

SMALL PERTURBATIONS ON MEANS AND LATERAL MEANS

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ABSTRACT. We begin by investigating how small changes in the definition of ordinary means affect their properties, particularly the property of being a mean. In this direction, we look for weakenings of the basic defining property of means. Consequently, we introduce weaker notions in two directions. We study these functions and provide many examples.

1. Introduction

In this paper, we extend the notion of a mean in several directions.

The original motivations typically arise from statistics, physics and numerical analysis. Another source of such generalization comes from slight modification of the defining formula of an ordinary mean. That is, we slightly modify the function so that it is no longer a mean; however, in some sense, it remains close to one.

In some cases, if we restrict the domain of the modified function, we may obtain a new mean. However, on the entire domain of the original function, it is not a mean but rather a weaker type of mean, which we call a lateral mean, satisfying only one of the defining inequalities of a mean.

A different type of generalization arises when none of the defining inequalities holds, yet the function value remains close to the boundary values in an appropriate sense.

As an application, it is worth mentioning that in certain cases lateral means can be used to approximate ordinary mean.

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In the first part of the paper, we define the notion of a lateral mean and provide properties that may be of independent interest.

We then present several examples highlighting the underlying motivation and investigate various properties of these lateral means.

In the next section, we study an operation called duality, which may be of interest in its own right. Here, however, we focus on how this method can be used to construct lateral means.

We subsequently examine how certain operations behave with respect to lateral means and how classical theorems on means can be generalized to this setting. For example, we analyze compounding and show how a two-variable lateral mean can be extended to three-variables.

Next, we address a generalization of monotonicity for lateral means.

In the final part of the paper, we propose a further possible generalization of means.

1.1. Basic notions and notations

An **n -variable mean** on $H \subset \mathbb{R}$ is a function $\mathcal{K} : H^n \rightarrow \mathbb{R}$ such that, for all

$$(a_1, \dots, a_n) \in H^n, \min\{a_1, \dots, a_n\} \leq \mathcal{K}(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\},$$

where $(n \in \mathbb{N} - \{1\})$. We sometimes use the adjective “ordinary” to distinguish such means from lateral means. In most cases, we assume that the range of $\mathcal{K}$ is contained in H ; however, this is not part of the definition.

- An n -variable function $\mathcal{K}$ is called **reflexive** if

$$\mathcal{K}(a, \dots, a) = a.$$

A mean $\mathcal{K}$ is called **strict** if

$$\min\{a_1, \dots, a_n\} < \mathcal{K}(a_1, \dots, a_n) < \max\{a_1, \dots, a_n\}$$

holds whenever $a_1, \dots, a_n$ are not all equal.

- It is called **monotone** if $a_1 \leq b_1, \dots, a_n \leq b_n$ implies that

$$\mathcal{K}(a_1, \dots, a_n) \leq \mathcal{K}(b_1, \dots, b_n).$$

- It is called **symmetric** if for every bijection $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ we have

$$\mathcal{K}(a_1, \dots, a_n) = \mathcal{K}(a_{\sigma(1)}, \dots, a_{\sigma(n)}).$$

- An n -variable symmetric mean is called **m -associative** ($1 \leq m \leq n$) if

$$\mathcal{K}(a_1, \dots, a_n) = \mathcal{K}(\mathcal{K}(a_1, \dots, a_m), \dots, \mathcal{K}(a_1, \dots, a_m), a_{m+1}, \dots, a_n),$$

where the first m terms are all equal to $\mathcal{K}(a_1, \dots, a_m)$.

- An algebraic expression $\mathcal{K}(a_1, \dots, a_n)$ of fixed parameters $a_1, \dots, a_n \in \mathbb{R}$ is **mean-like** if

$$\min\{a_1, \dots, a_n\} \leq \mathcal{K}(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\}.$$

Here, we do not consider $a_1, \dots, a_n$ as variables that can change, instead, we consider them as fixed values, and the expression is mean-like for these particular values (i.e., it need not be mean-like for other values). As an simple example, consider the expression $\frac{2+8}{3}$. This is an algebraic expression of 2 and 8, and it is mean-like for those parameters since $2 \leq \frac{2+8}{3} \leq 8$.

A **multiset** (or mset) is a pair $\langle H, m \rangle$, where H is a set and $m : H \rightarrow \mathbb{N}$. This generalizes the usual notion of a set by allowing multiple occurrences of its elements (see [4]). We say that $\langle K, l \rangle$ is a subset of $\langle H, m \rangle$ (denoted by $\langle K, l \rangle \subset \langle H, m \rangle$) if $K \subset H$ and $l(k) \leq m(k)$ for every $k \in K$.

If $\mathcal{K}$ is symmetric, then we may consider $\mathcal{K}$ as acting on finite multisets, since the order of the elements does not matter. That is, we write $\mathcal{K}(H)$ instead of $\mathcal{K}(a_1, \dots, a_n)$ whenever H is the multiset of $\{a_1, \dots, a_n\}$. We will apply this convention later.

DEFINITION 1.1. Let $H \subset \mathbb{R}$. If $\mathcal{K}$ is a function such that $\mathcal{K}$ is defined on H^n for all $n \in \mathbb{N}$, i.e.,

$$\text{Dom } \mathcal{K} = \bigcup_{k=1}^{\infty} H^k \quad \text{and} \quad \mathcal{K} : \bigcup_{k=1}^{\infty} H^k \rightarrow \mathbb{R},$$

then $\mathcal{K}$ is called a **multi-variable function**.

If $n \in \mathbb{N}$, then $\mathcal{K}|_n$ will denote $\mathcal{K}|_{H^n}$, i.e., it denotes its n -variable version.

DEFINITION 1.2. If $\mathcal{K}$ is a multivariable function such that $\forall n \in \mathbb{N} \mathcal{K}|_n$ is an n -variable mean, then $\mathcal{K}$ is called a **multivariable mean**.

Here, we also allow $\mathcal{K}|_1$ to be undefined.

Throughout this paper, the function $\mathcal{A}$ will denote the arithmetic mean of any (finite) number of variables.

Let $n \in \mathbb{N} - \{1\}$, $a_1, \dots, a_n \in \mathbb{R}$, and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and increasing function (i.e., a homeomorphism). The quasi-arithmetic mean by f is defined as

$$\mathcal{K}_f(a_1, \dots, a_n) = f^{-1} \left(\frac{f(a_1) + \dots + f(a_n)}{n} \right).$$

Finally, we introduce some usual notations:

- $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$,
- $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$.
- If $x \in \mathbb{R}$, then we use $|x|^+ = \max\{0, x\}$, $|x|^- = -\min\{0, x\}$.
- Set $S(x, \varepsilon) = \{y \in \mathbb{R}^n : d(x, y) < \varepsilon\}$ ($x \in \mathbb{R}^n, \varepsilon > 0$).
- If $x \in \mathbb{R}$, then $\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$, $\lceil x \rceil = \min\{z \in \mathbb{Z} : x \leq z\}$.

2. Definition and properties of lateral means

DEFINITION 2.1. Let $\mathcal{K} : H^n \rightarrow \mathbb{R}$ ($H \subset \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 2$). Then, $\mathcal{K}$ is called a **left-mean** if $\min\{a_1, \dots, a_n\} \leq \mathcal{K}(a_1, \dots, a_n)$ holds; it is called a **right-mean** if $\mathcal{K}(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\}$ holds ($a_1, \dots, a_n \in H$). $\mathcal{K}$ is called a **lateral mean** if it is either a left-mean or a right-mean.

PROPOSITION 2.2. $\mathcal{K}$ is a mean if and only if it is a left-mean and a right-mean.

REMARK 2.3. In some cases, we will assume that the range of $\mathcal{K}$ is a subset of H . Of course, in such cases, we will explicitly state this.

The properties of being reflexive, monotone, continuous, symmetric, and m -associative are defined in the same way as for ordinary means. The notion of multi-variable lateral mean is defined analogously.

DEFINITION 2.4. A right-mean (left-mean) is called **strict** if

$$\mathcal{K}(a_1, \dots, a_n) < \max\{a_1, \dots, a_n\} \quad (\min\{a_1, \dots, a_n\} < \mathcal{K}(a_1, \dots, a_n))$$

whenever $a_1, \dots, a_n$ are not all equal.

DEFINITION 2.5. An n -variable lateral mean $\mathcal{K}$ is called **semi-reflexive** if

$$b = \mathcal{K}(a, \dots, a) \quad \text{implies that} \quad \mathcal{K}(b, \dots, b) = b.$$

Clearly, every reflexive lateral mean is semi-reflexive.

PROPOSITION 2.6. If an n -variable symmetric mean is n -associative, then it is semi-reflexive.

DEFINITION 2.7. An n -variable lateral mean $\mathcal{K}$ is called **left-continuous** if

$$\lim_{\substack{x_1 \rightarrow a_1, x_1 \leq a_1 \\ \dots \\ x_n \rightarrow a_n, x_n \leq a_n}} \mathcal{K}(x_1, \dots, x_n) = \mathcal{K}(a_1, \dots, a_n).$$

Right continuity is defined analogously with $x_i \geq a_i$ ($1 \leq i \leq n$).

Clearly, a continuous lateral mean is both left- and right-continuous; however, the converse does not hold.

DEFINITION 2.8. A right-mean (left-mean) $\mathcal{K}$ is called **strong** if,

$$\forall (a_1, \dots, a_n) \in \text{Dom } \mathcal{K},$$

it satisfies:

$$\mathcal{K}(a_1, \dots, a_n) \leq \min\{a_1, \dots, a_n\} \quad (\max\{a_1, \dots, a_n\} \leq \mathcal{K}(a_1, \dots, a_n)).$$

DEFINITION 2.9. A lateral mean $\mathcal{K}$ is called **mean-continuous** if

$$\lim_{\substack{(x_1, \dots, x_n) \rightarrow (a, \dots, a) \\ (x_1, \dots, x_n) \neq (a, \dots, a)}} \mathcal{K}(x_1, \dots, x_n) = a.$$

Obviously, a mean is always mean-continuous; however, there exist lateral means that are not. Clearly, the limit is $\leq a$ for a right-mean and $\geq a$ for a left-mean.

PROPOSITION 2.10. *A left-continuous lateral mean is mean-continuous if and only if it is reflexive. (The same holds for right-continuity.)*

We can measure how far a lateral mean is from being a mean. Naturally, there may be many such measures; here, we present two types.

DEFINITION 2.11. Let $\mathcal{K}: H^n \rightarrow \mathbb{R}$ be a lateral mean ($H \subset \mathbb{R}, n \in \mathbb{N}, n \geq 2$). Set

$$\begin{aligned} \text{mdist}(\mathcal{K}) &= \sup \left\{ |\mathcal{K}(a_1, \dots, a_n) - \max\{a_1, \dots, a_n\}|^+ : a_1, \dots, a_n \in H \right\} \\ &\quad + \sup \left\{ |\min\{a_1, \dots, a_n\} - \mathcal{K}(a_1, \dots, a_n)|^+ : a_1, \dots, a_n \in H \right\}, \\ \text{mdistp}(\mathcal{K}) &= \sup \left\{ \frac{|\mathcal{K}(a_1, \dots, a_n) - \max\{a_1, \dots, a_n\}|^+}{\max\{a_1, \dots, a_n\} - \min\{a_1, \dots, a_n\}} : a_1, \dots, a_n \in H \text{ not all equal} \right\} \\ &\quad + \sup \left\{ \frac{|\min\{a_1, \dots, a_n\} - \mathcal{K}(a_1, \dots, a_n)|^+}{\max\{a_1, \dots, a_n\} - \min\{a_1, \dots, a_n\}} : a_1, \dots, a_n \in H \text{ not all equal} \right\}. \end{aligned}$$

Clearly, for the left-means, the second term is 0, whereas for the right-means, the first term is 0 for both $\text{mdist}(\mathcal{K})$ and $\text{mdistp}(\mathcal{K})$. Obviously, $\mathcal{K}$ is a mean if and only if $\text{mdist}(\mathcal{K}) = 0$.

DEFINITION 2.12. Let $\mathcal{K}: [a, b]^n \rightarrow \mathbb{R}$ be a lateral mean such that the function $\mathcal{K}$ is Lebesgue measurable ($n \in \mathbb{N}, n \geq 2$). Set

$$\begin{aligned} \text{mdista}(\mathcal{K}) &= \frac{\lambda\{(a_1, \dots, a_n) \in [a, b]^n : \mathcal{K}(a_1, \dots, a_n) > \max\{a_1, \dots, a_n\}\}}{(b-a)^n} \\ &\quad + \frac{\lambda\{(a_1, \dots, a_n) \in [a, b]^n : \mathcal{K}(a_1, \dots, a_n) < \min\{a_1, \dots, a_n\}\}}{(b-a)^n}, \end{aligned}$$

where λ is the Lebesgue measure.

