Tatra Mt. Math. Publ. ?? (2026), 1–26
DOI: 10.2478/tmmp-2026-0004

Tatra Mountains Mathematical Publications

GENERAL FRAMEWORK FOR INTUITIONISTIC VALUES AND RELATED FUZZY SETS BASED ON AUTOMORPHISMS

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ABSTRACT. This paper introduces a novel framework for understanding intuitionistic values and fuzzy sets through the automorphism of the unit interval $[0,1]$. By generalizing the concepts of classical intuitionistic sets using strong negations generated by these automorphisms, we define the φ-intuitionistic values and explore their arithmetic, establishing a comprehensive set of operations based mainly on additive generators of t-(co)norms and their properties. This approach provides greater flexibility to the modelling with these sets.

1. Introduction

The introduction of fuzzy set theory by Zadeh [26] provided a revolutionary framework for modelling uncertainty and imprecision, which was an important moment in the evolution of soft computing. Atanassov's generalization of this framework, intuitionistic fuzzy sets [3], marked a significant advancement by introducing an additional layer of uncertainty to capture hesitation, thereby enriching the applicability of fuzzy logic in complex decision-making processes. This process was further expanded through the development of Pythagorean fuzzy sets and q-rung orthopair fuzzy sets by Yager [22,23], who attempted to introduce more general frameworks for representing uncertainty. These extensions

2020 Mathematics Subject Classification: 03E72,06D72,08A72,03G10.
Keywords: strong negations, intuitionistic fuzzy sets, interval arithmetic.
Adam Šeliga was funded by the EU NextGenerationEU through the Recovery and Resilience Plan for Slovakia under the project No. 09I03-03-V04-00276. Matúš Ďubek acknowledges the support of the grants VEGA 1/0036/23, VEGA 2/0128/24 and Program na podporu mladých výskumníkov (Young Researchers Support Programme).

to fuzzy sets became popular in applications across diverse fields, including artificial intelligence, medical diagnosis, and engineering, see, e.g., the papers [1, 4, 6, 11, 25]. However, it was shown that frameworks derived from Atanassov's original intuitionistic fuzzy sets are isomorphic to these extensions [8] and are built on a misleading idea of expanding the original domain of the original intuitionistic values to provide more flexibility. Our contribution lies in leveraging automorphisms of the unit interval $[0, 1]$ which unifies all currently known types of intuitionistic fuzzy sets, but also shows that the area of original intuitionistic values can be restricted instead of expanded, without losing any properties of the original framework. This approach builds upon the original idea of Zadeh's fuzzy sets and emphasizes the relationship between membership and non-membership under different strong negations. It provides a robust mathematical framework for exploring intuitionistic values and their interrelationships, further advancing the theory and enhancing the practical utility of fuzzy sets in addressing real-world challenges characterized by vagueness and ambiguity.

Our work aims to extend the theoretical basics of intuitionistic fuzzy sets and to broaden their spectrum of applications in fields demanding uncertainty modelling, such as pattern recognition and intelligent systems. This work extends the following three papers:

(1) Klement and Mesiar in [8] discuss the isomorphisms between different lattices of truth values and their corresponding families of fuzzy sets. This article criticizes the repetitiveness in the research of intuitionistic fuzzy sets and their generalizations, but does not provide a general framework for them.
(2) Mesiar et al. in [12] focus on extending the fuzzy set theory by exploring the properties and operations of lattices isomorphic to closed subintervals of $[0, 1]$, including intuitionistic and Pythagorean fuzzy sets. It aims to provide a unified framework for connectives based on strict t-norms and their aggregation in various types of fuzzy sets, facilitating a broader application without the need to prove properties for each new type of connective or lattice. Compared to our results, we find that the results of the presented paper are more general due to its foundational use of automorphisms to define and explore the properties of intuitionistic values and fuzzy sets.
(3) Tao et al. in [19], which focuses on the application of intuitionistic fuzzy values within the context of multiple-attribute decision-making, leveraging copula functions to aggregate intuitionistic fuzzy information. This approach, while innovative in its use of copula and co-copula functions to develop new operations and aggregation operators, we find to be restrictive (observe, e.g., the convexity of the considered additive generator)

and the use of automorphisms broader generality offers a robust mathematical framework to articulate and explore intuitionistic values and their interrelations in a comprehensive matter.

2. Preliminaries

Definition 2.1 ([15])**.** An (order-)automorphism of a partially ordered set $(A, \leq)$ is a mapping $\varphi\colon A \to A$ that is an isomorphism and a self-map.

In this paper, we work with the unit interval $A = [0,1]$ and the classical (total) order $\leq$ on real numbers. In this setting, an automorphism of the unit interval $[0,1]$ is any bijective strictly increasing mapping $\varphi\colon [0,1] \to [0,1]$.

Definition 2.2 ([13])**.** A fuzzy negation (or a fuzzy complement) is a mapping $n\colon [0,1] \to [0,1]$ that is non-increasing and satisfies the boundary conditions $n(0) = 1$ and $n(1) = 0$. If a fuzzy negation is strictly decreasing and continuous, it is called a strict negation. Moreover, if a fuzzy negation n is also involutive, i.e., $n(n(x)) = x$ for all $x \in [0,1]$, it is called a strong negation.

Theorem 2.3 ([20])**.** *Every strong negation n can be generated by an automorphism φ with the formula*

$$n(x) = \varphi^{-1}\big(1 - \varphi(x)\big) \quad \textit{for all } x \in [0,1].$$

Theorem 2.3 implies that the equation

$$\varphi\big(n(x)\big) + \varphi(x) = 1$$

holds for every $x \in [0,1]$. For the sake of completeness, we also provide a method for generating automorphisms φ from a strong negation n:

(i) find a fixed point of n, i.e., a point $u \in]0,1[$ such that $n(u) = u$;

(ii) choose a strictly increasing continuous function φ on $[0,u]$ such that $\varphi(0) = 0$ and $\varphi(u) = 1/2$;

(iii) set $\varphi(x) = 1 - \varphi(n(x))$ for $x \in]u,1]$.

In step (iii) of the construction, the values of φ on $[0,u]$ chosen in step (ii) are used since the strict decreasingness of n implies $n(x) < n(u) = u$ for $x > u$. This shows that there are infinitely many different automorphisms for the same strong negation.

Example. Every automorphism φ for which $\varphi(x) + \varphi(1-x) = 1$ holds for all $x \in [0,1]$ generates the standard Zadeh's negation given by $n_{\mathrm{Z}}(x) = 1 - x$

for all $x \in [0,1]$ originally introduced in the paper [26]. To give examples of such automorphisms that generate n_Z, consider

$$\varphi_{Z1}(x) = x \quad \text{or} \quad \varphi_{Z2}(x) = \frac{1-\cos(\pi x)}{2} \quad \text{for all } \ x \in [0,1].$$

Definition 2.4 ([9])**.** A function $T\colon [0,1]^2 \to [0,1]$ is called a triangular norm (or a t-norm, for short) if it is commutative, associative, non-decreasing and it has neutral element 1.

Definition 2.5 ([9])**.** A function $S\colon [0,1]^2 \to [0,1]$ is called a triangular conorm (or a t-conorm) if it is commutative, associative, non-decreasing and its neutral element is 0.

Theorem 2.6 ([9])**.** *For every t-norm T, there is a t-conorm S given by*

$$S(x,y) = 1 - T(1-x, 1-y) \quad \textit{for all } \ x,y \in [0,1].$$

T-norms and t-conorms given by Theorem 2.6 are called *dual* to each other.

Remark 1. Due to the associativity property, triangular norms and conorms can be uniquely extended from binary to k-ary forms, where $k \geq 3$ is a natural number.

