

ON THE ASYMPTOTIC BEHAVIOUR OF THE INTEGRAL $T(x) = \int_0^{+\infty} \left| \left\{ \frac{x}{t} \right\} - \left\{ \frac{x}{t+1} \right\} \right| dt$

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ABSTRACT. In this paper, we estimate the integral $T(x)$ considered in the title, where $\{t\}$ denote the fractional parts of the real number t , and x is any positive real number.

1. Introduction and Results

The study of sums involving fractional parts aroused great interest among many researchers, particularly in the late 19th century. One of the key results of this period is the asymptotic formula for the sum of fractional parts, derived from Dirichlet's seminal work [1] on the divisor function $\tau(n)$:

$$\sum_{n \leq x} \left\{ \frac{x}{n} \right\} = (1 - \gamma)x + O\left(x^{1/2}\right), \quad (1)$$

where γ is the Euler–Mascheroni constant. This result paved the way for further investigations of these sums, leading to approximate formulas and error bound estimates for sums involving fractional parts. Recently, the refinement of error terms for (1) has been considerably advanced by Bourgain [4]; he showed

$$\sum_{n \leq x} \left\{ \frac{x}{n} \right\} = (1 - \gamma)x + O\left(x^{517/1648+\varepsilon}\right),$$

where $\varepsilon > 0$ is arbitrarily small.

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2020 Mathematics Subject Classification: 11N37.

Keywords: asymptotic behaviour, fractional part, integral, divisor function.

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The study of the variation between two successive terms,

$$\left\{\frac{x}{n}\right\} \quad \text{and} \quad \left\{\frac{x}{n+1}\right\},$$

was initially carried out by Aurel Wintner [6]. Later, Balazard in [2], established the following result

$$\sum_{n \geq 1} \left| \left\{\frac{x}{n}\right\} - \left\{\frac{x}{n+1}\right\} \right| = \frac{2}{\pi} \zeta(3/2) \sqrt{x} + O\left(x^{2/5}\right), \quad (x > 0). \quad (2)$$

Balazard et al. [3] extended the result (2) by studying a more general case. They established the following asymptotic formula

$$\sum_{n \geq 0} \left| \left\{\frac{x}{n+a}\right\} - \left\{\frac{x}{n+b}\right\} \right| = \frac{2}{\pi} \zeta(3/2) \sqrt{cx} + O\left(c^{2/9} x^{4/9}\right), \quad (3)$$

where $b > a > 0$, $c = b - a$, and the result holds uniformly for $x \geq 40c^5(1+b)^{27/2}$.

The connection between sums and integrals has been a fundamental topic, with the Riemann–Stieltjes integral serving as a particularly powerful tool in this context. One of its key advantages lies in its simplicity, as demonstrated by the following result:

$$\sum_{\alpha < n \leq \beta} f(n) = \int_{\alpha}^{\beta} f(t) dt + O(|f(\alpha)| + |f(\beta)|),$$

where $\alpha, \beta \in \mathbb{Z}$ and f is a real-valued monotonic function on $[\alpha, \beta]$. If the function f is not monotonic, the problem poses some difficulties. In this case, Wu and Shi [7] focused on refining the error term in the difference

$$\sum_{n \leq x} f(n) \left\{\frac{x}{n}\right\}^k - \int_1^x f(t) \left\{\frac{x}{t}\right\}^k dt.$$

Building on this, Mercier and Nowak [5] derived the following bound:

$$\sum_{n \leq x} f(n) \left\{\frac{x}{n}\right\}^k = \int_1^x f(t) \left\{\frac{x}{t}\right\}^k dt + O\left(f(x) x^{131/416} (\log x)^{26947/8320}\right),$$

where $f(t)$ is a real-valued, positive, nondecreasing function defined for $t \geq 1$, and k is an arbitrary positive integer. These technical difficulties led us to adopt Balazard’s direct approach [2]. Attempts to apply a result analogous to that of Mercier and Nowak are hindered by the highly oscillatory nature of the integrand $|\{x/t\} - \{x/(t+1)\}|$, which does not satisfy the monotonicity and regularity conditions required for classical estimates of the sum-to-integral error. Furthermore, the absolute value precludes cancellation techniques, rendering

classical bounds insufficient for precise error control. By faithfully reproducing Balazard's method, which consists of partitioning the integration domain according to the constant values of $\lfloor x/t \rfloor$ and $\lfloor x/(t+1) \rfloor$, we perform a precise interval-by-interval analysis that overcomes these obstacles and leads to the asymptotic formula of Theorem 1.1. Motivated by this result, we study the asymptotic behaviour of $T(x)$ given in the title.