Clearly, for the left-means, the second term is 0, whereas for the right-means, the first term is 0 for $\text{mdista}(\mathcal{K})$. If $\mathcal{K}$ is a mean, then $\text{mdista}(\mathcal{K}) = 0$. Note that the converse is not true: $\mathcal{K}$ can be a proper lateral mean with $\text{mdista}(\mathcal{K}) = 0$. However, with an obvious additional property, it holds.

PROPOSITION 2.13. *Let $\mathcal{K}: [a, b]^n \rightarrow \mathbb{R}$ be a continuous lateral mean ($n \in \mathbb{N}, n \geq 2$). Then, if $\text{mdista}(\mathcal{K}) = 0$, $\mathcal{K}$ is a mean.*

Proof. Suppose, indirectly, that $\mathcal{K}$ is not a mean. Then, there is $(a_1, \dots, a_n) \in [a, b]^n$ such that $\mathcal{K}(a_1, \dots, a_n) > \max\{a_1, \dots, a_n\}$ (the other case is similar). That is, $\mathcal{K}(a_1, \dots, a_n) - \max\{a_1, \dots, a_n\} > 0$. Obviously, $\mathcal{K} - \max$ is continuous as well; hence, there exists $\delta > 0$ such that $(b_1, \dots, b_n) \in S((a_1, \dots, a_n), \delta)$ implies $\mathcal{K}(b_1, \dots, b_n) - \max\{b_1, \dots, b_n\} > 0$. $\square$

REMARK 2.14. If a lateral mean is given, then we can easily construct an ordinary mean from it by simply taking

$$\tilde{\mathcal{K}}(a_1, \dots, a_n) = \min\left\{\max\{\min\{a_1, \dots, a_n\}, \mathcal{K}(a_1, \dots, a_n)\}, \max\{a_1, \dots, a_n\}\right\}.$$

This truncates the lateral mean on one side while preserving its behaviour on the other.

3. Examples

3.1. Lateral means from statistics

Bessel's correction in statistics provides an example for a lateral mean. Suppose, we have a statistical sample of n elements, $x_1, \dots, x_n \in \mathbb{R}$. Then, the sample mean is

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}.$$

The sample standard variance is

$$\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n},$$

while the unbiased sample standard variance is

$$\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}.$$

This suggests the following lateral means:

EXAMPLE 3.1.

$$\mathcal{B}^+(a_1, \dots, a_n) = \frac{\sum_{i=1}^n a_i}{n-1} \quad (a_i \in \mathbb{R}^+); \quad \mathcal{B}^-(a_1, \dots, a_n) = \frac{\sum_{i=1}^n a_i}{n-1} \quad (a_i \in \mathbb{R}^-).$$

Clearly,

$\min\{a_1, \dots, a_n\} \leq \mathcal{B}^+(a_1, \dots, a_n)$ and $\mathcal{B}^-(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\}$,
so they are lateral means. However,

$$\mathcal{B}^+(1, 2) = 3 \quad \text{and} \quad \mathcal{B}^-(-1, -2) = -3,$$

which shows that they are not means.

PROPOSITION 3.2. $\mathcal{B}^+$ and $\mathcal{B}^-$ are strict, monotone, symmetric, continuous; however, they are not reflexive.

EXAMPLE 3.3. There is a sequence (a_n) such that $\forall n \mathcal{B}^+(a_1, \dots, a_n) \not\leq \max\{a_1, \dots, a_n\}$. We present two such sequences.

1. Let $a_1 = 1$, $a_n = 2$ ($n > 1$). Then, $\mathcal{B}^+(a_1, \dots, a_n) = 2 + \frac{1}{n-1}$.
2. Let $a_1 = 1$, $a_n = 2 - \frac{1}{2^{n-1}}$ ($n > 1$). Then, $\mathcal{B}^+(a_1, \dots, a_n) = 2 + \frac{1}{(n-1)2^n}$.

Similar examples can be constructed for $\mathcal{B}^-$.

LEMMA 3.4. *Let $a_i \in \mathbb{R}^+$, $a_1 \leq \dots \leq a_n$. Then, $\mathcal{B}^+(a_1, \dots, a_n)$ is mean-like if and only if*

$$\mathcal{B}^+(a_1, \dots, a_{n-1}) \leq a_n.$$

Proof.

$$\mathcal{B}^+(a_1, \dots, a_n) \leq a_n$$

if and only if

$$\sum_{i=1}^n a_i \leq (n-1)a_n$$

if and only if

$$\sum_{i=1}^{n-1} a_i \leq (n-2)a_n$$

if and only if

$$\mathcal{B}^+(a_1, \dots, a_{n-1}) \leq a_n. \quad \square$$

PROPOSITION 3.5. *Let $a_i \in \mathbb{R}$, $a_1 \leq \dots \leq a_n$. If $a_1 \leq 0, a_n \geq 0$, then*

$$a_1 \leq \frac{\sum_{i=1}^n a_i}{n-1} \leq a_n.$$

Proof.

$$a_1 \leq \frac{\sum_{i=1}^{n-1} a_i}{n-1} \leq \frac{\sum_{i=1}^n a_i}{n-1} \leq \frac{\sum_{i=2}^n a_i}{n-1} \leq a_n. \quad \square$$

THEOREM 3.6. *If $a_n \in \mathbb{R}^+$, $a_n \rightarrow +\infty$, then $\exists n \in \mathbb{N}$ such that $\mathcal{B}^+(a_1, \dots, a_n)$ is mean-like.*

Proof. Suppose the contrary:

$$\forall n \in \mathbb{N} \max\{a_1, \dots, a_n\} < \frac{\sum_{i=1}^n a_i}{n-1}.$$

Take such n . Obviously,

$$a_n < \frac{\sum_{i=1}^n a_i}{n-1}$$

is equivalent to

$$a_n < \frac{\sum_{i=1}^{n-1} a_i}{n-2}.$$

Now, we get

$$\begin{aligned} (n-2)a_n &< \sum_{i=1}^{n-1} a_i, \\ (n-2) \sum_{i=1}^n a_i &= (n-2) \sum_{i=1}^{n-1} a_i + (n-2)a_n < (n-1) \sum_{i=1}^{n-1} a_i, \\ \frac{\sum_{i=1}^n a_i}{n-1} &< \frac{\sum_{i=1}^{n-1} a_i}{n-2}. \end{aligned}$$

However, by the indirect assumption,

$$a_{n+1} < \frac{\sum_{i=1}^{n-1} a_i}{n-2}, \quad \text{and thus,} \quad \forall m \geq n, \quad a_m < \frac{\sum_{i=1}^{n-1} a_i}{n-2},$$

which is a contradiction. $\square$

THEOREM 3.7. *If*

$$\mathcal{B}^+(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\},$$

then

$$\mathcal{B}^+(a_1, \dots, a_n, a_{n+1}) \leq \max\{a_1, \dots, a_n, a_{n+1}\} \quad (a_i \in \mathbb{R}^+).$$

Proof. Let $a_1 \leq \dots \leq a_n$. Then, $\frac{\sum_{i=1}^n a_i}{n-1} \leq a_n$ holds. There are two cases:

(1) If $a_n \leq a_{n+1}$, then $\frac{\sum_{i=1}^n a_i}{n-1} \leq a_n \leq a_{n+1}$ gives

$$\sum_{i=1}^n a_i \leq (n-1)a_{n+1} \quad \text{and} \quad \sum_{i=1}^{n+1} a_i \leq na_{n+1},$$

which yields

$$\frac{\sum_{i=1}^{n+1} a_i}{n} \leq a_{n+1}.$$

(2) If $a_{n+1} < a_n$, then $\max\{a_1, \dots, a_n, a_{n+1}\} = a_n$.

Now, summing

$$\sum_{i=1}^n a_i \leq (n-1)a_n \quad \text{and} \quad a_{n+1} < a_n$$

gives

$$\sum_{i=1}^{n+1} a_i \leq na_n,$$

hence

$$\frac{\sum_{i=1}^{n+1} a_i}{n} \leq a_n.$$

$\square$

COROLLARY 3.8. *If $a_n \in \mathbb{R}^+$, $a_n \rightarrow +\infty$, then $\exists N \in \mathbb{N}$ such that $n < N$ implies that $\mathcal{B}^+(a_1, \dots, a_n)$ is not mean-like, whereas $n \geq N$ implies that $\mathcal{B}^+(a_1, \dots, a_n)$ is mean-like.*

Proof. 3.6 and 3.7. □

When we consider the unbiased sample standard deviation

$$\sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}},$$

we can also define the following lateral mean:

EXAMPLE 3.9.

$$\mathcal{K}(a_1, \dots, a_n) = \sqrt{\frac{\sum_{i=1}^n a_i^2}{n-1}} \quad (a_i \in \mathbb{R}^+).$$

We will investigate this type of lateral means in Section 3.4 from a more general perspective.

In statistics, it is sometimes useful to discard the smallest and/or the largest elements of a sample in order to obtain a more usable sample. This motivates the definition of the following lateral means. We restrict our attention to the positive cases only.

EXAMPLE 3.10. Let $a_1, \dots, a_n \in \mathbb{R}^+$, $a_1 \leq \dots \leq a_n$, $n \geq 3$. Then, set

$$\begin{aligned} \mathcal{K}_1(a_1, \dots, a_n) &= \frac{\sum_{i=2}^n a_i}{n}, \\ \mathcal{K}_2(a_1, \dots, a_n) &= \frac{\sum_{i=1}^{n-1} a_i}{n}, \\ \mathcal{K}_3(a_1, \dots, a_n) &= \frac{\sum_{i=2}^{n-1} a_i}{n}. \end{aligned}$$

Obviously,

$$\mathcal{K}_3(a_1, \dots, a_n) \leq \mathcal{K}_2(a_1, \dots, a_n) \leq \mathcal{K}_1(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\},$$

hence all of them are lateral means. The example $\mathcal{K}_1(10, 11, 12) < 10$ demonstrates that they are not means.

3.2. Lateral means from numerical analysis

The following lateral mean is inspired by the approximation of ordinary means, i.e., the motivation comes from numerical analysis.

Suppose, we have n real numbers, $a_1, \dots, a_n \in \mathbb{R}$, and we want to calculate their arithmetic mean. However, among of the numbers, there may be some irrational numbers. So, the calculation is possible only approximately. To do this, we take approximations of all numbers, for example, we take the first three digits of each number and use them in the calculation. The more digits we include, the better the approximation we get.

EXAMPLE 3.11. Let

$$a_i \in \mathbb{R} \quad (1 \leq i \leq n) \quad \text{and} \quad a_i = x_i.x_{i,1}x_{i,2}x_{i,3}\dots,$$

where

$$x_i \in \mathbb{Z} \quad \text{and} \quad x_{i,k} \in \{0, \dots, 9\} \quad \text{for} \quad (k \in \mathbb{N}),$$

i.e., we take the decimal representation of each number. We also assume that $\forall i \forall N \exists k > N$ such that $x_{i,k} \neq 9$, i.e., the representation has no trailing 9's. Then, we define

$$\mathcal{A}_m(a_1, \dots, a_n) = \mathcal{A}(x_1.x_{1,1}\dots x_{1,m}; \dots; x_n.x_{n,1}\dots x_{n,m}) \quad (m \in \mathbb{N} \cup \{0\}),$$

that is, we take the first m digits of each number and calculate the arithmetic mean of these numbers. In this way, we get an approximation of $\mathcal{A}(a_1, \dots, a_n)$.

Let us emphasize

$$\mathcal{A}_0, \quad \text{that is,} \quad \mathcal{A}_0(a_1, \dots, a_n) = \mathcal{A}(\lfloor a_1 \rfloor, \dots, \lfloor a_n \rfloor),$$

i.e., we take the arithmetic mean of the floors of the parameter values.

Here, we provide a more precise, equivalent definition, which also allows some generalization.

DEFINITION 3.12. Let $m \in \mathbb{Z}$. Let

$$\text{Dom}\mathcal{A}_m = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid (n \in \mathbb{N}): \exists i \in \{1, \dots, n\} \text{ such that } \lfloor 10^m a_i \rfloor \neq 0\}.$$

If $(a_1, \dots, a_n) \in \text{Dom}\mathcal{A}_m$, then set

$$\mathcal{A}_m(a_1, \dots, a_n) = \mathcal{A}\left(\frac{\lfloor 10^m a_1 \rfloor}{10^m}, \dots, \frac{\lfloor 10^m a_n \rfloor}{10^m}\right).$$

We will also use the notation $\mathcal{A}_m^-$ for $\mathcal{A}_m$.

REMARK 3.13.

- 1) Notice that we excluded the tuples $(a_1, \dots, a_n)$, where $\forall i \lfloor 10^m a_i \rfloor = 0$, since for such tuples, $\mathcal{A}_m(a_1, \dots, a_n)$ would equal 0, which could cause difficulties and undesirable behaviour.
- 2) In particular, $\mathcal{A}_m$ is not defined for $(0, \dots, 0)$.
- 3) In the definition of $\mathcal{A}_m$ inside $\mathcal{A}$, we do not ignore zeros, i.e., we do not drop zeros from the tuple inside $\mathcal{A}$.
- 4) $\mathcal{A}_m$ may be defined for a single number; however, its result can be different, e.g., $\mathcal{A}_0(1.1) = 1$.