Definition 2.7 ([9])**.** A continuous t-norm T is called Archimedean if it satisfies the inequality $T(x,x) < x$ for every $x \in]0,1[$ and a continuous t-conorm is called Archimedean if it satisfies the inequality $S(x,x) > x$ for every $x \in]0,1[$. If a t-norm is Archimedean, so is its dual t-conorm, and vice versa.

Definition 2.8 ([9])**.** An additive generator is a mapping $t\colon [0,1] \to [0,\infty]$ which is continuous, strictly decreasing, and satisfies the boundary condition $t(1) = 0$.

Definition 2.9. Let t be an additive generator. Then, the pseudo-inverse of t is a mapping $t^{(-1)}\colon [0,\infty] \to [0,1]$ given by the formula

$$t^{(-1)}(x) = t^{-1}\Big(\min\big(x, t(0)\big)\Big) \quad \text{for all} \quad x \in [0,\infty].$$

Theorem 2.10 ([9])**.** *If a continuous t-norm T (a t-conorm S) is Archimedean, there exists an additive generator t such that*

$$T(x,y) = t^{(-1)}\big(t(x) + t(y)\big),$$
$$\Big(S(x,y) = 1 - t^{(-1)}\big(t(1-x) + t(1-y)\big)\Big) \quad \textit{holds for all} \quad (x,y) \in [0,1]^2.$$

Remark 2. If the additive generator t is unbounded, its pseudo-inverse $t^{(-1)}$ is equal to the standard inverse t^{-1}. Triangular norms and triangular conorms generated by the unbounded additive generators are called strict, and those generated by the bounded additive generators are called nilpotent.

Remark 3. Note that the multiplicative generators can be considered instead of the additive generators.

Definition 2.11 ([10])**.** Let $[a, b]$ and $[c, d]$ be two intervals. Then, $[a, b] \leq [c, d]$ if and only if $a \leq c$ and $b \leq d$.

Definition 2.12 ([12])**.** Let $\mathbb{I} = \{[a, b] \colon a, b \in [0, 1] \text{ and } a \leq b\}$. The definition of triangular norms and conorms can be modified to the set $\mathbb{I}$ by the following axioms: A triangular norm on $\mathbb{I}$ is a mapping $\mathbb{T} \colon \mathbb{I}^2 \to \mathbb{I}$ such that it is commutative, associative, the interval $[1, 1]$ acts as its neutral element and it is non-decreasing with respect to the partial order given by Definition 2.11. A triangular conorm on $\mathbb{I}$ is a mapping $\mathbb{S} \colon \mathbb{I}^2 \to \mathbb{I}$ such that it is commutative, associative, the interval $[0, 0]$ acts as its neutral element and it is non-decreasing with respect to the same partial order.

Definition 2.13 ([12])**.** A triangular norm $\mathbb{T}$ on $\mathbb{I}$ is called *representable*, if there exist t-norms $T_1, T_2 \colon [0, 1]^2 \to [0, 1]$ such that

$$T_1 \leq T_2 \quad \text{and} \quad \mathbb{T}([a, b], [c, d]) = [T_1(a, c), T_2(b, d)] \quad \text{for all} \quad [a, b], [c, d] \in \mathbb{I},$$

i.e., the left-endpoint of the output depends only on the left-endpoints of the inputs and the right-endpoint of the output depends only on the right-endpoints of the inputs. In this paper, we will consider only the case when the t-norms T_1 and T_2 coincide, i.e., $T_1 = T_2$. By analogous modifications, we will consider a representable interval t-conorm given by

$$\mathbb{S}([a, b], [c, d]) = [S(a, c), S(b, d)], \quad \text{where } S \text{ is a t-conorm.}$$

Definition 2.14 ([26])**.** Let $\mathbb{X}$ be a non-empty set filling the role of a universe of discourse. A fuzzy set A defined on the universe of discourse $\mathbb{X}$ is the set of the form $A = \{\langle x, \mu_A(x)\rangle \colon x \in \mathbb{X}\}$, where the function $\mu \colon \mathbb{X} \to [0, 1]$ is determining the degree of membership of the element x to the fuzzy set A.

3. φ-intuitionistic values and their arithmetic

Let A be a fuzzy set on the universe of discourse $\mathbb{X}$ and let μ represent the membership degree of the elements of $\mathbb{X}$ in the fuzzy set A. In the classical fuzzy set theory, a non-membership degree ν is given by $\nu(x) = n(\mu(x))$ for $x \in \mathbb{X}$, where n is a strong negation. If the standard negation proposed by Zadeh is considered, one can write this relationship as $\nu = 1 - \mu$, which can be rewritten as $\mu + \nu = 1$.

This concept can be additionally extended by allowing some degree of uncertainty, which can be introduced by changing the relationship between the degree of membership and non-membership to

$$\nu \leq 1 - \mu, \tag{1}$$

i.e., the degree of non-membership does not have to be equal to the complement of the degree of membership, but it must not exceed this value. Again, this relation can be rewritten as $\mu + \nu \leq 1$. This approach leads to the definition of the intuitionistic fuzzy sets proposed by Atanassov [3], defined by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathbb{X}\},$$

where $\mu_A(x) + \nu_A(x) \leq 1$ holds for all $x \in \mathbb{X}$.

These sets can be further generalized by considering different strong negations instead of the classical Zadeh's negation. In this paper, we will consider strong negations because they provide the possibility to use their generators in the form of automorphisms φ. This gives us a generalized version of the relationship (1) given by

$$\nu \leq \varphi^{-1}\big(1 - \varphi(\mu)\big).$$

Since φ is a strictly increasing continuous function, we can rewrite this equation in the form

$$\varphi(\mu) + \varphi(\nu) \leq 1.$$

This leads to the following definition.

Definition 3.1. Let $\varphi\colon [0,1] \to [0,1]$ be an automorphism. Then the set

$$\mathbb{IV}_\varphi = \big\{(u, v) : u, v \in [0,1] \text{ and } \varphi(u) + \varphi(v) \leq 1\big\}$$

is called the class of φ-intuitionistic values.

Remark 4. In the literature, intuitionistic values can also be found under the term "the intuitionistic membership grades,"for example, by Yager in [23] or in [24].

Remark 5. A similar approach to the construction of various types of intuitionistic fuzzy sets was introduced by Alcantud in [2] under the name "complemental fuzzy sets". In this framework, the degrees of membership and non-membership are obtained from a set of "c-pairs" S_c, which is a set of all elements pairwise bounded by a fixed strong negation. Although this framework shares the same foundations as the φ-intuitionistic values, our approach expands on this idea by using automorphisms. This enables users to choose different generators of the same strong negation, making the framework more comprehensive.

Remark 6. Since $\varphi\colon [0,1] \to [0,1]$ is bijective and the condition $\varphi(u)+\varphi(v) \leq 1$, or $0 \leq \varphi(u) \leq 1-\varphi(v) \leq 1$, is satisfied by every φ-intuitionistic value, the set of all φ-intuitionistic values is in one-to-one correspondence with the set of all closed sub-intervals of $[0,1]$ by $(u,v) \mapsto [\varphi(u), 1-\varphi(v)]$. We will refer to these intervals as *corresponding intervals*. This link shows the isomorphism of the bounded poset $(\mathbb{I}, \leq, [0,0], [1,1])$ and the poset $(\mathbb{IV}_\varphi, \preceq, (0,1), (1,0))$, where the partial order $\preceq$ on $\mathbb{IV}_\varphi$ is characterized by $(u_a, v_a) \preceq (u_b, v_b)$ if and only if

$$\varphi(u_a) \leq \varphi(u_b) \quad \text{and} \quad \varphi(1-v_a) \leq \varphi(1-v_b) \ \ (\text{i.e., } u_a \leq u_b \quad \text{and} \quad v_a \geq v_b).$$

Therefore, with no danger of confusion, we will denote the discussed partial order on $\preceq$ as $\leq$. A value with the maximal possible degree of membership $(1,0)$ corresponds to the singleton set $[1,1]$, a value with the maximal possible degree of non-membership $(0,1)$ corresponds to the singleton set $[0,0]$ and a value without any knowledge of membership or non-membership $(0,0)$, i.e., with maximal uncertainty corresponds to the interval $[0,1]$.