We now state our principal result.

THEOREM 1.1. *For all $x > 0$, we have*

$$T(x) = \frac{2}{\pi} \zeta(3/2) \sqrt{x} + O\left(x^{3/7}\right), \quad (4)$$

where ζ denotes the Riemann zeta function.

Certain formulas from Balazard's original article [2] are repeated here to ensure the clarity and coherence of our presentation.

2. Proof of Theorem 1.1

Let $x > 1$ be a real number. As $T(x)$ is already defined in the title, we proceed directly with its integral representation. This function can also be rewritten as

$$T(x) = \int_0^{+\infty} \left| \left\lfloor \frac{x}{t} \right\rfloor - \left\lfloor \frac{x}{t+1} \right\rfloor - \frac{x}{t(t+1)} \right| dt.$$

Setting $k = \lfloor \frac{x}{t} \rfloor$ and $h = \lfloor \frac{x}{t+1} \rfloor$, we obtain

$$t \in I(h, k; x) := \left(\frac{x}{k+1}, \frac{x}{k} \right] \cap \left(\frac{x}{h+1} - 1, \frac{x}{h} - 1 \right], \quad (5)$$

where $k \geq h \geq 0$. Consequently, $T(x)$ can be expressed as

$$T(x) = \sum_{k \geq h \geq 0} \left(\int_{t \in I(h, k; x)} \left| k - h - \frac{x}{t(t+1)} \right| dt \right).$$

Now, for a positive integer d , and in a manner similar to Balazard's construction, we define the set

$$E_d := \{(h, k) \in \mathbb{N}^2 : 0 \leq h \leq k, k - h = d\} = \{(k - d, k) : k \in \mathbb{N}, d \leq k\}. \quad (6)$$

The set E_d is defined in a manner analogous to that in Balazard [2]. We also introduce the auxiliary function

$$\begin{aligned} T_d(x) &= \sum_{(h,k) \in E_d} \left(\int_{t \in I(h,k;x)} \left| k - h - \frac{x}{t(t+1)} \right| dt \right) \\ &= \sum_{d \leq k} \left(\int_{t \in I(k-d,k;x)} \left| d - \frac{x}{t(t+1)} \right| dt \right). \end{aligned} \quad (7)$$

Thus, we obtain the final expression

$$T(x) = \sum_{d \geq 0} T_d(x).$$

To derive an asymptotic formula for the sum $T(x)$ defined above, we introduce a large positive real number D , and represent the sum $T(x)$ as follows:

$$T(x) = T_0(x) + \sum_{0 < d \leq D} T_d(x) + \sum_{d > D} T_d(x).$$

We then set

$$T_1(x) := T_1(x; D) = \sum_{0 < d \leq D} T_d(x)$$

and

$$T_2(x) := T_2(x; D) = \sum_{d > D} T_d(x),$$

so that

$$T(x) = T_0(x) + T_1(x) + T_2(x).$$

In our next plan, we will study separately the three partial sums $T_0(x)$, $T_1(x)$ and $T_2(x)$ by finding their asymptotic formulas, ultimately obtaining an asymptotic formula for the sum $T(x)$.