REMARK 3.14. Now, we have $\mathcal{A}_m$ defined for negative m as well, which can still have practical value. When calculating the arithmetic mean of large numbers, sometimes we just care about the first 3–4 digits and not the remaining ones. Moreover, if there are small values among the numbers, we sometimes may ignore those too, better to say, we may replace them with 0. These two operations together behave like $\mathcal{A}_m$ for some negative m .

PROPOSITION 3.15. *For $\text{Dom}\mathcal{A}_m$, the following statements hold:*

1. $\text{Dom}\mathcal{A}_m = \bigcup_{n=1}^{\infty} \mathbb{R}^n - [0, 10^{-m})^n$.
2. If $k < m$, then $\text{Dom}\mathcal{A}_k \subsetneq \text{Dom}\mathcal{A}_m$.
3. $\bigcup_{m=-\infty}^{+\infty} \text{Dom}\mathcal{A}_m = \bigcup_{n=1}^{\infty} \mathbb{R}^n - \{0\}^n$.
4. If $n \in \mathbb{N}$, then $\bigcup_{m=1}^{-\infty} \mathbb{R}^n - \text{Dom}\mathcal{A}_m = (\mathbb{R}^+ \cup \{0\})^n$.

Proof.

1. $(a_1, \dots, a_n) \notin \text{Dom}\mathcal{A}_m \iff \forall i \in \{1, \dots, n\} a_i \in [0, 10^{-m})$.
2. By (1), $\text{Dom}\mathcal{A}_k \cap \mathbb{R}^n \subsetneq \text{Dom}\mathcal{A}_m \cap \mathbb{R}^n$ holds for all $n \in \mathbb{R}$.
3. Consequence of (1).
4. Consequence of (1). □

PROPOSITION 3.16. *If $m \in \mathbb{Z}, n \in \mathbb{N}, (a_1, \dots, a_n) \in \text{Dom}\mathcal{A}_0$, then*

$$\mathcal{A}_m(a_1, \dots, a_n) = \frac{1}{10^m} \mathcal{A}_0(10^m a_1, \dots, 10^m a_n).$$

PROPOSITION 3.17. *Let $\mathcal{A}_m$ be the function defined in 3.12 ($m \in \mathbb{Z}$). Then, the following statements hold:*

1. $\mathcal{A}_m(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\}$, i.e., $\mathcal{A}_m$ is a right-mean ($(a_1, \dots, a_n) \in \text{Dom}\mathcal{A}_m$),
2. $\mathcal{A}_m(19 \cdot 10^{-m-1}, 21 \cdot 10^{-m-1}) = 15 \cdot 10^{-m-1}$ demonstrates that $\mathcal{A}_m$ is not a mean,
3. $\mathcal{A}_m \leq \mathcal{A}_{m+1}$,
4. $\mathcal{A}_m \leq \mathcal{A}$,
5. $\mathcal{A} - \mathcal{A}_m \leq \frac{1}{10^m}$,
6. $\lim_{m \rightarrow \infty} \mathcal{A}_m(a_1, \dots, a_n) = \mathcal{A}(a_1, \dots, a_n)$; moreover, the convergence is uniform ($(a_1, \dots, a_n) \in \text{Dom}\mathcal{A}_0$),
7. a) $\text{mdist}(\mathcal{A}_m) = \frac{1}{10^m}$,
 b) $\text{mdistp}(\mathcal{A}_m) = \infty$,
 c) for $\mathcal{A}_m|_2$ $\text{mdista}(\mathcal{A}_m|_{[1,2]^2}) = \frac{4}{10^m} (1 - \frac{1}{10^m})$.

Proof.

1., 3. and 4. comes from the monotonicity of $\mathcal{A}$.

5. follows from the shift invariance of $\mathcal{A}$.

6. is a consequence of 5.

To see

(7) a observe that

$$\min\{a_1, \dots, a_n\} - \mathcal{A}_m(a_1, \dots, a_n) \leq \mathcal{A}(a_1, \dots, a_n) - \mathcal{A}_m(a_1, \dots, a_n) \leq \frac{1}{10^m},$$

and for $a = \frac{1}{10^m} - \frac{1}{10^k}$ ($k > m$), we get

$$\min\{a, \dots, a\} - \mathcal{A}_m(a, \dots, a) = \frac{1}{10^m} - \frac{1}{10^k} \rightarrow \frac{1}{10^m} \quad \text{if } k \rightarrow \infty.$$

(7) b Let $a_1 = \dots = a_{n-1} = 1 - \frac{1}{10^{m+1}}$, and let $a_{n_k} \rightarrow 1 - \frac{1}{10^{m+1}}$, $a_{n_k} > 1 - \frac{1}{10^{m+1}}$.

Then,

$$\lim_{k \rightarrow \infty} |\min\{a_1, \dots, a_{n-1}, a_{n_k}\} - \mathcal{A}_m(a_1, \dots, a_{n-1}, a_{n_k})|^+ = \frac{1}{10^m} - \frac{1}{10^{m+1}},$$

whereas

$$\lim_{k \rightarrow \infty} \max\{a_1, \dots, a_{n-1}, a_{n_k}\} - \min\{a_1, \dots, a_{n-1}, a_{n_k}\} = 0,$$

which gives

$$\text{mdistp}(\mathcal{A}_m) = \infty.$$

(7) c Clearly, $\mathcal{A}_m(x, y) \geq \min\{x, y\}$ holds if and only if $|x - y| \geq \frac{2}{10^m}$. Then,

$$\lambda\{(x, y) \in [1, 2]^2 : \mathcal{A}_m(x, y) \geq \min\{x, y\}\} = \left(1 - \frac{2}{10^m}\right)^2,$$

which gives

$$\lambda\{(x, y) \in [1, 2]^2 : \mathcal{A}_m(x, y) < \min\{x, y\}\} = \frac{4}{10^m} \left(1 - \frac{1}{10^m}\right). \quad \square$$

PROPOSITION 3.18. *Let $a_n \rightarrow \alpha \in \overline{\mathbb{R}}$. Then,*

- (1) *If $m \in \mathbb{Z}$, $\alpha \in \mathbb{R}$, $10^m \alpha \notin \mathbb{Z}$, then $\lim_{n \rightarrow \infty} \mathcal{A}_m(a_1, \dots, a_n) = \mathcal{A}_m(\alpha)$.*
- (2) *If $m \in \mathbb{Z}$, $\alpha \in \mathbb{R}$, $\alpha \neq 10^{-m}$, $\forall n \alpha \leq a_n$, then $\lim_{n \rightarrow \infty} \mathcal{A}_m(a_1, \dots, a_n) = \mathcal{A}_m(\alpha)$.*
- (3) *If $\alpha = \pm\infty$, then $\lim_{n \rightarrow \infty} \mathcal{A}_m(a_1, \dots, a_n) = \alpha$ ($m \in \mathbb{Z}$).*
- (4) $\lim_{n \rightarrow \infty}^{m \rightarrow \infty} \mathcal{A}_m(a_1, \dots, a_n) = \alpha$.

Proof. We will refer to the well-known result that if $a_n \rightarrow \alpha \in \overline{\mathbb{R}}$, then $\lim_{m \rightarrow \infty} \mathcal{A}(a_1, \dots, a_m) = \alpha$.

(1) By definition,

$$\mathcal{A}_m(a_1, \dots, a_n) = \mathcal{A}\left(\frac{\lfloor 10^m a_1 \rfloor}{10^m}, \dots, \frac{\lfloor 10^m a_n \rfloor}{10^m}\right).$$

Clearly,

$$10^m a_n \rightarrow 10^m \alpha, \quad \text{and then,} \quad \lfloor 10^m a_n \rfloor \rightarrow \lfloor 10^m \alpha \rfloor,$$

which implies

$$t \frac{\lfloor 10^m a_n \rfloor}{10^m} \rightarrow \frac{\lfloor 10^m \alpha \rfloor}{10^m},$$

which proves the claim.

(2) $\frac{\lfloor 10^m a_n \rfloor}{10^m} \rightarrow \frac{\lfloor 10^m \alpha \rfloor}{10^m}$ holds in this case as well.

(3) $\frac{\lfloor 10^m a_n \rfloor}{10^m} \rightarrow \pm\infty$.

(4) By 3.17(4) and (5), $\mathcal{A} - \frac{1}{10^m} \leq \mathcal{A}_m \leq \mathcal{A}$. □

EXAMPLE 3.19. If $10^m \alpha \in \mathbb{Z}$ and if we omit the condition $\alpha \leq a_n$, then 3.18 (2) does not hold, e.g., let $a_n = 2 - \frac{1}{n}$. Then, $\mathcal{A}_0(a_1, \dots, a_n) = 1 \rightarrow 1 \neq \mathcal{A}_0(2) = 2$.

PROPOSITION 3.20. $\mathcal{A}_m$ is strict, monotone, symmetric, right-continuous, and semi-reflexive. It is not reflexive, not mean-continuous, and not left-continuous, hence not continuous either. The continuity points equals to the left-continuity points, which are $\{(a_1, \dots, a_n) : \forall i \ 10^{-m} a_i \notin \mathbb{Z}\}$.

Proof.

- $\mathcal{A}_m$ is strict because $\mathcal{A}_m \leq \mathcal{A} < \max$.
- $\mathcal{A}_m$ is monotone because $f(x) = \lfloor x \rfloor$ is a monotone function and $\mathcal{A}$ is a monotone mean.
- $\mathcal{A}_m$ is symmetric because so is $\mathcal{A}$.
- $\mathcal{A}_m$ is right-continuous because $f(x) = \lfloor x \rfloor$ is a right-continuous function and $\mathcal{A}$ is a continuous mean.
- $\mathcal{A}_m$ is semi-reflexive because if $b = \mathcal{A}_m(a, \dots, a) = \frac{\lfloor 10^m a \rfloor}{10^m}$, then

$$\mathcal{A}_m(b, \dots, b) = \frac{1}{10^m} \mathcal{A}(\lfloor 10^m a \rfloor, \dots, \lfloor 10^m a \rfloor) = b.$$

- $\mathcal{A}_m$ is not reflexive, as, e.g., $\mathcal{A}_m(2.1 \cdot 10^{-m}, \dots, 2.1 \cdot 10^{-m}) = 2 \cdot 10^{-m}$ shows.
- $\mathcal{A}_m$ is not mean-continuous as shown in 2.10, and fails to be reflexive.
- $\mathcal{A}_m$ is not left-continuous, as, e.g., take the sequence

$$a_k = 2 \cdot 10^{-m} - \frac{1}{k} \quad \text{and} \quad \lim_{k \rightarrow \infty} \mathcal{A}_m|_n(a_k, \dots, a_k) = 10^{-m}$$

while

$$\mathcal{A}_m|_n(2 \cdot 10^{-m}, \dots, 2 \cdot 10^{-m}) = 2 \cdot 10^{-m}.$$

- Regarding (left-)continuity points, if for any i $10^{-m}a_i \in \mathbb{Z}$, then $\mathcal{A}_m|_n$ is not left-continuous at $(a_1, \dots, a_n)$, since

$$\lim_{k \rightarrow \infty} \mathcal{A}_m|_n(a_1, \dots, a_{i-1}, a_i - \frac{1}{k}, a_{i+1}, \dots, a_n) \neq \mathcal{A}_m|_n(a_1, \dots, a_n).$$

- If $\forall i$ $10^{-m}a_i \notin \mathbb{Z}$ and $\forall i$ $\lim_{k \rightarrow \infty} a_k^{(i)} = a_i$, then $\lim_{k \rightarrow \infty} \frac{\lfloor 10^m a_k^{(i)} \rfloor}{10^m} = \frac{\lfloor 10^m a_i \rfloor}{10^m}$, which, together with the fact that $\mathcal{A}$ is continuous, shows that $\mathcal{A}_m|_n$ is continuous at such points. $\square$

We now examine k -associativity.