Using the automorphism φ in the previous definition, we can obtain different kinds of intuitionistic values used in theory and practice. For example, if $\varphi(t) = t$, then we obtain the classical intuitionistic values introduced by Atanassov [3]; if $\varphi(t) = t^2$, then we obtain the Pythagorean intuitionistic values introduced by Yager [23]; if $\varphi(t) = t^3$, then Fermatean intuitionistic values are obtained [18]. Note that more general bijections can be used, for example, $\varphi(t) = t^q$, when the q-rung intuitionistic values are obtained. This implies that the mentioned attempts to generalize intuitionistic values by increasing the area of their domain were in fact specific instances of a more general framework, not generalizations of the original Atanassov intuitionistic values. This makes them misleading because instead of giving the user more freedom in expressing their belief about membership grade as stated, e.g., in [22], they provide a different view on the data by tying the degrees of membership and non-membership with different instances of the Yager negation class.

Remark 7. The 'top' boundary of the set $\mathbb{IV}_\varphi$ for any automorphism $\varphi\colon [0,1] \to [0,1]$ consists of points $(u,v) \in [0,1]^2$ such that $\varphi(u) + \varphi(v) = 1$, i.e., $v = \varphi^{-1}(1-\varphi(u))$. This means that the top boundary is given by a mapping $u \mapsto n(u)$, i.e., by the strong negation generated by the automorphism φ.

Following the previous remark, we can see that the set $\mathbb{IV}_\varphi$ is invariant with respect to the choice of the automorphisms generating the same strong negation.

Remark 8. Let φ_1 and φ_2 be two automorphisms generating the same strong negation. Then $\mathbb{IV}_{\varphi_1} = \mathbb{IV}_{\varphi_2}$.

Moreover, based on the shape of the automorphism φ, we can obtain some information about the shape of the boundary of the set $\mathbb{IV}_\varphi$, i.e., the negation generated by the automorphism φ.

Theorem 3.2. *Let $\varphi\colon [0,1] \to [0,1]$ be an automorphism. If φ is convex (concave), then the negation n generated by φ is concave (convex).*

P r o o f. For the sake of simplicity, we will prove only the part when φ is convex. The second part, i.e., taking φ to be a concave function, can be proven analogously.

It is easy to notice that since φ is a strictly increasing bijection then, so is its inverse φ^{-1}. Since φ is convex, then φ^{-1} is concave: To prove this, consider any $u, v, \lambda \in [0,1]$ and set $a = \varphi^{-1}(\lambda u + (1-\lambda)v)$ and $b = \lambda\varphi^{-1}(u) + (1-\lambda)\varphi^{-1}(v)$. Then

$$\begin{aligned}\varphi(a) &= \varphi\Big(\varphi^{-1}\big(\lambda u + (1-\lambda)v\big) = \lambda u + (1-\lambda)v\Big)\\ &= \lambda\varphi\big(\varphi^{-1}(u)\big) + (1-\lambda)\varphi\big(\varphi^{-1}(v)\big)\\ &\geq \varphi\big(\lambda\varphi^{-1}(u) + (1-\lambda)\varphi^{-1}(v)\big) = \varphi(b)\end{aligned}$$

following the fact that φ is a convex function. Since φ is strictly increasing, the fact that $\varphi(a) \geq \varphi(b)$ implies that $a \geq b$ and thus

$$\varphi^{-1}\big(\lambda u + (1-\lambda)v\big) \geq \lambda\varphi^{-1}(u) + (1-\lambda)\varphi^{-1}(v)$$

for any $u, v, \lambda \in [0,1]$, i.e., φ^{-1} is a concave function.

Again, consider $u, v, \lambda \in [0,1]$ to be arbitrary. Since φ^{-1} is concave, we have the following inequality:

$$\varphi^{-1}\Big(\lambda\big(1-\varphi(u)\big) + (1-\lambda)\big(1-\varphi(v)\big)\Big) \geq \lambda n(u) + (1-\lambda)n(v).$$

Applying φ on both sides of the inequality and using the fact that φ is strictly increasing, we obtain

$$\lambda\big(1-\varphi(u)\big) + (1-\lambda)\big(1-\varphi(v)\big) \geq \varphi\big(\lambda n(u) + (1-\lambda)n(v)\big). \tag{2}$$

Now, set $a = n\big(\lambda u + (1-\lambda)v\big)$ and $b = \lambda n(u) + (1-\lambda)n(v)$. Then

$$\begin{aligned}\varphi(a) = \varphi\Big(n\big(\lambda u + (1-\lambda)v\big)\Big) &= \varphi\bigg(\varphi^{-1}\Big(1-\varphi\big(\lambda u + (1-\lambda)v\big)\Big)\bigg)\\ &= 1-\varphi\big(\lambda u + (1-\lambda)v\big).\end{aligned}$$

Since φ is convex, we obtain $\varphi\big(\lambda u + (1-\lambda)v\big) \leq \lambda\varphi(u) + (1-\lambda)\varphi(v)$ and combining this with the previous, we get the following result

$$\varphi(a) \geq 1 - \lambda\varphi(u) - (1-\lambda)\varphi(v) = \lambda\big(1-\varphi(u)\big) + (1-\lambda)\big(1-\varphi(v)\big).$$

When comparing this with the inequality (2), we obtain

$$\varphi(a) \geq \varphi\Big(\lambda n(u) + \big(1-\varphi(v)\big)\Big) = \varphi(b).$$

Again, following the increasingness of φ we have $a \geq b$, i.e.,

$$n\big(\lambda u + (1-\lambda)v\big) \geq \lambda n(u) + (1-\lambda)n(v)$$

for arbitrary $u, v, \lambda \in [0,1]$. This implies that n is a concave function. $\square$

Remark 9. Note that choosing a convex negation given by a concave automorphism results in a restriction of the domain of classical intuitionistic values. This contradicts the idea of improving classical intuitionistic sets by simply expanding their domain, like in the case of the Pythagorean fuzzy sets. For this reason, when considering which type of φ-intuitionistic values are the best for a specific problem, the focus should be on the relationship between degrees of membership and non-membership, not on the domain itself, as the area of their domain is just a consequence of their relationship. This task remains an open problem. Even within the framework of q-rung values, which is parametric and could in principle allow for an optimization-based approach, the parameter q is most often arbitrarily selected, as can be observed, e.g., in [7].

Example. The converse to the implication in the previous theorem is not valid the other way around, i.e., if the strong negation is convex (concave), it does not necessarily mean that the automorphism that generates it is concave (convex). As a counter-example, let us consider the standard Zadeh's negation n_{Z} that is both convex and concave and the automorphism $\varphi_{\mathrm{Z}2}$ from Example 2 that is neither convex nor concave.

Example. Let us consider an automorphism φ as a piecewise linear function that changes its slope at the point $[a, 1-a]$, where $a \in]0,1[$, defined by the formula

$$\varphi_a(x) = \begin{cases} \frac{1-a}{a}x & \text{if } x \leq a, \\ \frac{a}{1-a}x + \frac{1-2a}{1-a} & \text{if } x > a. \end{cases}$$

If $a = 1/2$: the class of φ-intuitionistic values defined by this automorphism coincides with the standard intuitionistic values.

If $a > 1/2$: the automorphism φ extends the domain of the standard intuitionistic values, and, interestingly.

If $a < 1/2$: the domain of the standard intuitionistic values is reduced, as can be observed in Figure 1.

