{\alpha_1}}(\Phi_m(g(y))) \leq H^{\frac{1}{\alpha_1}}(\Phi_{m-1}(g(y))),$$

and hence

$$\Phi_{m+1}(\ell) \leq \Phi_m(\ell).$$

Therefore,

$$\Phi_1(\ell) \geq \Phi_2(\ell) \geq \cdots \geq \Phi_m(\ell) \geq \mathcal{X}(\ell_1), \quad \ell \in [\ell_1, \infty)_{\mathbb{T}}.$$

As a consequence, we have

$$\lim_{m\to\infty} \Phi_m(\ell) = \Phi(\ell) \quad \text{exists and finite for } \ell \geq \ell_1.$$

By applying Lebesgue's dominated convergence theorem, we obtain

$$\Phi(\ell) = \mathcal{X}(\ell_1) + \int_{\ell_1}^{\ell}\int_{s}^{\infty} \widehat{P}(y) H^{\frac{1}{\alpha_1}}(\Phi(g(y)))\Delta y \Delta s \quad \text{for } \ell \geq \ell_1.$$

Taking Δ-derivative twice, we deduce that $\Phi(\ell)$ is a non-oscillatory solution of (3.32), which contradicts our initial assumption. Thus, the proof is complete. □

Theorem 3.5. *Let* (1.3) *and* (1.4) *be satisfied. If there exist numbers* $e_1, \Theta \in (0,1)_{\mathbb{R}}$ *such that both the delay equation* (3.16) *and*

$$\Gamma^{\Delta}(\ell) + \Theta \widehat{P}(\ell) H^{\frac{1}{\alpha_1}}(g(\ell)) H^{\frac{1}{\alpha_1}}(\Gamma(g(\ell))) = 0 \tag{3.37}$$

are oscillatory, then it follows that the conclusion Theorem 3.4 *remains valid.*

P r o o f. Following the proof of Theorem 3.1, we distinguish two cases, Case (I) and Case (II), for $\ell \in [\ell_1, \infty)_{\mathbb{T}}$. Since the proof of Case (I) is identical to that of Case (I) of Theorem 3.1, it is omitted.

Case (II): Clearly, there exist $\ell_1 \geq \ell_0$ and a constant $e \in (0,1)_{\mathbb{R}}$ such that

$$\mathcal{X}(g(\ell)) \geq eg(\ell)\mathcal{X}^{\Delta}(g(\ell)) \quad \text{for } \ell \in [\ell_1, \infty)_{\mathbb{T}}. \tag{3.38}$$

Following the approach used in the proof of Case (II) of Theorem 3.4, we obtain (3.33). Substituting (3.38) in (3.33), we derive

$$-\mathcal{X}^{\Delta\Delta}(\ell) \geq \widehat{P}(\ell)H^{\frac{1}{\alpha_1}}(\mathcal{X}(eg(\ell)\mathcal{X}^{\Delta}(g(\ell)))).$$

Setting $\Gamma(\ell) = \mathcal{X}^{\Delta}(\ell)$ and $\bar{e} = H^{\frac{1}{\alpha_1}}(e)$ and then using hypothesis (iii), we obtain

$$\Gamma^{\Delta}(\ell) + \bar{e}\widehat{P}(\ell)H^{\frac{1}{\alpha_1}}(g(\ell))H^{\frac{1}{\alpha_1}}(\Gamma(g(\ell))) \leq 0.$$

The remainder of the proof is analogous to that of Case (I) in Theorem 3.1 and is therefore omitted. This completes the proof □

For the next result, we introduce a function $\bar{P}(\ell)$ satisfying

$$\bar{P}(\ell) \geq \min\left\{\widehat{P}(\ell)H^{\frac{1}{\alpha_1}}(g(\ell)),\ P(\ell)H(\Psi[g(\ell), \ell_0])\right\} \quad \text{for } \ell \in [\ell_0, \infty)_{\mathbb{T}}.$$