PROPOSITION 3.21. *Let $1 \leq k \leq n$. Then,*

$$\mathcal{A}_m|_n(\mathcal{A}_m|_k(a_1, \dots, a_k), \dots, \mathcal{A}_m|_k(a_1, \dots, a_k), a_{k+1}, \dots, a_n) \leq \mathcal{A}_m|_n(a_1, \dots, a_n).$$

Proof. Clearly,

$$\begin{aligned} & \mathcal{A}_m(\mathcal{A}_m(a_1, \dots, a_k), \dots, \mathcal{A}_m(a_1, \dots, a_k), a_{k+1}, \dots, a_n) \\ &= \mathcal{A}\left(\frac{\lfloor \frac{\lfloor 10^m a_1 \rfloor, \dots, \lfloor 10^m a_k \rfloor}{k} \rfloor, \dots, \lfloor \frac{\lfloor 10^m a_1 \rfloor, \dots, \lfloor 10^m a_k \rfloor}{k} \rfloor, \lfloor 10^m a_{k+1} \rfloor, \dots, \lfloor 10^m a_n \rfloor\right) \\ &\leq \mathcal{A}\left(\frac{\lfloor \frac{\lfloor 10^m a_1 \rfloor, \dots, \lfloor 10^m a_k \rfloor}{k} \rfloor, \dots, \lfloor \frac{\lfloor 10^m a_1 \rfloor, \dots, \lfloor 10^m a_k \rfloor}{k} \rfloor, \lfloor 10^m a_{k+1} \rfloor, \dots, \lfloor 10^m a_n \rfloor\right) \\ &= \mathcal{A}\left(\frac{\lfloor 10^m a_1 \rfloor, \dots, \lfloor 10^m a_k \rfloor, \lfloor 10^m a_{k+1} \rfloor, \dots, \lfloor 10^m a_n \rfloor}{10^m n}\right) = \mathcal{A}_m(a_1, \dots, a_n). \quad \square \end{aligned}$$

EXAMPLE 3.22. $\mathcal{A}_m|_n$ is not k -associative in general ($1 \leq k \leq n$). Set $v = 10^{-m}$. Let

$$a_i = \begin{cases} 2v & \text{if } i = k, \\ nv & \text{if } i = n, \\ v, & \text{otherwise.} \end{cases}$$

Now,

$$\mathcal{A}_m|_n(a_1, \dots, a_n) = \mathcal{A}_m(v, \dots, v, 2v, v, \dots, v, nv) = \frac{n-2+2+n}{n}v = 2v.$$

While

$$\mathcal{A}_m|_k(a_1, \dots, a_k) = \mathcal{A}_m|_k(v, \dots, v, 2v) < 2v \quad \text{and} \quad \lfloor \mathcal{A}_m|_k(v, \dots, v, 2v) \rfloor = v,$$

therefore

$$\begin{aligned} \mathcal{A}_m|_n(\mathcal{A}_m|_k(v, \dots, v, 2v), \dots, \mathcal{A}_m|_k(v, \dots, v, 2v), v, \dots, v, nv) = \\ \mathcal{A}(v, \dots, v, nv) < 2v. \end{aligned}$$

PROPOSITION 3.23. *Ran $\mathcal{A}_m|_n$ is a closed and discrete set, moreover,*

$$\text{Ran } \mathcal{A}_m|_n = \left\{ \frac{1}{10^m} \frac{k}{n} : k \in \mathbb{Z} \right\}.$$

Proof. First, note that $\text{Ran } \mathcal{A}_m|_n = \frac{1}{10^m} \text{Ran } \mathcal{A}_0|_n$, because

$$\begin{aligned} \mathcal{A}_m(a_1, \dots, a_n) &= \mathcal{A}\left(\frac{\lfloor 10^m a_1 \rfloor}{10^m}, \dots, \frac{\lfloor 10^m a_n \rfloor}{10^m}\right) \\ &= \frac{1}{10^m} \mathcal{A}(\lfloor 10^m a_1 \rfloor, \dots, \lfloor 10^m a_n \rfloor) \\ &= \frac{1}{10^m} \mathcal{A}_0(10^m a_1, \dots, 10^m a_n). \end{aligned}$$

Evidently,

$$\text{Ran } \mathcal{A}_0|_n \subset \left\{ \frac{k}{n} : k \in \mathbb{Z} \right\}, \text{ because } \mathcal{A}_0(a_1, \dots, a_n) = \mathcal{A}(\lfloor a_1 \rfloor, \dots, \lfloor a_n \rfloor).$$

If $k \in \mathbb{Z}$, then $\mathcal{A}_0|_n(1, \dots, 1, k - n + 1) = \frac{k}{n}$, which gives the other inclusion. $\square$

Now, we provide a sufficient bound for $\mathcal{A}_m(a_1, \dots, a_n)$ to be mean-like.

PROPOSITION 3.24. *Let $a_1, \dots, a_n \in \mathbb{R}$ be fixed, $a_1 \leq \dots \leq a_n$, $a_1 < a_n$. Then, $m > \lg\left(\frac{n}{a_n - a_1}\right) + 1$ implies that $\mathcal{A}_m(a_1, \dots, a_n)$ is mean-like.*

Proof. First, note that

$$\frac{a_n - a_1}{n} = \frac{(n-1)a_1 + a_n}{n} - a_1 = \mathcal{A}(a_1, \dots, a_1, a_n) - a_1,$$

where there are $n-1$ pieces of a_1 inside $\mathcal{A}$.

Let $m > \lg\left(\frac{a_n - a_1}{n}\right) + 1$. Then, it is clear that

$$\mathcal{A}(a_1, \dots, a_{n-1}, a_n) - \frac{1}{10^m} \leq \mathcal{A}_m(a_1, \dots, a_{n-1}, a_n) \leq \mathcal{A}(a_1, \dots, a_{n-1}, a_n),$$

but

$$\mathcal{A}(a_1, \dots, a_1, a_n) - \frac{1}{10^m} \leq \mathcal{A}(a_1, \dots, a_{n-1}, a_n) - \frac{1}{10^m}.$$

Now,

$$\frac{1}{10^m} < \mathcal{A}(a_1, \dots, a_1, a_n) - a_1,$$

equivalent with $a_1 < \mathcal{A}(a_1, \dots, a_1, a_n) - \frac{1}{10^m}$, which completes the proof. $\square$

COROLLARY 3.25. *Let $n \in \mathbb{N} - \{1\}$ be fixed, $h > 0$. Let*

$$L = \{(a_1, \dots, a_n) \in \mathbb{R}^n : \max\{a_1, \dots, a_n\} - \min\{a_1, \dots, a_n\} \geq h\}.$$

Set $\hat{\mathcal{A}}_m = \mathcal{A}_m|_L$ ($m \in \mathbb{N} \cup \{0\}$). Then, there is $M \in \mathbb{N}$ such that $m > M$ implies that $\hat{\mathcal{A}}_m$ is a mean.

EXAMPLE 3.26. In Definition 3.12, we could replace “arithmetic mean” with any continuous (monotone) ordinary mean. For example, for the geometric mean $\mathcal{G}$, we can have the following. Let $m \in \mathbb{Z}$. Let

$$\text{Dom } \mathcal{G}_m = \{(a_1, \dots, a_n) \in (\mathbb{R}^+)^n (n \in \mathbb{N}) : \forall i \in \{1, \dots, n\}$$

such that

$$\lfloor 10^m a_i \rfloor \neq 0\}.$$

If $(a_1, \dots, a_n) \in \text{Dom } \mathcal{G}_m$, then the set

$$\mathcal{G}_m(a_1, \dots, a_n) = \mathcal{G}\left(\frac{\lfloor 10^m a_1 \rfloor}{10^m}, \dots, \frac{\lfloor 10^m a_n \rfloor}{10^m}\right).$$

Notice the difference in the domain: $\forall i$ instead of $\exists i$. The following example shows why that is needed.

Let us observe $(100, 1)$: $\mathcal{G}(100, 1) = 10$, and if we allowed $(100, 1)$ for $\mathcal{G}_{-1}$, we would obtain $\mathcal{G}_{-1}(100, 1) = 0$, which would lead to strange behaviour that we wish to avoid.

Now, we define some counterparts of $\mathcal{A}_m$.

DEFINITION 3.27. Let $m \in \mathbb{Z}$. Let

$$\text{Dom } \mathcal{A}_m^+ = \{(a_1, \dots, a_n) \in \mathbb{R}^n \ (n \in \mathbb{N}) : \exists i \in \{1, \dots, n\} \text{ such that } \lceil 10^m a_i \rceil \neq 0\}.$$

If $(a_1, \dots, a_n) \in \text{Dom } \mathcal{A}_m^+$, then set

$$\mathcal{A}_m^+(a_1, \dots, a_n) = \mathcal{A}\left(\frac{\lceil 10^m a_1 \rceil}{10^m}, \dots, \frac{\lceil 10^m a_n \rceil}{10^m}\right).$$

Similar statements for $\mathcal{A}_m^+$ can be proved as in 3.17. We omit the proof, as it is similar.

PROPOSITION 3.28. For $\mathcal{A}_m^+$, the following statements hold:

- (1) $\min \leq \mathcal{A}_m^+$, i.e., $\mathcal{A}_m^+$ is a left-mean,
- (2) $\mathcal{A}_m^+$ is not a mean,
- (3) $\mathcal{A}_{m+1}^+ \leq \mathcal{A}_m^+$,
- (4) $\mathcal{A} \leq \mathcal{A}_m^+$,
- (5) $\mathcal{A}_m^+ - \mathcal{A} \leq \frac{1}{10^m}$,
- (6) $\lim_{m \rightarrow \infty} \mathcal{A}_m^+(a_1, \dots, a_n) = \mathcal{A}(a_1, \dots, a_n)$, moreover, the convergence is uniform $((a_1, \dots, a_n) \in \text{Dom } \mathcal{A}_0^+)$,
- (7) a) $\text{mdist}(\mathcal{A}_m^+) = \frac{1}{10^m}$,
b) $\text{mdistp}(\mathcal{A}_m^+) = \infty$,
c) for 2-variable $\mathcal{A}_m^+$ $\text{mdista}(\mathcal{A}_m^+|_{[1,2]^2}) = \frac{4}{10^m} \left(1 - \frac{1}{10^m}\right)$.

DEFINITION 3.29. Set $\mathcal{A}_m^{-+} = \mathcal{A}_m^- + \frac{1}{10^m}$, $\mathcal{A}_m^{+-} = \mathcal{A}_m^+ - \frac{1}{10^m}$.

REMARK 3.30. Actually, $\mathcal{A}_m^{-+}$ corresponds to the algorithm where we truncate the numbers at the m^{th} digit, increase the m^{th} digit by 1, and calculate the arithmetic mean of those numbers. For $\mathcal{A}_m^{+-}$, similarly, we decrease the m^{th} digit by 1.

PROPOSITION 3.31.

$\mathcal{A}_m^-$ and $\mathcal{A}_m^{+-}$ are right-means. $\mathcal{A}_m^+$ and $\mathcal{A}_m^{-+}$ are left-means.

Proof. By 3.17(5), $\min \leq \mathcal{A} \leq \mathcal{A}_m^{-+}$. By 3.28(5), $\mathcal{A}_m^{+-} \leq \mathcal{A} \leq \max$. $\square$

PROPOSITION 3.32. $\mathcal{A}_m^{+-} = 10^{-m} \mathcal{A}_0^{+-}$, $\mathcal{A}_m^{-+} = 10^{-m} \mathcal{A}_0^{-+}$.

PROPOSITION 3.33. $\mathcal{A} - \frac{1}{10^m} \leq \mathcal{A}_m^{+-} \leq \mathcal{A}_m^- \leq \mathcal{A} \leq \mathcal{A}_m^+ \leq \mathcal{A}_m^{-+} \leq \mathcal{A} + \frac{1}{10^m}$.

Proof. Clearly, $\lceil x \rceil - 1 \leq \lfloor x \rfloor \leq \lceil x \rceil \leq \lfloor x \rfloor + 1$, hence

$$\begin{aligned} \mathcal{A} \left(\frac{10^m a_1}{10^m}, \dots, \frac{10^m a_n}{10^m} \right) - \frac{1}{10^m} &\leq \mathcal{A} \left(\frac{\lceil 10^m a_1 \rceil - 1}{10^m}, \dots, \frac{\lceil 10^m a_n \rceil - 1}{10^m} \right) \leq \\ &\mathcal{A} \left(\frac{\lfloor 10^m a_1 \rfloor}{10^m}, \dots, \frac{\lfloor 10^m a_n \rfloor}{10^m} \right) \leq \mathcal{A} \left(\frac{10^m a_1}{10^m}, \dots, \frac{10^m a_n}{10^m} \right) \leq \\ &\mathcal{A} \left(\frac{\lceil 10^m a_1 \rceil}{10^m}, \dots, \frac{\lceil 10^m a_n \rceil}{10^m} \right) \leq \mathcal{A} \left(\frac{\lfloor 10^m a_1 \rfloor + 1}{10^m}, \dots, \frac{\lfloor 10^m a_n \rfloor + 1}{10^m} \right). \quad \square \end{aligned}$$

EXAMPLE 3.34. We cannot generally expect equalities in 3.33. For example,

$$\mathcal{A}_0^{+-}(2.1, 3) < \mathcal{A}_0^-(2.1, 3) < \mathcal{A}(2.1, 3) < \mathcal{A}_0^+(2.1, 3) < \mathcal{A}_0^{-+}(2.1, 3)$$

since

$$\begin{aligned} \mathcal{A}_0^{+-}(2.1, 3) &= 2, & \mathcal{A}_0^-(2.1, 3) &= 2.5, & \mathcal{A}(2.1, 3) &= 2.55, \\ \mathcal{A}_0^+(2.1, 3) &= 3, & \mathcal{A}_0^{-+}(2.1, 3) &= 3.5. \end{aligned}$$

Multiplying the parameters by 10^{-m} provides an example for m instead of 0.