2.1. Exploring the behaviour of the sum $T_0(x)$

LEMMA 2.1. *For all $x > 0$, we have*

$$T_0(x) = \frac{2}{3}\sqrt{x} + O(1). \quad (8)$$

Proof. We note that in the case where $d = 0$, we may have $h = k$, which leads to

$$t \in I(k, k; x) = \left(\frac{x}{k+1}, \frac{x}{k} - 1 \right] \quad (\text{see (5)})$$

We note that the interval $I(k, k; x)$, is non-empty if and only if $k(k+1) \leq x$. Let us denote by $K = K(x)$ the greatest integer k such that $k(k+1) \leq x$, i.e.,

$$K = \left\lfloor \sqrt{x + 1/4} - 1/2 \right\rfloor.$$

Notice that

$$K = \sqrt{x} + O(1) \quad \text{and} \quad K \sim \sqrt{x}.$$

So, according to (7), we obtain the following expression for $T_0(x)$, i.e.,

$$\begin{aligned} T_0(x) &= \sum_{0 \leq k \leq K} \int_{x/(k+1)}^{x/k-1} \frac{x}{t(t+1)} dt \\ &= x \sum_{0 \leq k \leq K} \int_{x/(k+1)}^{x/k-1} \left(\frac{1}{t^2} - \frac{1}{t^3} + O\left(\frac{1}{t^4}\right) \right) dt. \end{aligned}$$

Simplifying this expression, we get

$$\begin{aligned} T_0(x) &= x \sum_{0 \leq k \leq K} \left(\frac{1}{x/k-1} - \frac{1}{x/(k+1)} - \frac{1}{2(x/k-1)^2} + \frac{1}{2(x/(k+1))^2} + \right. \\ &\quad \left. O\left(\frac{(k+1)^3}{x^3}\right) \right). \quad (9) \end{aligned}$$

To exploring the behaviour of $T_0(x)$, we begin by noting that

$$\begin{aligned} &x \sum_{0 \leq k \leq K} \left(\frac{1}{x/k-1} - \frac{1}{x/(k+1)} \right) \\ &= x \sum_{0 \leq k \leq K} \left(\frac{k}{x-k} - \frac{k+1}{x} \right) = K + O(1). \end{aligned} \quad (10)$$

Moreover, we have

$$\begin{aligned}
 x \sum_{0 \leq k \leq K} & \left(-\frac{1}{2(x/k-1)^2} + \frac{1}{2(x/(k+1))^2} + O\left(\frac{(k+1)^3}{x^3}\right) \right) \\
 &= -\frac{K^3 + O(K^2)}{3x} + O\left(\frac{K^2}{x}\right) + O\left(\frac{K^4}{x^2}\right) \\
 &= -\frac{1}{3}\sqrt{x} + O(1).
 \end{aligned} \tag{11}$$

The conjunction of (9), (10) and (11) gives (8). $\square$

2.2. Exploring the behaviour of the sum $T_1(x)$

The key to this step of the proof is to first specify the intervals $I(h, k; x)$ mentioned in (5) without using intersections. Specifically, if

$$k > h \geq 0 \quad \text{and} \quad x > 0,$$

we have

$$\frac{x}{k} \leq \frac{x}{h} - 1 \iff kh \leq k - h \leq x.$$

In particular, this leads to the implications

$$\frac{x}{k+1} \leq \frac{x}{h+1} - 1 \implies \frac{x}{k} \leq \frac{x}{h} - 1$$

and

$$\frac{x}{k} > \frac{x}{h} - 1 \implies \frac{x}{k+1} > \frac{x}{h+1} - 1.$$

These observations naturally lead us to consider only three distinct intervals, systematically linked to three distinct parts of the set E_d , defined in (6). So, for each particular interval, we will associate its corresponding subset, denoted as $E_{d,i} := E_{d,i}(x)$ ($i = 1, 2, 3$), knowing that $h = k - d$.