Example. Consider an automorphism φ on $[0,1]$ given by $\varphi(x) = \log_2(x+1)$. The corresponding negation n is given by

$$n(x) = \frac{1-x}{1+x}$$

and the set $\mathbb{IV}_\varphi$ consists of points $(u,v) \in [0,1]^2$ such that $(u+1)(v+1) \leq 2$. The visualization of this example can be found in Figure 2.

Definition 3.3. Let $\mathbb{X}$ be the universe of discourse and let $\varphi\colon [0,1] \to [0,1]$ be an automorphism. Then a φ-intuitionistic fuzzy set A defined on the universe of discourse $\mathbb{X}$ is a set of the form

$$A = \{\langle x, \mu(x), \nu(x)\rangle : \big(\mu(x), \nu(x)\big) \in \mathbb{IV}_\varphi \text{ for all } x \in \mathbb{X}\}. \tag{3}$$

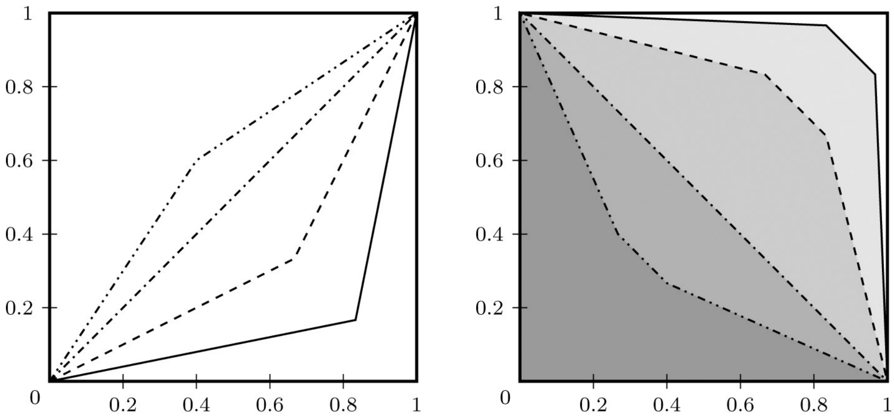

FIGURE 1. A visualization of the automorphisms φ_a from Example 3 for $a = 5/6$ (solid line), $a = 2/3$ (dashed line), $a = 1/2$ (dash-dotted line), and $a = 2/5$ (dash-dot-dotted line) on the left. The corresponding sets of φ-intuitionistic values $\mathbb{IV}_{\varphi_a}$ for these values of a are illustrated on the right.

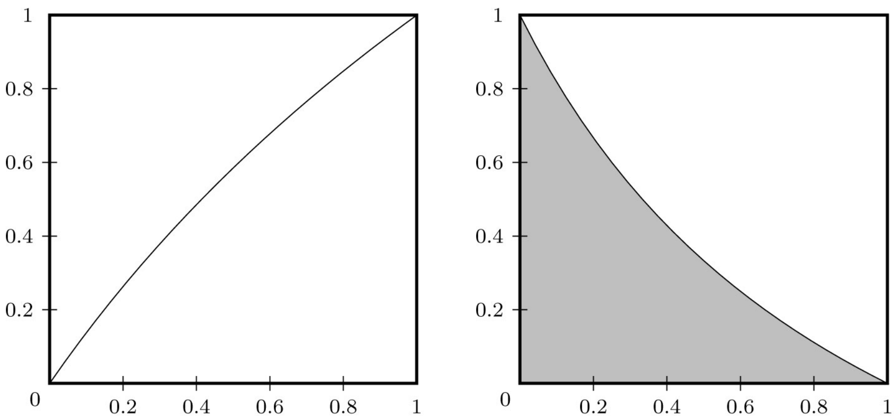

FIGURE 2. A visualization of the automorphism φ from Example 3 on the left and the visualization of the corresponding set of φ-intuitionistic values on the right.

Remark 10. Formula (3) implies that to every element of the universe of discourse $\mathbb{X}$ we assign a φ-intuitionistic value, i.e., the set A from the previous definition is in a one-to-one correspondence with the values of two functions $\mu, \nu \colon \mathbb{X} \to [0,1]$ such that $\varphi\big(\mu(x)\big) + \varphi\big(\nu(x)\big) \leq 1$ for all $x \in \mathbb{X}$. In the setting of the classical intuitionistic fuzzy sets and the Pythagorean fuzzy sets, the function μ is called *the membership function* and the function ν is called *the non-membership function.* We will adopt the same terminology for the φ-intuitionistic fuzzy sets.

In the following, we will omit the full notation of the membership and non-membership functions for simplicity, so whenever we consider a φ-intuitionistic fuzzy set A, we will denote μ, ν as $\mu_A,\ nu_A$ and A as (μ_A, ν_A).

Definition 3.4. Let φ be an automorphism. The degree of uncertainty $\mathrm{Hes}_\varphi(u, v)$ of a φ-intuitionistic value (u, v) is defined by

$$\mathrm{Hes}_\varphi(u, v) = \varphi^{-1}\big(1 - \varphi(u) - \varphi(v)\big).$$

Remark 11. The degree of uncertainty is captured using its corresponding interval by

$$\mathrm{Hes}_\varphi(u, v) = \varphi^{-1}\Big(w\big([\varphi(u), 1 - \varphi(v)]\big)\Big),$$

where w denotes the width of the interval given by $w([a, b]) = b - a$.

Example. The degree of uncertainty is not invariant with respect to the choice of automorphism φ. Consider the classical intuitionistic values built upon the standard Zadeh's negation, which are obtained both from the automorphisms φ_{Z1} and φ_{Z2}, see Example 2. Consider an intuitionistic value $(1/5, 3/5)$. Its degree of uncertainty with respect to automorphism φ_{Z1} is

$$\mathrm{Hes}_{\varphi_{\mathrm{Z1}}}(1/5, 3/5) = \frac{1}{5},$$

but its degree of uncertainty with respect to the automorphism φ_{Z2} is

$$\begin{aligned}
\mathrm{Hes}_{\varphi_{\mathrm{Z2}}}(1/5, 3/5) &= \varphi_{\mathrm{Z2}}^{-1}\left(1 - \frac{2 - \cos(\pi/5) - \cos(3\pi/5)}{2}\right) \\
&= \varphi_{\mathrm{Z2}}^{-1}\left(1 - \frac{-2 + \frac{1}{4}(1 - \sqrt{5} + \frac{1}{4}(1 + \sqrt{5}))}{2}\right) \\
&= \varphi_{\mathrm{Z2}}^{-1}\left(\frac{1}{4}\right) = \frac{1}{3}.
\end{aligned}$$

Remark 12. The previous example shows an advantage of our proposed method: Even though we work with the same set of intuitionistic values, the behavior of these values depends on the choice of the automorphism generating them. This provides an additional degree of freedom in applications.

Remark 13. Note that the equation

$$\varphi(u) + \varphi(v) + \varphi\big(\mathrm{Hes}_\varphi(u, v)\big) = 1$$

is satisfied for every φ-intuitionistic value (u, v). Moreover, if $(u, v) \in \mathbb{IV}_\varphi$, then $\mathrm{Hes}_\varphi(u, v) = \mathrm{Hes}_\varphi(v, u)$ whatever the choice of φ.