PROPOSITION 3.35. $\mathcal{A}_m^{-+}, \mathcal{A}_m^{+-}$ and $\mathcal{A}_m^+$ are strict, monotone, symmetric. They are not reflexive, not mean-continuous, and not continuous either.

$\mathcal{A}_m^{-+}$ is right-continuous, $\mathcal{A}_m^{+-}$ and $\mathcal{A}_m^+$ are left-continuous.

$\mathcal{A}_m^{-+}$ is not left-continuous, $\mathcal{A}_m^{+-}$ and $\mathcal{A}_m^+$ are not right-continuous.

$\mathcal{A}_m^+$ is semi-reflexive, while $\mathcal{A}_m^{-+}$ and $\mathcal{A}_m^{+-}$ are not.

Proof. Everything can be proved similarly as in 3.20. The only new result is non-semi-reflexivity:

$$\mathcal{A}_m^{-+}(2.1 \cdot 10^{-m}, \dots, 2.1 \cdot 10^{-m}) = 3 \cdot 10^{-m}$$

and

$$\mathcal{A}_m^{-+}(3 \cdot 10^{-m}, \dots, 3 \cdot 10^{-m}) = 4 \cdot 10^{-m}.$$

A similar example can be found for $\mathcal{A}_m^{+-}$. $\square$

Let us further generalize the method in Definition 3.12.

PROPOSITION 3.36. *Let $\mathcal{K}$ be a continuous mean of n variables. For each $x \in \mathbb{R}$, we fix a sequence (x_i) such that $x_i \leq x$ and $\lim_{i \rightarrow \infty} x_i = x$. Let*

$$\mathcal{K}_m(a_1, \dots, a_n) = \mathcal{K}(a_{1,m}, \dots, a_{n,m}) \quad (m \in \mathbb{N}, a_1, \dots, a_n \in \mathbb{R}),$$

where the sequence $(a_{i,m})$ belongs to a_i ($1 \leq i \leq n$). Then, $\mathcal{K}_m$ is a lateral mean.

Proof. Let $a_{j,m} = \max\{a_{1,m}, \dots, a_{n,m}\}$. Then, $a_{j,m} \leq a_j \leq \max\{a_1, \dots, a_n\}$, which gives that $\mathcal{K}_m(a_1, \dots, a_n) \leq a_{j,m} \leq \max\{a_1, \dots, a_n\}$. $\square$

PROPOSITION 3.37. *Let $a_1, \dots, a_n \in \mathbb{R}$ be fixed and $\mathcal{K}$ be a strict, continuous mean. For each $i \in \{1, \dots, n\}$, let $(a_{i,m})$ be a sequence such that*

$$\lim_{m \rightarrow \infty} a_{i,m} = a_i \quad (i \in \{1, \dots, n\}).$$

Let $\mathcal{K}_m(a_1, \dots, a_n) = \mathcal{K}(a_{1,m}, \dots, a_{n,m})$ ($m \in \mathbb{N}$). Then, there is $M \in \mathbb{N}$ such that $m > M$ implies that $\mathcal{K}_m(a_1, \dots, a_n)$ is mean-like.

Proof. Let $a_1 = \min\{a_1, \dots, a_n\}$, $a_n = \max\{a_1, \dots, a_n\}$, $k = \mathcal{K}(a_1, \dots, a_n)$. $\mathcal{K}$ is strict, hence $a_1 < k < a_n$. Choose $\epsilon > 0$ such that $a_1 < k - \epsilon < k + \epsilon < a_n$. Then, by the continuity of $\mathcal{K}$, find $M \in \mathbb{N}$ such that $m > M$ implies that $\mathcal{K}(a_{1,m}, \dots, a_{n,m}) \in (k - \epsilon, k + \epsilon)$. $\square$

REMARK 3.38. The two variable min shows that if we omit the condition of strictness, then the previous proposition fails to be true.

3.3. A lateral mean from physics

The motivation of the next example comes from physics. Suppose we have an electrical network with two resistors connected into a parallel circuit. To find the total resistance of the components, we use the following formula:

$$R_{\text{total}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}, \tag{1}$$

where R_1, R_2 are the resistance of the two components.

EXAMPLE 3.39. For $a, b \in \mathbb{R}^+$, set

$$\mathcal{K}(a, b) = \frac{1}{\frac{1}{a} + \frac{1}{b}}.$$

This is a type of perturbation of the harmonic mean for two variables.

PROPOSITION 3.40. *Let $\mathcal{K}$ be the mean defined in 3.39. Then, the following statements hold:*

1. $\mathcal{K}(a, b) \leq \min\{a, b\} \leq \max\{a, b\}$, hence it is a strong right-mean,
2. $\mathcal{K}$ is not reflexive ($\mathcal{K}(a, a) = \frac{a}{2}$), it is symmetric, continuous, monotone,
3. $\lim_{x \rightarrow +\infty} \mathcal{K}(a, x) = a$,
4. $\lim_{x \rightarrow 0+0} \mathcal{K}(a, x) = 0$.

Of course, we can generalize this lateral mean for several variables in two ways. First, according to this physics example, when applying the formula 1 for several transistors (the second generalization appears in 3.46):

$$\mathcal{K}(a_1, \dots, a_n) = \frac{1}{\frac{1}{a_1} + \dots + \frac{1}{a_n}} \quad (a_1, \dots, a_n \in \mathbb{R}^+).$$

Similarly to the two variable case, it can be readily seen that it is a strong right-mean. It is not reflexive, but it is symmetric, continuous and monotone.

Here, we mention one more property.

PROPOSITION 3.41. *Let a sequence (a_n) be given such that $\underline{\lim} a_n > 0$. Then,*

$$\lim_{n \rightarrow \infty} \mathcal{K}(a_1, \dots, a_n) = \lim_{n \rightarrow \infty} \frac{1}{\sum_{i=1}^n \frac{1}{a_i}} = 0.$$

3.4. Quasi-arithmetic lateral means

DEFINITION 3.42. Let $n \in \mathbb{N} - \{1\}$, $a_1, \dots, a_n \in \mathbb{R}$, $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and increasing. Let $\mathcal{L}$ be a right-mean (left-mean) defined on the subsets of $\mathbb{R}^n$ for each $n \in \mathbb{N} - \{1\}$. Set

$$\begin{aligned} \text{Dom } \mathcal{K}_{f, \mathcal{L}} &= \{(a_1, \dots, a_n) \in \mathbb{R}^n \ (n \in \mathbb{N}) : (f(a_1), \dots, f(a_n)) \in \text{Dom } \mathcal{L}\}. \\ \mathcal{K}_{f, \mathcal{L}}(a_1, \dots, a_n) &= f^{-1}\left(\mathcal{L}(f(a_1), \dots, f(a_n))\right) \quad (m \in \mathbb{Z}). \end{aligned}$$

PROPOSITION 3.43. *If $\mathcal{L}$ is a right-mean (left-mean), then so is $\mathcal{K}_{f, \mathcal{L}}$.*

Proof. We will prove the “right” part; the other is similar. Let

$$a_j = \max\{a_1, \dots, a_n\}.$$

Then,

$$f(a_j) = \max\{f(a_1), \dots, f(a_n)\} \quad \text{and} \quad \mathcal{L}(f(a_1), \dots, f(a_n)) \leq f(a_j),$$

which gives that

$$\mathcal{K}_{f, \mathcal{L}}(a_1, \dots, a_n) \leq a_j. \quad \square$$

PROPOSITION 3.44. *Let $\mathcal{L}$ be a lateral mean. If $\mathcal{L}$ is strict, monotone, symmetric and continuous, then so is $\mathcal{K}_{f, \mathcal{L}}$, respectively. Moreover, $\mathcal{L}$ is reflexive if and only if $\mathcal{K}_{f, \mathcal{L}}$ is reflexive.*

Proof. All properties can be readily checked. $\square$

We will mention two special cases:

- First, we can easily generalize the lateral means 3.1 and 3.39.

EXAMPLE 3.45. Let $n \in \mathbb{N} - \{1\}$, $a_1, \dots, a_n \in \mathbb{R}^+$, $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be continuous and increasing. Set

$$\begin{aligned}\mathcal{K}_{f, \mathcal{B}^+}(a_1, \dots, a_n) &= f^{-1}\left(\mathcal{B}^+(f(a_1), \dots, f(a_n))\right) \\ &= f^{-1}\left(\frac{f(a_1) + \dots + f(a_n)}{n-1}\right).\end{aligned}$$

- In particular, starting from power means, we get:

EXAMPLE 3.46. Let $n \in \mathbb{N} - \{1\}$, $a_1, \dots, a_n \in \mathbb{R}^+$, $x \in \mathbb{R}$. Set

$$\begin{aligned}\mathcal{K}_x(a_1, \dots, a_n) &= \sqrt[x]{\frac{a_1^x + \dots + a_n^x}{n-1}}, \\ \mathcal{K}_0(a_1, \dots, a_n) &= \sqrt[n-1]{a_1 \cdots a_n}.\end{aligned}$$

PROPOSITION 3.47. *The following statements hold:*

1. $\min\{a_1, \dots, a_n\} \leq \mathcal{K}_{f, \mathcal{B}^+}(a_1, \dots, a_n)$, i.e., $\mathcal{K}_{f, \mathcal{B}^+}$ is a lateral mean
2. $\mathcal{K}_{f, \mathcal{B}^+}$ is strict, monotone, symmetric, and continuous, but it is not reflexive.

PROOF. $\mathcal{B}^+$ is a left-mean, and we apply 3.43 and 3.44. $\square$

REMARK 3.48.

- a) $\lim_{x \rightarrow 0} \mathcal{K}_x = \mathcal{K}_0$ does not hold; moreover, $\lim_{x \rightarrow 0} \mathcal{K}_x(a_1, \dots, a_n)$ does not exist in general.
- b) $x < y \not\Rightarrow \mathcal{K}_x \leq \mathcal{K}_y$, e.g., $\sqrt{2 \cdot 3 \cdot 4} \not\leq \frac{2+3+4}{2} \not\leq \sqrt{\frac{2^2+3^2+4^2}{2}}$.

PROPOSITION 3.49. *If $n = 2$, $0 < v < u$, then $\mathcal{K}_u(a, b) < \mathcal{K}_v(a, b)$ ($a, b \in \mathbb{R}^+$).*

PROOF. Let $c = a^v$, $d = b^v$, $k = \frac{u}{v}$. Then, we need to show that $(c+d)^k > c^k + d^k$. Change c to x and consider it as a variable, while fixing d . Then, when $x = 0$, the equality holds. Derivating by x , we get $k(x+d)^{k-1} > kx^{k-1}$, which obviously holds. $\square$

For quasi-arithmetic means, there is another way how to define the associated lateral means in a manner similar to Definition of 3.12.

DEFINITION 3.50. Let $n \in \mathbb{N} - \{1\}$, $a_1, \dots, a_n \in \mathbb{R}$, $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous and increasing. Set

$\text{Dom } \mathcal{K}_{f, m} = \{(a_1, \dots, a_n) \in \mathbb{R}^n (n \in \mathbb{N}) : \exists i \in \{1, \dots, n\} \text{ such that } \lfloor 10^m f(a_i) \rfloor \neq 0\}$.

$$\mathcal{K}_{f, m}(a_1, \dots, a_n) = f^{-1}\left(\mathcal{A}_m(f(a_1), \dots, f(a_n))\right) \quad (m \in \mathbb{Z}).$$

PROPOSITION 3.51. *Let $n \in \mathbb{N} - \{1\}$, $a_1, \dots, a_n \in \mathbb{R}$, $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and increasing. Then, $\mathcal{K}_{f,m}$ is a right-mean that is strict, monotone, symmetric, and continuous, but not reflexive.*

Proof. $\mathcal{A}_m$ is a right-mean, and we apply 3.43 and 3.44. $\square$

3.5. Various other lateral means

EXAMPLE 3.52. Let an ordinary symmetric mean $\mathcal{M}$ be given, defined on arbitrarily many positive numbers, i.e., $\mathcal{M} : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined for all $n \in \mathbb{N}$. Or, we can rephrase this by saying that $\mathcal{M}$ is defined on all finite multi-subsets of $\mathbb{R}$. Set

$$\mathcal{M}'(\{a_i : 1 \leq i \leq n\}) = \mathcal{M}(\{a_i > 0 : 1 \leq i \leq n\}) \quad \text{and} \quad \mathcal{M}'(\emptyset) = 0,$$

i.e., we apply $\mathcal{M}$ to the positive elements only. For example, $\mathcal{M}'(\{-1, -2\}) = 0$. Obviously, $\min\{a_1, \dots, a_n\} \leq \mathcal{M}'(a_1, \dots, a_n)$. Hence, it is a lateral mean. It can be readily seen that if $\mathcal{M}$ is continuous, then $\mathcal{M}'$ is right-continuous. Left-continuity holds as long as 0 is not among the elements. If it is, then $\mathcal{M}'$ is not left-continuous.