Furthermore, we assume that

$$H^{\frac{1}{\alpha_1}}(z) := \big(H(z)\big)^{\frac{1}{\alpha_1}} = H(z^{\frac{1}{\alpha_1}}) \quad \text{for } z \neq 0.$$

Remark 3.6. A typical example of a nonlinear function satisfying $H^{\frac{1}{\alpha_1}}(z) = H\big(z^{\frac{1}{\alpha_1}}\big)$, $z \neq 0$, is

$$H(z) = \operatorname{sgn}(z)\,|z|^{\beta}, \quad \beta > 0.$$

Theorem 3.7. *Let* (1.3) *and* (1.4) *be satisfied. If, for every constant* $\Theta \in (0,1)_{\mathbb{R}}$,

$$\zeta^{\Delta}(\ell) + \Theta\bar{P}(\ell)H(\zeta^{\frac{1}{\alpha_1}}(g(\ell))) = 0 \tag{3.39}$$

is oscillatory, then (1.2) *oscillates.*

P r o o f. The details of the proof are omitted, as they follow by a straightforward argument. □

4. Applications

Next, we examine a particular case of (1.2), namely,

$$\left[\psi(\ell)\left(\mathcal{X}^{\Delta\Delta}(\ell)\right)^{\alpha_1}\right]^{\Delta\Delta} + P(\ell)\mathcal{X}^{\beta}(\ell - \tau) = 0, \tag{4.40}$$

where $\tau > 0$ is a constant delay, $g(\ell) = \ell - \tau$ and β is the quotient of positive odd integers.

For this equation, we have

$$\tilde{P}(\ell) \geq \min\left\{\widehat{P}(\ell)H^{\frac{1}{\alpha_1}}(g(\ell)),\ P(\ell)H(\Psi[g(\ell),\ell_0])\right\},$$

where

$$H^{\frac{1}{\alpha_1}}(g(\ell)) = g^{\frac{\beta}{\alpha_1}}(\ell),$$

$$\widehat{P}(\ell) = \left(\frac{1}{\psi(\ell)}\int_{\ell}^{\infty}\int_{s}^{\infty} P(x)\Delta x\Delta s\right)^{\frac{1}{\alpha_1}},$$

$$\text{and}\quad H(\Psi[g(\ell),\ell_0]) = \left(\int_{\ell_0}^{\ell-\tau}\int_{\ell_0}^{s}\left(\frac{x}{\psi(x)}\right)^{\frac{1}{\alpha_1}}\Delta x\Delta s\right)^{\beta}.$$

For the continuous case $\mathbb{T} = \mathbb{R}$, the following result is an immediate consequence of Theorem 3.7, together with [19, Theorem 1] and the results established in [5].

Corollary 4.1. *If* (1.3) *holds, then equation* (4.40) *is oscillatory under either of the following conditions:*

(1) *If* $\alpha_1 = \beta$, *then*

$$\lim_{\ell\to\infty}\int_{\ell_0}^{\ell}\tilde{P}(x)\ln\left(e\int_{\ell}^{\ell+\tau}\tilde{P}(y)\mathrm{d}y\right)\mathrm{d}x > 0,$$

with $\int_{\ell}^{\ell+\tau}\tilde{P}(y)\mathrm{d}y > 0$.

(2) *If* $\alpha_1 < \beta$, *then*

$$\lim_{\ell\to\infty}\int_{\ell_0}^{\ell}\tilde{P}(x)\mathrm{d}x = \infty.$$

The next result is an immediate consequence of Theorem 3.7 and [23, Theorem 3(i)].