Definition 3.5. Let A, B be two φ-intuitionistic fuzzy sets defined on the same universe of discourse $\mathbb{X}$. Then $A \subseteq B$ holds if and only if

$$\mu_A \leq \mu_B \quad \text{and} \quad \nu_A \geq \nu_B \quad \text{for all} \quad x \in \mathbb{X}.$$

It is convenient to define operations on the φ-intuitionistic fuzzy sets using their corresponding intervals (see Remark 10). Let A, B be two φ-intuitionistic fuzzy sets and let

$$\star\colon \mathbb{I}^k \to \mathbb{I}, k \in \mathbb{N}$$

be an interval operation. Then the operation $\star$ for the φ-intuitionistic fuzzy sets can be introduced as follows:

(i) Choose an automorphism φ;

(ii) Find the corresponding intervals of A and B which are given by

$$[\varphi(\mu_A), 1 - \varphi(\nu_A)] \quad \text{and} \quad [\varphi(\mu_B), 1 - \varphi(\nu_B)];$$

(iii) Apply the operation $\star$ on these intervals resulting in the interval $[c_1, c_2]$;

(iv) Transform the resulted interval back to a φ-intuitionistic fuzzy set by using the formula

$$\big(\varphi^{-1}(c_1), \varphi^{-1}(1 - c_2)\big).$$

In this paper, we will consider operations obtained by this principle based on strict representable t-norms and t-conorms, but also the standard set operations proposed by Atanassov in [3]. This leads us to the following definition:

Definition 3.6. Let $A_1, \dots, A_k$ be a set of φ-intuitionistic fuzzy sets, and let $\star\colon \mathbb{I}^k \to \mathbb{I}$, where $k \in \mathbb{N}$ is a natural number, be an interval operation. Then the operation $\star$ for the φ-intuitionistic fuzzy sets A_i for $i = 1, 2, \dots, k$ is given by formula

$$\mathop{\bigstar}_{i=1}^{k} A_i = h\left(\mathop{\bigstar}_{i=1}^{k} [\varphi(\mu_{A_i}), 1 - \varphi(\nu_{A_i})]\right),$$

where $h\colon \mathbb{I} \to \mathbb{IV}_\varphi$ denotes the function

$$h([c_1, c_2]) = \big(\varphi^{-1}(c_1), \varphi^{-1}(1 - c_2)\big).$$

Theorem 3.7. *Let A, B be two φ-intuitionistic fuzzy sets, λ be a positive real number, t be an additive generator of a strict t-norm T, and S be its dual t-conorm. Following Definition 3.6, we obtain the following φ-intuitionistic fuzzy set operations:*

$$A^c = (\nu_A, \mu_A),$$

$$A \cap B = (\min\{\mu_A, \mu_B\}, \max\{\nu_A, \nu_B\}),$$

$$A \cup B = (\max\{\mu_A, \mu_B\}, \min\{\nu_A, \nu_B\}),$$

$$A \oplus B = \left(\varphi^{-1}\Big(S\big(\varphi(\mu_A), \varphi(\mu_B)\big)\Big),\ \varphi^{-1}\Big(T\big(\varphi(\nu_A), \varphi(\nu_B)\big)\Big)\right), \tag{4}$$

$$\begin{aligned}
A \odot B &= \Big(\varphi^{-1}\Big(T\big(\varphi(\mu_A), \varphi(\mu_B)\big)\Big),\ \varphi^{-1}\Big(S\big(\varphi(\nu_A), \varphi(\nu_B)\big)\Big)\Big),\\
\lambda \cdot A &= \left(\varphi^{-1}\left(1 - t^{-1}\Big(\lambda t\big(1-\varphi(\mu_A)\big)\Big)\right),\ \varphi^{-1}\left(t^{-1}Bl\big(\lambda t\big(\varphi(\nu_A)\big)\big)\right)\right), \qquad (5)\\
A^{\lambda} &= \left(\varphi^{-1}\left(t^{-1}\Big(\lambda t\big(\varphi(\mu_A)\big)\Big)\right),\ \varphi^{-1}\left(1 - t^{-1}\Big(\lambda t\big(1-\varphi(\nu_A)\big)\Big)\right)\right),
\end{aligned}$$

where a t-conorm $\mathbb{S}$ and t-norm $\mathbb{T}$, were used to define pseudo-addition $\oplus$ and pseudo-multiplication $\odot$, respectively.

P r o o f. Let us start by proving the equation (4):

$$\begin{aligned}
A \oplus B &\equiv \mathbb{S}\big([\varphi(\mu_A), 1-\varphi(\nu_A)],\ [\varphi(\mu_B), 1-\varphi(\nu_B)]\big)\\
&= \big[S\big(\varphi(\mu_A), \varphi(\mu_B)\big),\ S\big(1-\varphi(\nu_A), 1-\varphi(\nu_B)\big)\big]\\
&\equiv \Big(\varphi^{-1}\Big(S\big(\varphi(\mu_A), \varphi(\mu_B)\big)\Big),\ \varphi^{-1}\Big(1 - S\big(1-\varphi(\nu_A), 1-\varphi(\nu_B)\big)\Big)\Big)\\
&= \Big(\varphi^{-1}\Big(S\big(\varphi(\mu_A), \varphi(\mu_B)\big)\Big),\ \varphi^{-1}\Big(T\big(\varphi(\nu_A), \varphi(\nu_B)\big)\Big)\Big).
\end{aligned}$$

The proof of the equality 5 is as follows: Firstly, note that

$$\begin{aligned}
&2 \cdot [\mu_A, \nu_A]\\
&= \mathbb{S}([\mu_A, \nu_A],\ [\mu_A, \nu_A])\\
&= \big[1 - t^{-1}\big(t(1-\mu_A) + t(1-\mu_A)\big),\ 1 - t^{-1}\big(t(1-\nu_A) + t(1-\nu_A)\big)\big]\\
&= \big[1 - t^{-1}\big(2t(1-\mu_A)\big),\ 1 - t^{-1}\big(2t(1-\nu_A)\big)\big].
\end{aligned}$$

By expanding this formula to an arbitrary number of inputs, we obtain

$$\begin{aligned}
k \cdot [\mu_A, \nu_A] &= \sum_{i=1}^{k} [\mu_A, \nu_A] \qquad (6)\\
&= \big[1 - t^{-1}\big(kt(1-\mu_A)\big),\ 1 - t^{-1}\big(kt(1-\nu_A)\big)\big]. \qquad (7)
\end{aligned}$$

where k is a natural number. Next, consider the following formula

$$\frac{1}{k} \cdot [\mu_A, \nu_A] = \left[1 - t^{-1}\left(\frac{1}{k}t(1-\mu_A)\right),\ 1 - t^{-1}\left(\frac{1}{k}t(1-\nu_A)\right)\right]$$

for exponentiation with a reciprocal of k. This operation is well-defined, since

$$\begin{aligned}
&\frac{1}{k}\cdot(k\cdot[\mu_A,\nu_A]) = \\
&\frac{1}{k}\cdot\big[1-t^{-1}\big(kt(1-\mu_A)\big),1-t^{-1}\big(kt(1-\nu_A)\big)\big] = \\
&\bigg[1-t^{-1}\bigg(kt\bigg(1-\Big(1-t^{-1}\big(\tfrac{1}{k}t(1-\mu_A)\big)\Big)\bigg)\bigg), \\
&\qquad\qquad\qquad 1-t^{-1}\bigg(kt\bigg(1-\Big(1-t^{-1}\big(\tfrac{1}{k}t(1-\nu_A)\big)\Big)\bigg)\bigg)\bigg] = \\
&\bigg[1-t^{-1}\Big(kt\Big(t^{-1}\big(\tfrac{1}{k}t(1-\mu_A)\big)\Big)\Big),\ 1-t^{-1}\Big(kt\Big(t^{-1}\big(\tfrac{1}{k}t(1-\nu_A)\big)\Big)\Big)\bigg] = \\
&\bigg[1-t^{-1}\Big(kt\Big(t^{-1}\big(\tfrac{1}{k}t(1-\mu_A)\big)\Big)\Big),\ 1-t^{-1}\Big(kt\Big(t^{-1}\big(\tfrac{1}{k}t(1-\nu_A)\big)\Big)\Big)\bigg] = \\
&\big[1-t^{-1}\big(k\tfrac{1}{k}t(1-\mu_A)\big),\ 1-t^{-1}\big(k\tfrac{1}{k}t(1-\nu_A)\big)\big] = \\
&\big[1-t^{-1}\big(t(1-\mu_A)\big),\ 1-t^{-1}\big(t(1-\nu_A)\big)\big] = \\
&[1-(1-\mu_A),\ 1-(1-\nu_A)] = [\mu_A,\nu_A].
\end{aligned}$$

Then, formula (6) can be extended to the case of rational numbers.