Monotonicity does not hold either

$$\mathcal{A}'(\{-1, 2, 3\}) \not\leq \mathcal{A}'(\{1, 2, 3\}).$$

EXAMPLE 3.53. Let $\mathcal{K}(a, b) = \frac{\sqrt{a^2 + b^2}}{2}$ ($a, b \in \mathbb{R}^+$). Obviously, $\mathcal{K}(a, b) \leq \max\{a, b\}$, hence it is a lateral mean. It is strict, monotone, symmetric, and continuous, but it is not reflexive.

There is a smaller domain where $\mathcal{K}$ becomes a mean. One must restrict the domain to the values $\{(a, b) \in (\mathbb{R}^+)^2 : \sqrt{3}a \leq b \text{ or } \sqrt{3}b \leq a\}$.

EXAMPLE 3.54. Let $a_1, \dots, a_n \in \mathbb{R}^+ \cup \{0\}$. Then set

$$\mathcal{K}(a_1, \dots, a_n) = \frac{a_1 + a_1 a_2 + a_1 a_2 a_3 + \dots + a_1 \dots a_n}{n}.$$

If $1 \leq a_i$ ($1 \leq i \leq n$), then $\min\{a_1, \dots, a_n\} \leq \mathcal{K}(a_1, \dots, a_n)$. Actually, if $a_i = \min\{a_1, \dots, a_n\}$, then we get

$$a_i = \frac{na_i}{n} \leq \frac{a_i + a_i^2 + \dots + a_i^n}{n} \leq \mathcal{K}(a_1, \dots, a_n).$$

Similarly, we get that if $1 \geq a_i$ ($1 \leq i \leq n$), then $\mathcal{K}(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\}$. Therefore, $\mathcal{K}|_{[1, +\infty)}$ is a right-mean, while $\mathcal{K}|_{[0, 1]}$ is a left-mean. Clearly, they are not means, simply because they are not reflexive.

EXAMPLE 3.55. Let $a_1, \dots, a_n \in \mathbb{R}^+ \cup \{0\}$. Then set

$$\mathcal{K}(a_1, \dots, a_n) = \sqrt[n]{\frac{a_1 + a_1 a_2 + a_1 a_2 a_3 + \dots + a_1 \dots a_n}{n}}.$$

If $1 \geq a_i$ ($1 \leq i \leq n$), then $\min\{a_1, \dots, a_n\} \leq \mathcal{K}(a_1, \dots, a_n)$. Actually, if $a_i = \min\{a_1, \dots, a_n\}$, then we get

$$a_i = \sqrt[n]{\frac{na_i^n}{n}} \leq \sqrt[n]{\frac{a_i + a_i^2 + \dots + a_i^n}{n}} \leq \mathcal{K}(a_1, \dots, a_n).$$

Similarly, we get that if $1 \leq a_i$ ($1 \leq i \leq n$), then $\mathcal{K}(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\}$. Therefore, $\mathcal{K}|_{[1, +\infty)}$ is a left-mean, while $\mathcal{K}|_{[0, 1]}$ is a right-mean. Clearly, it is not a mean, simply because it is not reflexive.

EXAMPLE 3.56.

- a) Let $\mathcal{K}(a, b) = \frac{a+b}{2+\max\{a,b\}-\min\{a,b\}}$ ($a, b \in \mathbb{R}^+$). Obviously, $\mathcal{K}(a, b) \leq \max\{a, b\}$, hence it is a lateral mean. $\mathcal{K}(2, 4) = 1.5$ shows that it is not a mean. It is strict, symmetric, continuous and reflexive, hence it is mean-continuous as well. It is not monotone as $\frac{2+3}{3} \not\leq \frac{2+4}{4}$ witnesses.

- b) If $a, b \in \mathbb{R}^+$, let

$$\mathcal{K}(a, b) = \begin{cases} \frac{a+b}{2+\max\{a,b\}-\min\{a,b\}} & \text{if } a \neq b, \\ a, & \text{otherwise.} \end{cases}$$

Obviously, $\mathcal{K}(a, b) \leq \max\{a, b\}$, hence it is a lateral mean. $\mathcal{K}(10, 11) = 7$ shows that it is not a mean. It is strict, symmetric, reflexive, not continuous, and not mean-continuous.

EXAMPLE 3.57. It is easy to construct a non-continuous, non-reflexive, but mean-continuous quasi means. For example, if $a, b \in [1, +\infty)$, let

$$\mathcal{K}(a, b) = \begin{cases} \frac{a+b}{2} & \text{if } a \neq b, \\ a + \frac{1}{a}, & \text{otherwise.} \end{cases}$$

4. Duality and lateral means

There is a kind of algebraic duality between certain means that are defined by addition and multiplication only (or other operations derived from these). If we have such a mean, then, if in the definition of the mean we replace addition with multiplication and vice versa, we may end up with a new mean, which is called the dual of the original mean. Unfortunately, this operation is not well-defined because the same mean can be represented in many different formulas. All of these can be investigated to see if the dual is a mean or not. In some cases, this operation results in lateral means.

First, let us give some non proper explanation for why we replace $\frac{a}{n}$ with $\sqrt[n]{a}$. Let $x = \frac{a}{n}$. Then $nx = a$, that is, $x + \dots + x = a$. Replacing “+” with “.”, we get x^n , hence $\frac{a}{n} \rightsquigarrow \sqrt[n]{a}$. A similar reasoning shows that $\sqrt[n]{a} \rightsquigarrow \frac{a}{n}$.

EXAMPLE 4.1. If we consider the arithmetic mean and replace addition with multiplication, and division by n with n^{th} root, then we end up with the geometric mean. And vice versa: from the geometric mean, we get the arithmetic mean by replacing multiplication with addition, and the n^{th} root with division by n :

$$\frac{a_1 + \cdots + a_n}{n} \longleftrightarrow \sqrt[n]{a_1 \cdots a_n}.$$

EXAMPLE 4.2. If we start with the x^{th} power mean ($x \in \mathbb{R}$) and replace addition with multiplication, division by n with n^{th} root, and the power to x with multiplication by x , we again end up with the geometric mean:

$$\sqrt[x]{\frac{a_1^x + \cdots + a_n^x}{n}} \rightsquigarrow \frac{\sqrt[n]{x a_1 \cdots x a_n}}{x} = \sqrt[n]{a_1 \cdots a_n}.$$

We present an example demonstrating that different but equivalent formulas can end up with different results.

EXAMPLE 4.3. Let $a_1, a_2, a_3 \in \mathbb{R}^+$.

$$\begin{aligned} \sqrt[6]{\frac{(a_1^2 + a_2^2)(a_2^2 + a_3^2)(a_3^2 + a_1^2)}{8}} &= \sqrt[3]{\sqrt{\frac{a_1^2 + a_2^2}{2}} \sqrt{\frac{a_2^2 + a_3^2}{2}} \sqrt{\frac{a_3^2 + a_1^2}{2}}} \rightsquigarrow \\ &\quad \frac{\sqrt{a_1 a_2} + \sqrt{a_2 a_3} + \sqrt{a_3 a_1}}{3}, \end{aligned}$$

which is a mean. While

$$\begin{aligned} \sqrt[6]{\frac{(a_1^2 + a_2^2)(a_2^2 + a_3^2)(a_3^2 + a_1^2)}{8}} &\rightsquigarrow \frac{\sqrt[3]{(2a_1 2a_2) + (2a_2 2a_3) + (2a_3 2a_1)}}{6} = \\ &\quad \frac{1}{2^{\frac{3}{4}}} \frac{a_1 a_2 + a_2 a_3 + a_3 a_1}{3} \end{aligned}$$

is only a lateral mean.

5. Operations on lateral means

PROPOSITION 5.1. *Let $\mathcal{K}_0, \dots, \mathcal{K}_n$ be n -variable left-means (right-means). Set*

$$\mathcal{K}(a_1, \dots, a_n) = \mathcal{K}_0(\mathcal{K}_1(a_1, \dots, a_n), \dots, \mathcal{K}_n(a_1, \dots, a_n)) \quad (a_1, \dots, a_n \in \mathbb{R}).$$

Then, $\mathcal{K}$ is a left-mean (right-mean) as well.

Proof. We show the “right” part, the other one is similar.

Let $a_n = \max\{a_1, \dots, a_n\}$ and let j be chosen such that

$$\mathcal{K}_j(a_1, \dots, a_n) = \max\{\mathcal{K}_1(a_1, \dots, a_n), \dots, \mathcal{K}_n(a_1, \dots, a_n)\}.$$

$$\mathcal{K}_0(\mathcal{K}_1(a_1, \dots, a_n), \dots, \mathcal{K}_n(a_1, \dots, a_n)) \leq \mathcal{K}_j(a_1, \dots, a_n) \leq a_n. \quad \square$$

We now present a generalization of compounding for lateral means. We formulate it for right-means; a similar statement can be given for left-means.

THEOREM 5.2. *Let $\mathcal{K}, \mathcal{M} : (\mathbb{R}^+ \cup \{0\})^2 \rightarrow \mathbb{R}^+ \cup \{0\}$ be 2-variable right-means such that $\mathcal{K} \leq \mathcal{M}$. Moreover, let $\mathcal{M}$ be strict and continuous. Let*

$$a, b \in \mathbb{R}^+ \cup \{0\}, \quad a \leq b.$$

Define $a_0 = a$, $b_0 = b$, $a_{n+1} = \mathcal{K}(a_n, b_n)$, $b_{n+1} = \mathcal{M}(a_n, b_n)$ ($n \in \mathbb{N}$). Then, $(a_n), (b_n)$ converge to the same limit, say d . Consequently, $\mathcal{N}(a, b) = d$ becomes a strict right-mean. Moreover, if $\mathcal{K}$ is also continuous, then $\mathcal{N}$ is upper semi-continuous.

Proof. It can be readily seen that $\forall n \quad a_n \leq b_n$, $b_{n+1} < b_n$. Hence, (b_n) is decreasing and bounded below; therefore, it converges, say to d . Let c be an accumulation point of (a_n) . Let $a_{n_k} \rightarrow c$. Suppose that $c < d$. Let $e = \mathcal{M}(c, d) < d$. Let $\epsilon > 0$ be chosen such that $e + \epsilon < d$. Then, by the continuity of $\mathcal{M}$, there is K such that $n_k > K$ implies $\mathcal{M}(a_{n_k}, b_{n_k}) \in (e - \epsilon, e + \epsilon)$. This implies that $b_{n_k+1} < d$, which is a contradiction. Clearly, $d < b$; hence $\mathcal{N}(a, b)$ is a strict right-mean.

To see the upper semi-continuity of $\mathcal{N}$, first observe that for the sequences defined above and for every $n \in \mathbb{N}$, both a_n and b_n depend continuously on a and b . That is, $a_n = a_n(a, b)$ can be regarded as a function of a, b ; in other words, it is a two-variable continuous function.

Now, let

$$x, y \in \mathbb{R}^+ \cup \{0\}, \quad x \leq y, \quad x_n \rightarrow x, \quad y_n \rightarrow y, \quad d = \mathcal{N}(x, y).$$

Suppose indirectly that $d_n = \mathcal{N}(x_n, y_n) \not\rightarrow d$. Then, changing to a subsequence, we can assume that $\mathcal{N}(x_n, y_n) \rightarrow c > d$. Let $\epsilon = \frac{c-d}{3}$. Then, there is N such that $b_N(x, y) < d + \epsilon$. By the continuity of b_N at (x, y) , there is M such that $m > M$ implies $b_N(x_m, y_m) < d + 2\epsilon$. However, $d_m < b_N(x_m, y_m)$, which yields a contradiction. $\square$

We now provide a sufficient condition for $\mathcal{N}$ to be continuous.

PROPOSITION 5.3. *Let $x, y \in \mathbb{R}^+ \cup \{0\}$. Using the notation and conditions of 5.2, assume that $\forall \epsilon > 0 \exists \delta > 0 \exists M \in \mathbb{N}$ such that $m \geq M$, $|x - x'| < \delta$, $|y - y'| < \delta$ implies $|a_m(x', y') - \mathcal{N}(x', y')| < \epsilon$. Then, $\mathcal{N}$ is continuous at (x, y) .*

Proof. We use the notation from the proof of 5.2. We need to prove the lower semi-continuity of $\mathcal{N}$. Suppose indirectly that $d_n = \mathcal{N}(x_n, y_n) \not\rightarrow d$. Then, changing to a subsequence, we may assume that $\mathcal{N}(x_n, y_n) \rightarrow c < d$. Let $\epsilon = \frac{c-d}{4}$. By the condition, choose $\delta > 0, M \in \mathbb{N}$ such that

$$m \geq M, |x - x'| < \delta, |y - y'| < \delta \quad \text{implies} \quad |a_m(x', y') - \mathcal{N}(x', y')| < \epsilon.$$

Then, there exists $N > M$ such that $a_N(x, y) > d - \epsilon$. By the continuity of a_N at (x, y) , there is N' such that $k > N'$ implies

$$|x - x_k| < \delta, |y - y_k| < \delta \quad \text{and} \quad a_N(x_k, y_k) > d - 2\epsilon.$$

But then, $d_k > d - 3\epsilon$, which is a contradiction. $\square$

EXAMPLE 5.4. We provide lateral means that satisfy the conditions of 5.2, but the compounding lateral mean fails to be continuous. Let

$$\mathcal{K} = \mathcal{A}_0^-|_2, \quad \mathcal{M} = \mathcal{A}_1^-|_2.$$

Let

$$a = 1, b = 2 \quad \text{and} \quad c_n \rightarrow 1, \quad 0 < c_n < 1, \quad d_n \rightarrow 2, \quad 1 < d_n < 2.$$

Let us denote the compounding mean by $\mathcal{N}$.