In the case, where $\frac{x}{k+1} \leq \frac{x}{h+1} - 1$, we obtain

$$I(k-d, k; x) = \left(\frac{x}{k-d+1} - 1, \frac{x}{k} \right], \tag{12}$$

and

$$E_{d,1} = \{(k-d, k) : d \leq k, (k-d+1)(k+1) \leq xd\}.$$

In the case where $\frac{x}{k} > \frac{x}{h} - 1$, we obtain

$$I(k-d, k; x) = \left(\frac{x}{k+1}, \frac{x}{k-d} - 1 \right], \tag{13}$$

and

$$E_{d,2} = \{(k-d, k) : d \leq k, (k-d)k > xd\}.$$

Moreover, when

$$I(k-d, k; x) = \left(\frac{x}{k+1}, \frac{x}{k} \right], \quad (14)$$

we obtain

$$E_{d,3} = \{(k-d, k) : d \leq k, (k-d)k \leq xd < (k-d+1)(k+1)\}.$$

Now, for $x > 0$ and $k > d \geq 0$, we define $K_d := K_d(x)$ as the largest integer k satisfying $(k-d)k \leq dx$. So, we have $K_0 = 0$, and in general

$$K_d = \left\lfloor \frac{d + \sqrt{d^2 + 4dx}}{2} \right\rfloor.$$

The use of the function K_d allows us to give fluidity to the quadratic conditions associated with the number k , which contribute to the definition of the subsets

$$E_{d,i} \quad (i = 1, 2, 3),$$

i.e.,

$$E_{d,1} = \{(k-d, k) : d \leq k, k \leq K_d - 1\}, \quad (15)$$

$$E_{d,2} = \{(k-d, k) : d \leq k, k > K_d\}, \quad (16)$$

$$E_{d,3} = \{(k-d, k) : d \leq k, k = K_d\}. \quad (17)$$

Summarising the above results allows us to write $T_d(x)$ defined in (7), as follows

$$T_d(x) = T_{d,1}(x) + T_{d,2}(x) + T_{d,3}(x),$$

where

$$T_{d,i}(x) = \sum_{(k-d,k) \in E_{d,i}} \left(\int_{t \in I(k-d,k;x)} \left| d - \frac{x}{t(t+1)} \right| dt \right) \quad (i=1,2,3). \quad (18)$$

In the process of finding asymptotic formulas for the auxiliary functions $T_{d,i}(x)$ ($i = 1, 2, 3$), we must first consider how the absolute value is omitted in the expression of each function. For this, we define the function $N_d := N_d(x)$ as the largest real number verifying $t(t+1) \leq x/d$, such that $d > 0$. This leads to the following expression

$$N_d = \frac{-1 + \sqrt{1 + 4x/d}}{2}.$$

Compared to Balazard's article, it should be noted that N_d is defined here as a real number, not as an integer.

The obtained number N_d can be expressed in relation to the number K_d according to [2, Proposition 4, p. 10], as follows

$$\frac{x}{K_d + 1} \leq N_d \leq \frac{x}{K_d}, \quad (19)$$

in order to pave the way for the upcoming work.

We will now find the final expressions of the auxiliary functions

$$T_{d,i}(x) \quad (i = 1, 2, 3),$$

which will allow us to find the expression of the function $T_d(x)$, and finally work on determining its asymptotic formula. Throughout paragraphs 2.2.1 to 2.2.4, let us note that $d > 0$.

2.2.1. Obtaining the expression of the sum $T_{d,1}(x)$

Firstly, it is important to note that the definition of the sum $T_{d,1}(x)$, given by the formula (18) for $i = 1$, linked to the set $E_{d,1}$, defined in (15), as well as to the interval given in (12). The latter is non-empty if and only if $\frac{x}{k} > 1$, or, in other words, if and only if $k > K_d - 1$. Moreover, we know that if $k \leq K_d - 1$, it follows from (19) and the definition of K_d , that the variable t satisfies the inequality

$$t \geq \frac{x}{K_d} \geq N_d.$$

Therefore,

$$\begin{aligned} T_{d,1}(x) &= \sum_{K_d-1 < k \leq K_d-1} \int_{x/(k-d+1)-1}^{x/k} \left(d - \frac{x}{t(t+1)} \right) dt \\ &= \sum_{K_d-1 < k \leq K_d-1} \left(d \left(\frac{x}{k} - \frac{x}{k-d+1} + 1 \right) + x \log \frac{1 - (k-d+1)/x}{1 + k/x} \right). \end{aligned}$$

We continue, this time focusing on the telescopic identity

$$\sum_{a < k \leq b} (u_k - u_{k-r}) = \sum_{b-r < k \leq b} u_k - \sum_{a-r < k \leq a} u_k.$$

This identity will also be applied in the following paragraphs.