Finally, let $k\in\mathbb{R}_{\geq 0}$ and let $(q_n)_{n\in\mathbb{N}}$ be a sequence of rational numbers such that $q_n\to k$. We define

$$k\cdot[\mu_A,\nu_A] := \Big[\lim_{n\to\infty}\Big(1-t^{-1}\big(q_n\,t(1-\mu_A)\big)\Big),\lim_{n\to\infty}\Big(1-t^{-1}\big(q_n\,t(1-\nu_A)\big)\Big)\Big].$$

Since t^{-1} is continuous and $q_n\,t(1-x)\to k\,t(1-x)$ for any $x\in[0,1]$, the limit exists and does not depend on the particular choice of the sequence (q_n). Hence,

$$k\cdot[\mu_A,\nu_A] = \big[\,1-t^{-1}\big(k\,t(1-\mu_A)\big),\ 1-t^{-1}\big(k\,t(1-\nu_A)\big)\big].$$

This means that for φ-intuitionistic fuzzy sets, scalar multiplication is obtained by

$$\begin{aligned}
k\cdot(\mu_A,\nu_A) &\equiv k\cdot[\varphi(\mu_A),1-\varphi(\nu_A)] \\
&= \Big[1-t^{-1}\Big(kt\big(1-\varphi(\mu_A)\big)\Big),\ 1-t^{-1}\Big(kt\big(\varphi(\nu_A)\big)\Big)\Big] \\
&\equiv \bigg(\varphi^{-1}\bigg(1-t^{-1}\Big(kt\big(1-\varphi(\mu_A)\big)\Big)\bigg),\ \varphi^{-1}\bigg(t^{-1}\Big(kt\big(\varphi(\nu_A)\big)\Big)\bigg)\bigg). \quad \square
\end{aligned}$$

Proving the formulas for pseudo-multiplication and exponentiation is unnecessary, as they are analogous to the given ones.

Since these operations depend on the choice of the the automorphism φ and the additive generator of strict t-norm t, we should have denoted operations $\oplus, \odot$ etc. as $\oplus_{\varphi,t}, \odot_{\varphi,t}$, but, since φ and t are always considered as fixed, we adapt the original notation introduced by Atanassov in [3] and rely on the clarity of these notations from the context.

Remark 14. The operations that were originally introduced by Atanassov in [3] dealing with the classical intuitionistic fuzzy sets are based on the additive generator t given by $t(x) = -\ln(x)$ for all $x \in [0,1]$ leading to the t-norm $T_P(x,y) = xy$, i.e., the product t-norm, and the t-conorm $S_P(x,y) = x+y-xy$, i.e., the probabilistic sum. The use of this t-norm and this t-conorm is usually the first step in defining the operations for new types of intuitionistic fuzzy sets, see, e.g., [18, 22, 24].

Remark 15. It is only a technical matter to verify that the operations $\cap, \cup, \oplus, \odot$, and the complement are well-defined, i.e., the result of these operations is again a φ-intuitionistic fuzzy set. Consider two φ-intuitionistic fuzzy sets A, B. Then the facts that $\varphi(\mu_A) \leq 1 - \varphi(\nu_A)$ and $\varphi(\mu_B) \leq 1 - \varphi(\nu_B)$ imply:

- $\max\{\varphi(\nu_A), \varphi(\nu_B)\} \leq 1 - \min\{\varphi(\mu_A), \varphi(\mu_B)\}$, which can be rewritten into the form $\min\{\mu_A, \mu_B\} + \max\{\nu_A, \nu_B\} \leq 1$ implying that $A \cap B$ is a φ-intuitionistic fuzzy set;
- $\min\{\varphi(\nu_A), \varphi(\nu_B)\} \leq 1 - \max\{\varphi(\mu_A), \varphi(\mu_B)\}$, which can be expressed in the form $\max\{\mu_A, \mu_B\} + \min\{\nu_A, \nu_B\} \leq 1$ implying that $A \cup B$ is a φ-intuitionistic fuzzy set;
- $S(\varphi(\mu_A), \varphi(\mu_B)) \leq S(1 - \varphi(\nu_A), 1 - \varphi(\nu_B))$, which can be transformed into $S\big(\varphi(\mu_A), \varphi(\mu_B)\big) + T\big(\varphi(\nu_A), \varphi(\nu_B)\big) \leq 1$ implying that $A \oplus B$ is a φ-intuitionistic fuzzy set;
- $T\big(\varphi(\mu_A), \varphi(\mu_B)\big) \leq T\big(1 - \varphi(\nu_A), 1 - \varphi(\nu_B)\big)$, which can be articulated as $T\big(\varphi(\mu_A), \varphi(\mu_B)\big) + S\big(\varphi(\nu_A), \varphi(\nu_B)\big) \leq 1$ implying that $A \odot B$ is a φ-intuitionistic fuzzy set.

Additionally, it is easy to notice that the complement of a φ-intuitionistic fuzzy set is also a φ-intuitionistic fuzzy set, as swapping the values of degree of membership and non-membership does not interfere with the required inequality.

Example. Consider the generators φ_{Z1} and φ_{Z2} of the standard Zadeh's negation introduced in Example 2 and the additive generator of a t-(co)norm given by $t(x) = -\ln(x)$ for all $x \in [0,1]$, see Remark 14. Consider the φ-intuitionistic fuzzy sets $A, B, C_1, \ldots, C_n$ and real numbers $w_1, \ldots, w_n \geq 0$ summing to 1. Then some of the operations introduced in Theorem 3.7 together with the weighted average based on the automorphisms φ_{Z1} and φ_{Z2} are summarized in Table 1.

Table 1. The result of some of the operations together with the weighted average based on two different automorphisms $\varphi_{\mathrm{Z}1}$ and $\varphi_{\mathrm{Z}2}$ leading to the same set of φ-intuitionistic values/fuzzy sets, see Example 3.

Operations based on the automorphism $\varphi_{\mathrm{Z}1}$.

$A \oplus B$	$\big(\mu_A + \mu_B - \mu_A\mu_B, \nu_A\nu_B\big)$,
$A \odot B$	$\big(\mu_A\mu_B, \nu_A + \nu_B - \nu_A\nu_B\big)$,
$\lambda \cdot A$	$\big(1 - (1-\mu_A)^k, \nu_A^k\big)$,
A^λ	$\big(\mu_A^k, 1 - (1-\nu_A)^k\big)$,
$\sum_{i=1}^{n} w_i \cdot C_i$	$\left(1 - \prod_{i=1}^{n}(1-\mu_{C_i})^{w_i}, \prod_{i=1}^{n}(\nu_{C_i})^{w_i}\right)$.