Corollary 4.2. *Let* (1.3) *hold and* $\alpha_1 > \beta$. *Then equation* (4.40) *is oscillatory provided that there exists* $\delta > \tau^{-1}\ln\left(\frac{\alpha_1}{\beta}\right)$ *such that*

$$\liminf_{\ell\to\infty}\left(\tilde{P}(\ell)\exp(-\mathrm{e}^{-\delta\ell})\right) > 0.$$

For the discrete case $\mathbb{T} = \mathbb{Z}$, the following result follows immediately from Theorem 3.7, together with [6, Theorem 2.2] and [22, Theorem 1].

Corollary 4.3. *If* (1.3) *holds, then equation* (4.40) *is oscillatory under either of the following conditions:*

(1) *If* $\alpha_1 = \beta$, *then*

$$\liminf_{\ell\to\infty}\tilde{P}(\ell) > \frac{\tau^{\tau}}{(\tau+1)^{\tau+1}}.$$

(2) *If* $0 < \beta < \alpha_1$, *then*

$$\sum_{x=\ell_0}^{\infty} \tilde{P}(x) = \infty.$$

In a similar manner, the next result can be derived by applying Theorem 3.7 in conjunction with [21, Theorem 3.1].

Corollary 4.4. *If* (1.3) *holds and* $\alpha_1 > \beta$, *and there exists a sequence* $\Omega(\ell)\}$ *satisfying*

$$\Delta\Omega(\ell) > 0 \quad \textit{and} \quad \lim_{\ell\to\infty} \Omega(\ell) = \infty,$$

then equation (4.40) *is oscillatory provided that*

$$\limsup_{\ell\to\infty} \frac{\alpha_1 \Delta\Omega(\ell-\tau)}{\beta\Delta\Omega(\ell)} < 1 \quad \textit{and} \quad \liminf_{\ell\to\infty} \left[\tilde{P}(\ell)\frac{e^{-\Omega(\ell)}}{\Delta\Omega(\ell)}\right] > 0.$$

5. Conclusion

This paper presents new oscillation criteria for equation (1.2) by developing a comparison method that links the oscillatory behaviour of the considered higher-order dynamic equation to that of suitable lower-order equations. As applications of the main results, four corollaries were obtained, and an illustrative example is provided to demonstrate the effectiveness of the obtained criteria.

The significance of the proposed approach lies in its flexibility. Since numerous oscillation results are already available for first- and second-order dynamic equations, the developed comparison framework offers a systematic mechanism for deriving further oscillation criteria for equation (1.2). We believe that this technique can be adapted to a broader class of nonlinear dynamic equations on time scales and may stimulate further developments in the oscillation theory of higher-order functional dynamic equations.

Finally, we conclude by highlighting some directions for future research:

(1) The present study achieves the desired oscillation results through comparison with nonlinear lower-order equations. However, the analysis of nonlinear comparison equations may itself be challenging. Therefore, an important direction for future research is to establish comparison principles involving linear lower-order equations while preserving the essential nonlinear features of the original higher-order equation.

(2) Another interesting problem is to develop analogous oscillation criteria in the case where the integral condition (1.3) is finite, which remains open and deserves further investigation.

Acknowledgement. We would like to thank the anonymous reviewers for the helpful suggestions that allowed us to improved the presentation of our manuscript.