So, we find that

$$\begin{aligned} T_{d,1}(x) &= d \sum_{K_d-d < k \leq K_d-1} \frac{x}{k} - d \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} \frac{x}{k} \\ &\quad + d(K_d - K_{d-1} - 1) \end{aligned} \tag{20}$$

$$+ x \sum_{K_{d-1} < k \leq K_d-1} \log \frac{1 - (k-d+1)/x}{1 + k/x}. \tag{21}$$

2.2.2. Obtaining the expression of the sum $T_{d,2}(x)$

We recall that the sum $T_{d,2}(x)$, defined by formula (18) for $i = 2$, is associated with the set $E_{d,2}$, introduced in (16), as well as with the interval specified in (13). The latter is non-empty if and only if $\frac{x}{k+1} \leq \frac{x}{k-d} - 1$, which is equivalent to the condition $k \leq K_{d+1} - 1$. Furthermore, when $k > K_d$, it follows from (19), and the definition of K_d , that the variable t satisfies the inequality

$$t \leq \frac{x}{K_d + 1 - d} - 1 \leq \frac{x}{K_d + 1} \leq N_d.$$

Therefore,

$$\begin{aligned} T_{d,2}(x) &= \sum_{K_d < k \leq K_{d+1}-1} \int_{x/(k+1)}^{x/(k-d)-1} \left(\frac{x}{t(t+1)} - d \right) dt \\ &= x \sum_{K_d < k \leq K_{d+1}-1} \log \left(1 - \frac{k-d}{x} \right) \left(1 + \frac{k+1}{x} \right) \\ &\quad - d \sum_{K_d < k \leq K_{d+1}-1} \left(\frac{x}{k-d} - \frac{x}{k+1} - 1 \right). \end{aligned}$$

More explicitly, we obtain

$$\begin{aligned} T_{d,2}(x) &= x \sum_{K_d < k \leq K_{d+1}-1} \log \left(1 - \frac{k-d}{x} \right) \left(1 + \frac{k+1}{x} \right) \\ &\quad + d \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} \frac{x}{k} - d \sum_{K_d-d < k \leq K_{d+1}} \frac{x}{k} \\ &\quad + d(K_{d+1} - K_d - 1). \end{aligned} \tag{22}$$

2.2.3. Obtaining the expression of the sum $T_{d,3}(x)$

Currently, the function $T_{d,3}(x)$ is defined through relation (18) for $i = 3$, and is associated with the set defined in (17). Through this, and from (14), the interval of real variable t is given by $\left(\frac{x}{K_{d+1}}, \frac{x}{K_d} \right]$. Therefore,

$$\begin{aligned} T_{d,3}(x) &= \int_{x/(K_{d+1})}^{x/K_d} \left| d - \frac{x}{t(t+1)} \right| dt \\ &= \int_{x/(K_{d+1})}^{N_d} \left(\frac{x}{t(t+1)} - d \right) dt + \int_{N_d}^{x/K_d} \left(d - \frac{x}{t(t+1)} \right) dt. \end{aligned}$$

More precisely,

$$\begin{aligned}
 T_{d,3}(x) &= x \log \frac{1 + K_d/x}{1 + (K_d + 1)/x} \\
 &\quad - 2x \log \left(1 + \frac{1}{N_d} \right) - d \left(2N_d - \frac{x}{K_d + 1} - \frac{x}{K_d} \right). \tag{23}
 \end{aligned}$$

2.2.4. Final expression and behaviour of $T_d(x)$

Adding (20)–(23) and simplifying yields

$$\begin{aligned}
 T_d(x) &= T_{d,1}(x) + T_{d,2}(x) + T_{d,3}(x) \\
 &= d \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} \frac{x}{k} - d \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} \frac{x}{k} \\
 &\quad + x \sum_{K_{d-1} < k \leq K_{d+1}} \log \frac{1 + k/x}{1 - (k - d)/x} \\
 &\quad - x \log \left(1 - \frac{K_{d+1} - d}{x} \right) - x \log \left(1 - \frac{K_{d-1} - d + 1}{x} \right) \\
 &\quad + d(K_{d+1} - K_{d-1} - 2N_d - 2) - 2x \log \left(1 + \frac{1}{N_d} \right). \tag{24}
 \end{aligned}$$

Finding the behaviour of the sum $T_d(x)$ relies on the following lemmas.