We state that $1 \leq \mathcal{N}(1, 2)$. Clearly, $1 \leq a_0, b_0$ (using the notation of 5.2). Suppose that $1 \leq a_n, b_n$. Then, $1 \leq a_{n+1}, b_{n+1}$. Hence, we get $1 \leq \lim b_n$. We show that $\lim_{n \rightarrow \infty} \mathcal{N}(c_n, d_n) < 1$; moreover, $\forall n \mathcal{N}(c_n, d_n) < 0.75$. If n is fixed, then denote the associated sequences by

$$\left(a_m^{(c_n, d_n)} \right), \left(b_m^{(c_n, d_n)} \right).$$

One can readily get that

$$\begin{aligned} a_1^{(c_n, d_n)} &= 0.5, & b_1^{(c_n, d_n)} &\leq 1.5; \\ a_2^{(c_n, d_n)} &\leq 0.5, & b_2^{(c_n, d_n)} &\leq 1; \\ a_3^{(c_n, d_n)} &\leq 0.5, & b_3^{(c_n, d_n)} &\leq 0.75. \end{aligned}$$

Therefore,

$$\lim_{m \rightarrow \infty} b_m^{(c_n, d_n)} < 0.75.$$

DEFINITION 5.5. Let $\mathcal{K}: H^2 \rightarrow H$ be a symmetric right-mean. If $a, b \in H, a \leq b$, then let

$$\mathcal{K}^{(0)}(a, b) = \mathcal{K}^{(0)}(b, a) = \mathcal{K}(a, b),$$

$$\mathcal{K}^{(n+1)}(a, b) = \mathcal{K}^{(n+1)}(b, a) = \mathcal{K}(\mathcal{K}^{(n)}(a, b), b) \quad (n \in \mathbb{N})$$

DEFINITION 5.6. Let $\mathcal{K}: H^2 \rightarrow H$ be a symmetric right-mean. $\mathcal{K}$ is called right-idempotent if $\mathcal{K}^{(1)} = \mathcal{K}$.

Of course, similar notions can be defined for the “left” version.

PROPOSITION 5.7. Let $\mathcal{K}: H^2 \rightarrow H$ be a symmetric right-mean. Then, the following statements hold:

- (1) For all $n \in \mathbb{N}$, $\mathcal{K}^{(n)}$ is a right-mean as well.
- (2) If $\mathcal{K}$ is right-idempotent, then $\forall n \mathcal{K}^{(n)} = \mathcal{K}$.
- (3) If $\forall a \in H \forall b \in H, \lim_{n \rightarrow \infty} \mathcal{K}^{(n)}(a, b)$ exists and is finite, denote it by $\tilde{\mathcal{K}}(a, b)$. If $\mathcal{K}$ is continuous, then $\tilde{\mathcal{K}}$ is a right-idempotent right-mean.

Proof.

(1) and (2) are trivial.

(3): $\tilde{\mathcal{K}}$ is obviously a right-mean, since more generally, the limit of right-means is a right-mean.

If $a, b \in H$, let $y = \tilde{\mathcal{K}}(a, b)$. By definition, $\mathcal{K}^{(n)}(a, b) \rightarrow y$, which gives that $\mathcal{K}(\mathcal{K}^{(n)}(a, b), b) \rightarrow \mathcal{K}(y, b)$ since $\mathcal{K}$ is continuous. But $\mathcal{K}(\mathcal{K}^{(n)}(a, b), b) = \mathcal{K}^{(n+1)}(a, b) \rightarrow y$ as well, and thus, $\mathcal{K}(y, b) = y$, which also gives $\tilde{\mathcal{K}}(y, b) = y$. Hence, $\tilde{\mathcal{K}}(\tilde{\mathcal{K}}(a, b), b) = \tilde{\mathcal{K}}(y, b) = y = \tilde{\mathcal{K}}(a, b)$. $\square$

EXAMPLE 5.8. The continuity of $\mathcal{K}$ does not imply that $\tilde{\mathcal{K}}$ is continuous, e.g., let $\mathcal{K}(a, b) = \min\{a^2, b\}$ ($a, b \in [0, 2]$). Then, $\tilde{\mathcal{K}}(1, 2) = 1$ and $\tilde{\mathcal{K}}(1 - \epsilon, 2) = 0$ for all $\epsilon > 0$.

In [8], we investigated how a k -variable mean can be extended to an n -variable mean for $2 \leq k < n$. Here, we partially generalize that process for lateral means. We formulate the following theorems for right-means; one can easily get them for left-means as well.

PROPOSITION 5.9. *Let $H \subset \mathbb{R}$ be bounded from below. Let $\mathcal{K}: H^2 \rightarrow H$ be a strict, continuous, monotone right-mean. Let $a, b, c \in H$, $a \leq b \leq c$. Let*

$$\begin{aligned} a_0 &= a, \quad b_0 = b, \quad c_0 = c, \\ a_{n+1} &= \mathcal{K}(a_n, b_n), \quad b_{n+1} = \mathcal{K}(a_n, c_n), \quad c_{n+1} = \mathcal{K}(b_n, c_n) \quad (n \in \mathbb{N}). \end{aligned}$$

Now, all three sequences converge to the same limit.

Proof. First, note that monotonicity implies that $\forall n \ a_n \leq b_n \leq c_n$. Since $\mathcal{K}$ is a right-mean, the sequence (c_n) is decreasing, and therefore, it converges by boundedness; let $c_n \rightarrow d$. Assume that $b_{n_k} \rightarrow e < d$. Then, $c_{n_k+1} = \mathcal{K}(b_{n_k}, c_{n_k})$, but continuity yields that $\mathcal{K}(b_{n_k}, c_{n_k}) \rightarrow \mathcal{K}(e, d) < d$ by strictness, which is a contradiction.

Similarly, assume that $a_{n_k} \rightarrow e < d$. Then, $b_{n_k+1} = \mathcal{K}(a_{n_k}, c_{n_k})$, but continuity yields that $\mathcal{K}(a_{n_k}, c_{n_k}) \rightarrow \mathcal{K}(e, d) < d$ by strictness, which is also a contradiction. $\square$

EXAMPLE 5.10. If the boundedness property is omitted, then 5.9 fails to be true. For example, let $\mathcal{K}(a, b) = \frac{a+b}{2} - 1$. It can be readily seen that $c_n \rightarrow -\infty$.

PROPOSITION 5.11. *Let $\mathcal{K}: H^2 \rightarrow H$ be a strict, continuous, monotone right-mean. Using the notations of 5.9, if $c_n \rightarrow -\infty$, then $a_n \rightarrow -\infty$ and $b_n \rightarrow -\infty$ as well.*

PROPOSITION 5.12. *Let $\mathcal{K}: H^2 \rightarrow H$ be a strict, continuous, monotone, semi-reflexive right-mean. Using the notations of 5.9, all three sequences converge to the same limit.*

Proof. We show by induction that $\mathcal{K}(a, a) \leq a_n, b_n, c_n$, which is sufficient by reference to the proof of 5.9. Monotonicity gives that $\mathcal{K}(a, a) \leq a_1, b_1, c_1$. Then, suppose that $\mathcal{K}(a, a) \leq a_n, b_n, c_n$. Then,

$$\mathcal{K}(a, a) = \mathcal{K}(\mathcal{K}(a, a), \mathcal{K}(a, a)) \leq \mathcal{K}(a_n, b_n), \mathcal{K}(a_n, c_n), \mathcal{K}(b_n, c_n). \quad \square$$

THEOREM 5.13. *Let $\mathcal{K} : H^2 \rightarrow H$ be a strict, continuous, monotone right-mean that is either semi-reflexive or such that H is bounded from below. Using the notations of 5.9, let $\tilde{\mathcal{K}}(a, b, c)$ denote the common limit point of the sequences. Then, $\tilde{\mathcal{K}}$ is a strict, upper semi-continuous, monotone right-mean.*

Proof.

- $\tilde{\mathcal{K}}$ is a right-mean because $\tilde{\mathcal{K}}(a, b, c) \leq c_1 \leq c$.
- $\tilde{\mathcal{K}}$ is a strict since $\lim c_n \leq c_1 < c$.
- $\tilde{\mathcal{K}}$ is a monotone: Let $a' \leq a, b' \leq b, c' \leq c$, and let the associated sequences be $(a'_n), (b'_n), (c'_n)$. A straightforward induction shows that $a'_n \leq a_n, b'_n \leq b_n, c'_n \leq c_n$, which gives that $\lim c'_n \leq \lim c_n$.
- $\tilde{\mathcal{K}}$ is upper semi-continuous: First, note that for a fixed $n \in \mathbb{N}$, c_n can be considered as a function of a, b, c . Clearly, $c_n(a, b, c)$ is a continuous 3-variable function. Now, if $\varepsilon > 0$ is given, then $\exists n \in \mathbb{N}$ such that

$$c_n < \tilde{\mathcal{K}}(a, b, c) + \frac{\varepsilon}{2}, \quad c_n = c_n(a, b, c).$$

Furthermore, $\exists \delta > 0$ such that $|a' - a| < \delta, |b' - b| < \delta, |c' - c| < \delta$ implies that $|c_n(a', b', c') - c_n| < \frac{\varepsilon}{2}$. Therefore, $c_n(a', b', c') < \tilde{\mathcal{K}}(a, b, c) + \varepsilon$. Since $\tilde{\mathcal{K}}(a', b', c') \leq c_n(a', b', c')$ holds, and ε is arbitrary, it follows that

$$\tilde{\mathcal{K}}(a', b', c') \leq \tilde{\mathcal{K}}(a, b, c). \quad \square$$

EXAMPLE 5.14. Let us calculate what we get for $\mathcal{A}_0^-$ and 1.1, 2.1, 3.1 using the method described in 5.9.

$$\begin{array}{ll} \text{1.step: } 1.5, 2, 2.5; & \text{2.step: } 1.5, 1.5, 2; \\ \text{3.step: } 1, 1.5, 1.5; & \text{4.step: } 1, 1, 1. \end{array}$$

Hence, this extension does not give back $\mathcal{A}_0^-(1.1, 2.1, 3.1) = 2$, in contrast to $\mathcal{A}$, where it does give back the 3-variable version (see [8]).

EXAMPLE 5.15. $\mathcal{A}_0^-$ is a lateral mean that satisfies the conditions of 5.13, but the extension lateral mean $\tilde{\mathcal{A}}_0^-$ fails to be continuous.

Let $a = 1, b = 2, c = 3$. Then, easy calculation shows that $\tilde{\mathcal{A}}_0^-(1, 2, 3) = 1$.

Let $(a_n), (b_n), (c_n)$ be sequences such that

$$a_n \rightarrow 1, \quad 0 < a_n < 1, \quad b_n \rightarrow 2, \quad 1 < b_n < 2, \quad c_n \rightarrow 3, \quad 2 < c_n < 3.$$

Let us calculate $\tilde{\mathcal{A}}_0^-(a_n, b_n, c_n)$ for a fixed n .

$$\begin{array}{ll} \text{1.step: } 0.5, 1, 1.5; & \text{2.step: } 0.5, 0.5, 1; \\ \text{3.step: } 0, 0.5, 0.5; & \text{4.step: } 0, 0, 0. \end{array}$$

Hence, $\forall n \ \tilde{\mathcal{A}}_0^-(a_n, b_n, c_n) = 0$ and $\lim_{n \rightarrow \infty} \tilde{\mathcal{A}}_0^-(a_n, b_n, c_n) = 0 \neq \tilde{\mathcal{A}}_0^-(1, 2, 3)$.

The phenomenon in Example 5.14 is not an accidental. For $\mathcal{A}_m^-, \mathcal{A}_m^+$, these sequences become constant after finitely many steps, as the next statement shows.