Operations based on the automorphism $\varphi_{\mathrm{Z}2}$

$A \oplus B$	$\mu = \frac{1}{\pi}\arccos\Big(\frac{\cos(\pi\mu_A)+\cos(\mu_B)\cos(\pi\mu_A)+\cos(\mu_B)-1}{2}\Big)$,
	$\nu = \frac{1}{\pi}\arccos\Big(1 + \frac{(1-\cos(\pi\nu_A))(1-\cos(\pi\nu_B))}{2}\Big)$.
$A \odot B$	$\mu = \frac{1}{\pi}\arccos\Big(1 + \frac{(1-\cos(\pi\mu_A))(1-\cos(\pi\mu_B))}{2}\Big)$,
	$\nu = \frac{1}{\pi}\arccos\Big(\frac{\cos(\pi\nu_A)+\cos(\nu_B)\cos(\pi\nu_A)+\cos(\nu_B)-1}{2}\Big)$.
$\lambda \cdot A$	$\mu = \frac{1}{\pi}\arccos\Big((\frac{1}{2})^{\lambda-1}\big(1+\cos(\pi\mu_A)\big)^{\lambda} - 1\Big)$,
	$\nu = \frac{1}{\pi}\arccos\Big(1 - (\frac{1}{2})^{\lambda-1}\big(1-\cos(\pi\nu_A)\big)^{\lambda}\Big)$.
A^λ	$\mu = \frac{1}{\pi}\arccos\Big(1 - (\frac{1}{2})^{\lambda-1}\big(1-\cos(\pi\mu_A)\big)^{\lambda}\Big)$,
	$\nu = \frac{1}{\pi}\arccos\Big((\frac{1}{2})^{\lambda-1}\big(1+\cos(\pi\nu_A)\big)^{\lambda} - 1\Big)$.
$\sum_{i=1}^{n} w_i \cdot C_i$	$\mu = \dfrac{1}{\pi}\arccos\left(2\prod_{i=1}^{n}\Big(\frac{1}{2}\big(1+\cos(\pi\mu_{C_i})\big)\Big)^{w_i} - 1\right)$,
	$\nu = \dfrac{1}{\pi}\arccos\left(1 - 2\prod_{i=1}^{n}\Big(\frac{1}{2}\big(1+\cos(\pi\mu_{C_i})\big)\Big)^{w_i}\right)$.

Remark 16. Note that the choice of the type of the φ-intuitionistic fuzzy set, i.e., the choice of the automorphism φ, does not influence the complement and the standard intersection and union operations, although they are commonly studied for different types of intuitionistic fuzzy sets, see, e.g., [14, 23].

Remark 17. The operations introduced in Theorem 3.7 are only those that are commonly studied in the field of the intuitionistic fuzzy sets and values, but other operations can also be introduced thanks to the isomorphism with the corresponding intervals. As an example, consider a ternary median operator $\mathrm{Med}\colon \mathbb{I}^3 \to \mathbb{I}$ given by

$$\mathrm{Med}([a_1, a_2], [b_1, b_2], [c_1, c_3]) = [\mathrm{med}(a_1, b_1, c_1), \mathrm{med}(a_2, b_2, c_2)],$$

where $\mathrm{med}(a, b, c)$ denotes the standard median of a, b, c. This operation can be introduced for the φ-intuitionistic values, considering $a, b, c \in \mathbb{IV}_\varphi$, by the formula

$$\mu_{\mathrm{Med}(a,b,c)} = \varphi^{-1}\Big(\mathrm{med}\big(\varphi(\mu_a),\ \varphi(\mu_b), \varphi(\mu_c)\big)\Big),$$
$$\nu_{\mathrm{Med}(a,b,c)} = \varphi^{-1}\Big(1 - \mathrm{med}\big(\varphi(1 - \nu_a),\ \varphi(1 - \nu_b),\ \varphi(1 - \nu_c)\big)\Big).$$

Since strictly increasing φ does not interfere with the ordering of the inputs of med and φ is a generator of a strong negation, this formula can be simplified to

$$\mathrm{Med}(a, b, c) = \Big(\mathrm{med}(\mu_a, \mu_b, \mu_c),\ n\big(1 - \mathrm{med}(\nu_a, \nu_b, \nu_c)\big)\Big).$$

where n denotes the strong negation generated by φ.

In the following, we present and prove several properties of the operations introduced in Theorem 3.7 following Definition 3.6. A lot of papers include proofs of these properties for different types of intuitionistic fuzzy sets and for different operations based on a choice of a specific t-norm and a t-conorm, see, e.g., operations over intuitionistic fuzzy sets based on the Einstein t-norm and t-conorm [21], based on the Dombi t-norm and t-conorm [16], based on the Aczel-Alsina t-norm and t-conorm [17], or operations over the Pythagorean intuitionistic fuzzy sets based on the Einstein t-norm and t-conorm [5]. The general approach of the φ-intuitionistic fuzzy sets includes all of these mentioned types of intuitionistic fuzzy sets and their operations without the need to prove their properties individually. Furthermore, this approach is not limited to them.

The proofs of the theorems that follow are given only partially. Some of the proofs are omitted since they are analogous to the given ones. Some of the proofs of the properties of the fuzzy sets that are not dependent on the automorphism φ can be found in the literature, e.g., in [14] or [18], but the statements are included for the sake of completeness. Also, in these theorems, we will always deal with a fixed automorphism φ, so we will omit its indication in the symbols for the considered operations.

Theorem 3.8. *Let A, B be two φ-intuitionistic fuzzy sets and let $\lambda, \lambda_1, \lambda_2 > 0$. Then*

$$A \cap B = B \cap A, \tag{8}$$

$$A \cup B = B \cup A, \tag{9}$$

$$A \oplus B = B \oplus A, \tag{10}$$

$$A \odot B = B \odot A, \tag{11}$$

$$\lambda \cdot (A \oplus B) = (\lambda \cdot A) \oplus (\lambda \cdot B), \tag{12}$$

$$(A \odot B)^{\lambda} = A^{\lambda} \odot B^{\lambda}, \tag{13}$$

$$(\lambda_1 \cdot A) \oplus (\lambda_2 \cdot A) = (\lambda_1 + \lambda_2) \cdot A, \tag{14}$$

$$A^{\lambda_1} \odot A^{\lambda_2} = A^{(\lambda_1 + \lambda_2)}. \tag{15}$$

Proof. The proofs for the equalities (8) and (9) can be found in the mentioned literature (see, e.g., [14] or [18]). The proof of the equality (10) is as follows:

$$\begin{aligned}
A \oplus B &= \Big(\varphi^{-1}\Big(S\big(\varphi(\mu_A), \varphi(\mu_B)\big)\Big),\ \varphi^{-1}\Big(T\big(\varphi(\nu_A), \varphi(\nu_B)\big)\Big)\Big)\\
&= \Big(\varphi^{-1}\Big(S\big(\varphi(\mu_B), \varphi(\mu_A)\big)\Big),\ \varphi^{-1}\Big(T\big(\varphi(\nu_B), \varphi(\nu_A)\big)\Big)\Big)\\
&= B \oplus A
\end{aligned}$$

thanks to the commutativity of the t-norm T and a the t-conorm S. Equality (11) can be proven analogously.

The proof of the equality (12) is as follows:

$$\begin{aligned}
&\mu_{\lambda(A \oplus B)}\\
&= \varphi^{-1}\left(1 - t^{-1}\left(\lambda t\left(1 - \varphi\left(\varphi^{-1}\Big(S\big(\varphi(\mu_A),\ \varphi(\mu_B)\big)\Big)\right)\right)\right)\right)\\
&= \varphi^{-1}\left(1 - t^{-1}\Big(\lambda t\Big(1 - S\big(\varphi(\mu_A),\ \varphi(\mu_B)\big)\Big)\Big)\right)\\
&= \varphi^{-1}\left(1 - t^{-1}\left(\lambda t\left(1 - \left(1 - t^{(-1)}\Big(t\big(1 - \varphi(\mu_A)\big) + t\big(1 - \varphi(\mu_B)\big)\Big)\right)\right)\right)\right)\\
&= \varphi^{-1}\left(1 - t^{-1}\Big(\lambda\Big(t\big(1 - \varphi(\mu_A)\big) + t\big(1 - \varphi(\mu_B)\big)\Big)\Big)\right)\\
&= \varphi^{-1}\Big(1 - t^{(-1)}\Big(\lambda t\big(1 - \varphi(\mu_A)\big)\Big) + \lambda t\big(1 - \varphi(\mu_B)\big)\Big)
\end{aligned}$$

$$= \varphi^{-1}\left(1 - t^{(-1)}\left(t\Big(1 - \Big(1 - t^{-1}\big(\lambda t\big(1 - \varphi(\mu_A)\big)\big)\Big)\Big)\right.\right.$$