LEMMA 2.2. *For any integer $d > 0$, and $K_d - d < k \leq K_d$, we have*

$$\sum_{K_d - d < k \leq K_d} \frac{x}{k} = \sqrt{dx} + O(d).$$

Proof. Note that if

$$K_d - d < k \leq K_d, \quad \text{we have} \quad k = \sqrt{dx} + O(d) \quad (\text{see [2, p. 8]}).$$

Therefore,

$$\sum_{K_d - d < k \leq K_d} \frac{x}{k} = \sum_{K_d - d < k \leq K_d} \left(\sqrt{\frac{x}{d}} + O(1) \right) = \sqrt{dx} + O(d).$$

□

LEMMA 2.3. *For any integer $d > 0$, we have*

$$\begin{aligned}
 x \sum_{K_{d-1} < k \leq K_{d+1}} \log \frac{1 + k/x}{1 - (k - d)/x} &= \frac{(2d - 1)\sqrt{d + 1} - (2d + 1)\sqrt{d - 1}}{3} \sqrt{x} \\
 &\quad + O\left(d^{3/2}x^{-1/2}\right) + O(d). \tag{25}
 \end{aligned}$$

Proof. On the one hand, we have

$$\sum_{K_{d-1} < k \leq K_{d+1}} \log \left(1 + \frac{k}{x} \right) = \sum_{K_{d-1} < k \leq K_{d+1}} \left(\frac{k}{x} - \frac{k^2}{2x^2} + O \left(\frac{k^3}{x^3} \right) \right),$$

and, on the other hand, we have

$$\sum_{K_{d-1} < k \leq K_{d+1}} \log \left(1 - \frac{k-d}{x} \right) = \sum_{K_{d-1} < k \leq K_{d+1}} \left(-\frac{k-d}{x} - \frac{(k-d)^2}{2x^2} + O \left(\frac{k^3}{x^3} \right) \right).$$

We now use the fact that

$$K_{d+1} - K_{d-1} = \left(\sqrt{d+1} - \sqrt{d-1} \right) \sqrt{x} + O(1) \quad ([2, \text{p. } 8]), \quad (26)$$

which leads to

$$K_{d+1} - K_{d-1} \ll \sqrt{x/d}, \quad \text{and} \quad \sum_{K_{d-1} < k \leq K_{d+1}} k^3 \ll \sqrt{x/d} \cdot (dx)^{3/2} = dx^2.$$

Hence, we conclude that

$$\begin{aligned} x \sum_{K_{d-1} < k \leq K_{d+1}} \log \frac{1 + k/x}{1 - (k-d)/x} &= \sum_{K_{d-1} < k \leq K_{d+1}} \left(d - \frac{k^2}{x} + \frac{2kd - d^2}{2x} + O \left(\frac{k^3}{x^2} \right) \right) \\ &= \frac{(2d-1)\sqrt{d+1} - (2d+1)\sqrt{d-1}}{3} \sqrt{x} \\ &\quad + O \left(d^{3/2} x^{-1/2} \right) + O(d). \end{aligned}$$

□

LEMMA 2.4. *For any integer $d > 0$ and $x > 0$, we have*

$$T_d(x) = f(d)\sqrt{x} + O(d^2),$$

where

$$f(d) = \frac{(8d+2)\sqrt{d+1} - (8d-2)\sqrt{d-1}}{3} - 4\sqrt{d}.$$

Proof. We apply Lemma 2.2, first changing d to $d+1$, then d to $d-1$, and obtain

$$\begin{aligned} d \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} \frac{x}{k} - d \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} \frac{x}{k} \\ = d\sqrt{(d+1)x} + O(d^2) - d\sqrt{(d-1)x} + O(d^2). \end{aligned}$$