PROPOSITION 5.16. *Let $\mathcal{K} : H^2 \rightarrow H$ be a strict, continuous, monotone, semi-reflexive right-mean such that $\text{Ran } \mathcal{K}$ is closed and discrete. Using the notations of 5.9, there exist*

$$\exists N \in \mathbb{N} \ \exists d \in \mathbb{R} \quad \text{such that} \quad n \geq N \quad \text{implies that} \quad a_n = b_n = c_n = d.$$

PROOF. If for an $n \in \mathbb{N}$, not all values a_n, b_n, c_n are equal, then in the next step, one of the elements $a_{n+1}, b_{n+1}, c_{n+1}$ will strictly decrease by the strictness of $\mathcal{K}$. By monotonicity and semi-reflexivity, simple induction gives $\mathcal{K}(a, a) \leq a_n, b_n, c_n$. Since $\text{Ran } \mathcal{K}$ is closed and discrete, there are only finitely many values of $\text{Ran } \mathcal{K}$ between $\mathcal{K}(a, a)$ and c , which establishes the claim. $\square$

6. On quasi-monotonicity

We are going to generalize the standard notion of monotonicity.

DEFINITION 6.1. A two variable right-mean $\mathcal{K} : H^2 \rightarrow H$ is called **quasi-monotone** if $a \leq b$ ($a, b \in H$), then the sequence (a_n) defined inductively $a_0 = a$, $a_{n+1} = \mathcal{K}(a_n, b)$ is monotone.

A two variable left-mean $\mathcal{K} : H^2 \rightarrow H$ is quasi-monotone if $a \leq b$ ($a, b \in H$), then the sequence (b_n) defined inductively $b_0 = b$, $b_{n+1} = \mathcal{K}(a, b_n)$ is monotone.

PROPOSITION 6.2. *If $\mathcal{K} : H^2 \rightarrow H$ is a mean, then it is quasi-monotone.*

PROOF. A mean is both a right-mean and a left-mean, so we need to check quasi-monotonicity in both cases. We will show the “right” part; the other case is similar.

If $a, b \in H$, $a \leq b$, then $\forall n \in \mathbb{N}$, $a_n \leq \mathcal{K}(a_n, b) = a_{n+1} \leq b$, hence (a_n) is monotone. $\square$

PROPOSITION 6.3. *If a lateral mean $\mathcal{K} : H^2 \rightarrow H$ is monotone, then it is quasi-monotone as well.*

PROOF. We prove the statement for right-means; the argument for left-means is similar.

Suppose that $a_1 \leq a_0$. Then, $a_2 = \mathcal{K}(a_1, b) \leq \mathcal{K}(a_0, b) = a_1$ by monotonicity. By induction, it follows that (a_n) is decreasing.

If $a_0 \leq a_1$, a similar argument shows that (a_n) is increasing. $\square$

EXAMPLE 6.4. The converse is not true. Let $\mathcal{K} : [0, 1]^2 \rightarrow [0, 1]$ be symmetric and defined as follows: if $a, b \in [0, 1]$, $a \leq b$, then the set

$$\mathcal{K}(a, b) = \begin{cases} \frac{a}{2} & \text{if } b = 1 \text{ and } a = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ b, & \text{otherwise.} \end{cases}$$

It can be readily seen that $\mathcal{K}$ is a quasi-monotone right-mean. It is not monotone, as $1 = \mathcal{K}(0.4, 1) \not\leq \mathcal{K}(0.5, 1) = 0.25$ demonstrates.

EXAMPLE 6.5. The previous example was not continuous. We now provide a continuous counter example as well.

Let $\mathcal{K} : [0, 1]^2 \rightarrow [0, 1]$ be symmetric and defined as follows. If $a, b \in [0, 1]$, $a \leq b$, then the set

$$\mathcal{K}(a, b) = \begin{cases} b \cdot 2 \left(\frac{a}{b}\right)^2 & \text{if } a \leq \frac{b}{2}, \\ b \left(1 - 2 \left(1 - \frac{a}{b}\right)^2\right), & \text{otherwise.} \end{cases}$$

It can be readily checked that $\mathcal{K}$ is a continuous, quasi-monotone right-mean. It is not monotone, as $0, 2 = \mathcal{K}(0.2, 0, 4) \not\leq \mathcal{K}(0.2, 1) = 0.08$ demonstrates.

DEFINITION 6.6. A lateral mean $\mathcal{K} : H^2 \rightarrow H$ is called **right-injective** if

$$\forall b \in H \ \mathcal{K}(x, b) = \mathcal{K}(y, b) \text{ implies that } x = y.$$

It is left-injective if

$$\forall a \in H \ \mathcal{K}(a, x) = \mathcal{K}(a, y) \text{ implies that } x = y.$$

EXAMPLE 6.7. The example in 6.5 is clearly right-injective but not left-injective, as, for instance, $\mathcal{K}(0, 0.4) = \mathcal{K}(0, 1) = 0$ shows.

PROPOSITION 6.8. A lateral mean $\mathcal{K} : H^2 \rightarrow H$ is right-injective if and only if

$$\forall b \in H \ f_b^{\mathcal{K}}(x) = \mathcal{K}(x, b) \quad (0 \leq x \leq b)$$

is an injective function.

PROPOSITION 6.9. Let $\mathcal{K} : [0, 1]^2 \rightarrow [0, 1]$ be a continuous, quasi-monotone right-mean. Set $f_b(x) = \mathcal{K}(x, b)$ ($0 \leq x \leq b$, $b \in [0, 1]$). Then, the following statements hold:

(1) $Z_b = \{x \in [0, b] : f_b(x) = x\}$ is a closed set.

(2) Let us use the notation $[0, b] - Z_b = \bigcup_{n=1}^{\infty} N_i$, where $N_i = (a_i, b_i)$.
Then, either

$$\forall x \in N_i \ f_b(x) > x \quad \text{or} \quad \forall x \in N_i \ f_b(x) < x.$$

Set $s(i) = 1$ in the first case and $s(i) = -1$ in the second one.

(3) If $f_b(N_i) \cap N_j \neq \emptyset$, then $s(i) = s(j)$.

(4) If, moreover, $\mathcal{K}$ is right-injective, then $\forall i \ f_b|_{N_i}$ is a monotone function, and $f_b(N_i) = N_i$.

Proof.

1. This is obvious from the continuity of f_b .

2. The function $f_b(x) - x$ is continuous and never 0 on N_i .

3. If $\sup N_i \leq \inf N_j$ and $f_b(N_i) \cap N_j \neq \emptyset$, then $s(i) = 1$, and there exists $x \in N_i$ such that $f_b(x) \in N_j$. Then, we get $x < f_b(x)$, hence, by the quasi-monotonicity, $f_b(x) < f_b(f_b(x))$, which implies $s(j) = 1$ as well.

The other case is similar.

4. This is obvious. □

EXAMPLE 6.10. Using the notation of 6.9, $b \mapsto Z_b$ is not continuous in the Hausdorff-metric in general, for example, let $\mathcal{K}(a, b) = a - ab + a^2b$ ($a, b \in [0, 1]$). Then clearly,

$$Z_0 = [0, 1] \quad \text{and} \quad Z_b = \{0, 1\} \text{ if } b > 0.$$

7. Further generalizations

We can generalize even further. For example, consider the following function.

DEFINITION 7.1. Set $\mathcal{A}_m^* = \frac{\mathcal{A}_m^+ + \mathcal{A}_m^-}{2}$ ($m \in \mathbb{Z}$).

An obvious calculation shows that

$$\begin{aligned} \mathcal{A}_m^*(2 \cdot 10^{-m}, 2.1 \cdot 10^{-m}) &= \frac{2.5 \cdot 10^{-m} + 2 \cdot 10^{-m}}{2} = 2.25 \cdot 10^{-m} \\ &> \max\{2 \cdot 10^{-m}, 2.1 \cdot 10^{-m}\}, \\ \mathcal{A}_m^*(1.9 \cdot 10^{-m}, 2 \cdot 10^{-m}) &= \frac{2 \cdot 10^{-m} + 1.5 \cdot 10^{-m}}{2} = 1.75 \cdot 10^{-m} \\ &< \min\{1.9 \cdot 10^{-m}, 2 \cdot 10^{-m}\}. \end{aligned}$$

Therefore, $\mathcal{A}_m^*$ is not a lateral mean ($m \in \mathbb{Z}$). However, as m increases, it approaches being a mean in some sense. In that direction, we introduce the following terminology.

DEFINITION 7.2. $\mathcal{K} : H^n \rightarrow \mathbb{R}$ is called an **a-lateral mean** if $\exists K \geq 0$ such that

$$\mathcal{K}(a_1, \dots, a_n) \in [\min\{a_1, \dots, a_n\} - K, \max\{a_1, \dots, a_n\} + K].$$

If $\mathcal{K}$ is a multi-variable lateral mean, then it is an a-lateral mean if $\exists K \geq 0$ such that for all n the above condition holds.

PROPOSITION 7.3. $\mathcal{A}_m^*$ is an a-lateral mean with $K = \frac{1}{2 \cdot 10^m}$.

Proof. By 3.17 (5) and 3.28 (5),

$$\frac{\mathcal{A}_m^+ + \mathcal{A}_m^-}{2} \leq \frac{\mathcal{A} + \frac{1}{10^m} + \mathcal{A}}{2} = \mathcal{A} + \frac{1}{2 \cdot 10^m} \leq \max + \frac{1}{2 \cdot 10^m}.$$

The other inequality is similar. □

PROPOSITION 7.4.

$$\lim_{m \rightarrow \infty} \mathcal{A}_m^*(a_1, \dots, a_n) = \mathcal{A}(a_1, \dots, a_n)$$

and the convergence is uniform

$$(a_1, \dots, a_n) \in \text{Dom } \mathcal{A}_0^+ \cap \text{Dom } \mathcal{A}_0^-.$$

Proof. 3.17 (6) and 3.28 (6). □

PROPOSITION 7.5. $\mathcal{K}$ is an a-lateral mean if and only if $\text{mdist}(\mathcal{K}) < \infty$.

Now, let us examine another function similar to a lateral mean. It is related to $\mathcal{B}^+$ and $\mathcal{B}^-$, with the difference that we do not restrict its domain.

DEFINITION 7.6. Set $\mathcal{B}(a_1, \dots, a_n) = \frac{\sum_{i=1}^n a_i}{n-1}$ ($n \in \mathbb{N}$, $n \geq 2$, $a_i \in \mathbb{R}$).

Clearly, it is not a lateral mean, as $2 < \mathcal{B}(1, 2)$ and $\mathcal{B}(-1, -2) < -1$ show.

DEFINITION 7.7. $\mathcal{K} : H^n \rightarrow \mathbb{R}$ is called an **m-lateral mean** if $\exists K \geq 0$ such that

$$\mathcal{K}(a_1, \dots, a_n) \in [\min\{a_1, \dots, a_n\} - K \max\{|a_1|, \dots, |a_n|\}, \max\{a_1, \dots, a_n\} + K \max\{|a_1|, \dots, |a_n|\}].$$

If $\mathcal{K}$ is a multi-variable lateral mean, then it is an m-lateral mean if $\exists K \geq 0$ such that for all n the above condition holds. □

PROPOSITION 7.8. $\mathcal{B}|_n$ is an m-lateral mean with $K = \frac{1}{n-1}$.

Proof. Let $a_n = \max\{a_1, \dots, a_n\}$ and $0 \leq a_n$. Then,

$$\mathcal{B}(a_1, \dots, a_n) \leq \frac{n}{n-1} a_n = a_n + \frac{1}{n-1} a_n \leq a_n + \frac{1}{n-1} \max\{|a_1|, \dots, |a_n|\}.$$

If $a_n < 0$, then $a_n + \frac{1}{n-1} a_n \leq a_n$.

Similar arguments work for the other inequality. □

COROLLARY 7.9. $\mathcal{B}$ is an m-lateral mean with $K = 1$.

PROPOSITION 7.10. *If $\text{mdistp}(\mathcal{K}) < \infty$, then $\mathcal{K}$ is an m -lateral mean.*

Proof.

Let $(a_1, \dots, a_n) \in \text{Dom } \mathcal{K}$. Set $m = \min\{a_1, \dots, a_n\}$, $M = \max\{a_1, \dots, a_n\}$, $N = \max\{|a_1|, \dots, |a_n|\}$. Suppose that $\exists K$ such that

$$\mathcal{K}(a_1, \dots, a_n) - M \leq K(M - m).$$

Then,

$$\mathcal{K}(a_1, \dots, a_n) \leq M + KM - Km \leq M + KN + KN = M + (K + 1)N. \quad \square$$

Obvious examples show that a lateral mean is not necessarily an a-lateral mean or an m-lateral mean. Conversely, an a-lateral mean or an m-lateral mean is not necessarily a lateral mean.

EXAMPLE 7.11. An a-lateral mean is not necessarily an m-lateral mean. For example, let $\mathcal{K}(a, b) = \max\{a, b\} + 1$ ($a, b \in \mathbb{R}$). If $a = 0, b > 0$, then $K = \frac{1}{b}$ shows that it is not an m-lateral mean.

An m-lateral mean is not necessarily an a-lateral mean. For example, let

$$\mathcal{K}(a, b) = 2 \cdot \max\{a, b\} \quad (a, b \geq 1).$$

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