REFERENCES

[1] AGARWAL, R. P.—BOHNER, M.—LI, T.—ZHANG, C.: *Hille and Nehari type criteria for third order delay dynamic equations*, J. Difference Equ. Appl. **19** (2013), 1563–1579.
[2] AGARWAL, R. P.—BOHNER, M.—LI, T.—ZHANG, C.: *Oscillation theorems for fourth-order half-linear delay dynamic equations with damping*, Mediterr J. Math. **11** (2014), 463–475.
[3] BOHNER, M.—PETERSON, A.: *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhauser, Boston, 2001.
[4] BOHNER, M.—PETERSON, A.: *Advances in Dynamic Equations on Time Scales*, Birkhauser, Boston, 2003.
[5] ERBE, L. H.—KONG, Q.—ZHANG, B. G.: *Oscillation theory for functional differential equations*, Marcel Dekker, New York, 1995.
[6] ERBE, L. H.—ZHANG, B. G.: *Oscillation of discrete analogues of delay equations*, Differ. Integr. Equ. **2** (1989), 300–309.
[7] ERBE, L. H.—KARPUZ, B.—PETERSON, A.: *Kamenev-type oscillation criteria for higher order neutral delay dynamic equations*, Int. J. Difference Equ. **6** (2011), 1–16.
[8] ERBE, L. H.—MERT, R.: *Oscillation of even order nonlinear delay dynamic equations on time scales*, Czechoslovak J. Math. **63** (2013), 265–279.
[9] ERINGEN, A. C.: *Nonlocal Continuum Field Theories*, Springer, 2002.
[10] GRACE, S. R.—AGARWAL, R. P.—PINELAS, S.: *Oscillation criteria for certain fourth-order nonlinear difference equations*, Commun. Appl. Anal. **4** (2010), 337–342.
[11] GRACE, S. R.—HASSAN, T. S.: *Oscillation criteria for higher order nonlinear dynamic equations*, Math. Nachr. **287** (2014), 1659–1673.
[12] GRACE, S. R.—BOHNER, M.—SUN, S.: *Oscillation of fourth-order dynamic equations*, Hacet. J. Math. Stat. **39** (2010), 545–553.
[13] GRACE, S. R.—AGARWAL, R. P.—WANG, C.: *New oscillation results for non-canonical higher order nonlinear neutral dynamic equations*, Bol. Soc. Mat. Mex. **27** (2021), Art. 65.
[14] GRACE, S. R.—CHHATRIA, G. N.: *Oscillation of higher order nonlinear dynamic equations with a nonlinear neutral term*, Math. Meth. Appl. Sci. **46** (2023), 2373–2388.
[15] HASSAN, A.—ODINAEV, I.—HASSAN, T. S.: *Oscillatory behavior of noncanonical quasilinear second-order dynamic equations on time scales*, J. Math. **2023** (2023), Art. 5585174.
[16] WOINOWSKY-KRIEGER, A.: *The Behavior of Elastic Beams Under Load*, McGraw-Hill, 1950.
[17] LADDE, G. S.—LAKSHMIKANTHAM, V.—ZHANG, B. G.: *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York, 1987.
[18] LADDE, G. S.—LAKSHMIKANTHAM, V.: *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, 2000.
[19] LI, B.: *Oscillation of first order delay differential equations*, Proc. Am. Math. Soc. **124** (1996), 3729–3737.
[20] OGATA, K.: *Modern Control Engineering*, Prentice Hall, 2010.
[21] OCALAN, O.: *Oscillation analysis of solutions of first order sublinear and superlinear difference equations*, Rocky Mountain. J. Math., accepted.

[22] TANG, X. H.: *Oscillation for nonlinear delay difference equations*, Tamkang J. Math. **32** (2001), 275–280.
[23] TANG, X. H.: *Oscillation for first order superlinear delay differential equations*, J. London Math. Soc. **65** (2002), 115–122.
[24] ZHANG, C.—LI, T.—AGARWAL, R. P.—BOHNER, M.: *Oscillation results for fourth-order nonlinear dynamic equations*, Appl. Math. Lett. **25** (2012), 2058–2065.
[25] ZHANG, C.—AGARWAL, R. P.—BOHNER, M.—LI, T.: *Oscillation of fourth-order delay dynamic equations*, Sci. China Math. **58** (2015), 143–160.
[26] ZHOU, Y.—HE, J. W.—AHMAD, B.—ALSAEDI, A.: *Necessary and sufficient conditions for oscillation of fourth-order dynamic equations on time scales*, Adv. Differ. Equ. **2019** (2019), Art. 308.

Received May 12, 2025
Revised June 22, 2026
Accepted July 15, 2026
Publ. online November 30, 2026

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