For the penultimate step of (24), we have

$$-x \log \left(1 - \frac{K_{d+1} - d}{x} \right) - x \log \left(1 - \frac{K_{d-1} - d + 1}{x} \right) = \left(\sqrt{d+1} + \sqrt{d-1} \right) \sqrt{x} + O(d). \quad (27)$$

For the last expression of (24), using (26), and the fact that

$$\frac{K_d}{x} \leq \frac{1}{N_d} \leq \frac{K_d + 1}{x} \quad \text{and} \quad N_d \leq \sqrt{x/d},$$

we find

$$\begin{aligned} & d(K_{d+1} - K_{d-1} - 2N_d - 2) - 2x \log \left(1 + \frac{1}{N_d} \right) \\ &= d \left(\sqrt{d+1} - \sqrt{d-1} - 2/\sqrt{d} \right) \sqrt{x} + O(d) \\ &\quad - 2x \left(\frac{1}{N_d} + O \left(\frac{1}{N_d^2} \right) \right) \\ &= d \left(\sqrt{d+1} - \sqrt{d-1} \right) \sqrt{x} - 4\sqrt{d}\sqrt{x} + O(d). \end{aligned} \quad (28)$$

By inserting (25)–(28) into (24), we obtain

$$T_d(x) = f(d)\sqrt{x} + O(d^2),$$

where

$$f(d) = \frac{(8d+2)\sqrt{d+1} - (8d-2)\sqrt{d-1}}{3} - 4\sqrt{d}.$$

□

2.3. Exploring the behaviour of the sum $T_2(x)$

LEMMA 2.5. *Let D be a large positive real number. We have*

$$T_2(x) \leq \frac{\sqrt{x}}{D-1}.$$

Proof. Let D be a real number greater than 1. If

$$d = \left\lfloor \frac{x}{t} \right\rfloor - \left\lfloor \frac{x}{t+1} \right\rfloor > D,$$

then it follows that

$$\frac{x}{t^2} > \left\{ \frac{x}{t} \right\} - \left\{ \frac{x}{t+1} \right\} > D-1.$$

This implies the inequality

$$t < \sqrt{x/(D-1)}.$$

As a result, the corresponding sum satisfies the bound

$$T_2(x) = \sum_{d>D} T_d(x) \leq \int_0^{\sqrt{x/(D-1)}} dt = \frac{\sqrt{x}}{D-1}.$$

□

2.4. Completion of the proof of Theorem 1.1

For $x > 1$ and $D \geq 2$, we have

$$T(x) = \sum_{i=0}^2 T_i(x) = T_0(x) + \sum_{d=1}^D T_d(x) + \sum_{d>D} T_d(x).$$

Building on the results established in Lemmas 2.1, 2.4, and 2.5, it follows that

$$T(x) = \frac{2}{3}\sqrt{x} + O(1) + \sum_{d=1}^D (f(d)\sqrt{x} + O(d^2)) + O\left(\frac{\sqrt{x}}{D}\right).$$

We observe that

$$\sum_{d=1}^D f(d) = \sum_{d=1}^{\infty} f(d) - \sum_{d>D} f(d),$$

Furthermore, since $f(d) = O(d^{-3/2})$, it follows from [2, p. 16] that the series $\sum_{d=1}^{\infty} f(d)$ is convergent and satisfies

$$\sum_{d=1}^{\infty} f(d) = -\frac{2}{3} + \frac{2}{\pi}\zeta(3/2).$$

Thus, we obtain

$$T(x) = \frac{2}{\pi}\zeta(3/2)\sqrt{x} + O(D^3) + O\left(\frac{\sqrt{x}}{D}\right).$$

Finally, the result is obtained by setting $D = x^{1/7}$.

Acknowledgement. The first author would like to thank to Professor Michel Balazard for his valuable suggestions concerning the integral considered in this paper.

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Received February 23, 2026

Revised May 15, 2026

Accepted June 25, 2026

Publ. online July 20, 2026

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