$$\left.\left. + t\Big(1 - \Big(1 - t^{-1}\big(\lambda t\big(1 - \varphi(\mu_B)\big)\big)\Big)\Big)\right)\right)$$

$$= \varphi^{-1}\left(S\Big(1 - t^{-1}\big(\lambda t\big(1 - \varphi(\mu_A)\big)\big),\ 1 - t^{-1}\big(\lambda t\big(1 - \varphi(\mu_B)\big)\big)\Big)\right)$$

$$= \varphi^{-1}\left(S\left(\varphi\Big(\varphi^{-1}\Big(1 - t^{-1}\big(\lambda t\big(1 - \varphi(\mu_A)\big)\big)\Big)\Big),\right.\right.$$

$$\left.\left.\varphi\Big(\varphi^{-1}\Big(1 - t^{-1}\big(\lambda t\big(1 - \varphi(\mu_B)\big)\big)\Big)\Big)\right)\right) = \mu_{\lambda\cdot A\oplus\lambda\cdot B}.$$

$$\nu_{\lambda(A\oplus B)}$$

$$= \varphi^{-1}\left(t^{-1}\left(\lambda t\Big(\varphi\Big(\varphi^{-1}\big(T\big(\varphi(\nu_A), \varphi(\nu_B)\big)\big)\Big)\Big)\right)\right)$$

$$= \varphi^{-1}\left(t^{-1}\Big(\lambda t\Big(T\big(\varphi(\nu_A),\ \varphi(\nu_B)\big)\Big)\Big)\right)$$

$$= \varphi^{-1}\left(t^{-1}\left(\lambda t\Big(t^{(-1)}\Big(t\big(\varphi(\nu_A)\big) + t\big(\varphi(\nu_B)\big)\Big)\Big)\right)\right)$$

$$= \varphi^{-1}\left(t^{-1}\Big(\lambda\Big(t\big(\varphi(\nu_A)\big) + t\big(\varphi(\nu_B)\big)\Big)\Big)\right)$$

$$= \varphi^{-1}\Big(t^{(-1)}\Big(\lambda t\big(\varphi(\nu_A)\big) + \lambda t\big(\varphi(\nu_B)\big)\Big)\Big)$$

$$= \varphi^{-1}\left(t^{(-1)}\left(t\Big(t^{-1}\big(\lambda t\big(\varphi(\nu_A)\big)\big)\Big) + t\Big(t^{-1}\big(\lambda t\big(\varphi(\nu_B)\big)\big)\Big)\right)\right)$$

$$= \varphi^{-1}\left(T\Big(t^{-1}\big(\lambda t\big(\varphi(\nu_A)\big)\big),\ t^{-1}\big(\lambda t\big(\varphi(\nu_B)\big)\big)\Big)\right)$$

$$= \varphi^{-1}\left(T\left(\varphi\Big(\varphi^{-1}\Big(t^{-1}\big(\lambda t\big(\varphi(\nu_A)\big)\big)\Big)\Big),\ \varphi\Big(\varphi^{-1}\Big(t^{-1}\big(\lambda t\big(\varphi(\nu_B)\big)\big)\Big)\Big)\right)\right)$$

$$= \nu_{\lambda\cdot A\oplus\lambda\cdot B},$$

and the proof for the equality (13) can be done analogously.

Lastly, the proof for equality (14) is as follows:

$$
\begin{aligned}
&\mu_{\lambda_1 A \oplus \lambda_2 A} \\
&= \varphi^{-1}\Bigg(S\Bigg(\varphi\bigg(\varphi^{-1}\Big(1 - t^{-1}\big(\lambda_1 t(1-\varphi(\mu_A))\big)\Big)\bigg), \\
&\qquad\qquad \varphi\bigg(\varphi^{-1}\Big(1 - t^{-1}\big(\lambda_2 t(1-\varphi(\mu_A))\big)\Big)\bigg)\Bigg)\Bigg) \\
&= \varphi^{-1}\bigg(S\Big(1 - t^{-1}\big(\lambda_1 t(1-\varphi(\mu_A))\big),\ 1 - t^{-1} Bl(\lambda_2 t(1-\varphi(\mu_A)))\Big)\bigg) \\
&= \varphi^{-1}\Bigg(1 - t^{(-1)}\Bigg(t\bigg(1 - \Big(1 - t^{-1}\big(\lambda_1 t(1-\varphi(\mu_A))\big)\Big)\bigg) \\
&\qquad\qquad + t\bigg(1 - \Big(1 - t^{-1}\big(\lambda_2 t(1-\varphi(\mu_A))\big)\Big)\bigg)\Bigg)\Bigg) \\
&= \varphi^{-1}\Big(1 - t^{(-1)}\big(\lambda_1 t(1-\varphi(\mu_A)) + \lambda_2 t(1-\varphi(\mu_A))\big)\Big) \\
&= \varphi^{-1}\Big(1 - t^{-1}\big((\lambda_1+\lambda_2) t(1-\varphi(\mu_A))\big)\Big) \ = \ \nu_{(\lambda_1+\lambda_2)A}\,,
\end{aligned}
$$

$$
\begin{aligned}
&\nu_{\lambda_1 A \oplus \lambda_2 A} \\
&= \varphi^{-1}\Bigg(T\Bigg(\varphi\bigg(\varphi^{-1}\Big(t^{-1}\big(\lambda_1 t(\varphi(\nu_A))\big)\Big)\bigg),\ \varphi\bigg(\varphi^{-1}\Big(t^{-1}\big(\lambda_2 t(\varphi(\nu_A))\big)\Big)\bigg)\Bigg)\Bigg) \\
&= \varphi^{-1}\bigg(T\Big(t^{-1}\big(\lambda_1 t(\varphi(\nu_A))\big),\ t^{-1}\big(\lambda_2 t(\varphi(\nu_A))\big)\Big)\bigg) \\
&= \varphi^{-1}\Bigg(t^{(-1)}\bigg(t\Big(t^{-1}\big(\lambda_1 t(\varphi(\nu_A))\big)\Big) + t\Big(t^{-1}\big(\lambda_2 t(\varphi(\nu_A))\big)\Big)\bigg)\Bigg) \\
&= \varphi^{-1}\Big(t^{(-1)}\big(\lambda_1 t(\varphi(\nu_A)) + \lambda_2 t(\varphi(\nu_A))\big)\Big) \\
&= \varphi^{-1}\Big(t^{-1}\big((\lambda_1+\lambda_2) t(\varphi(\nu_A))\big)\Big) \ = \ \nu_{(\lambda_1+\lambda_2)A}
\end{aligned}
$$

with the proof for equality (15) being analogous. □

Theorem 3.9. *Let A, B be two φ-intuitionistic fuzzy sets and $\lambda > 0$, then*

$$(A \cap B)^c = A^c \cup B^c, \tag{16}$$

$$(A \cup B)^c = A^c \cap B^c, \tag{17}$$

$$(A \oplus B)^c = A^c \odot B^c, \tag{18}$$

$$(A \odot B)^c = A^c \oplus B^c, \tag{19}$$

$$\lambda \cdot (A^c) = (A^\lambda)^c, \tag{20}$$

$$(A^c)^\lambda = (\lambda \cdot A)^c. \tag{21}$$

P r o o f. We will prove only the equalities (18) and (20), as proofs for (16) and (17) can be found in the mentioned literature (see, e.g., [14] or [18]) and proofs for (19) and (21) are analogous to (18) and (20), respectively. Note that

$$\begin{aligned}(A \oplus B)^c &= \Big(\varphi^{-1}\Big(S\big(\varphi(\mu_A), \varphi(\mu_B)\big)\Big), \ \varphi^{-1}\Big(T\big(\varphi(\nu_A), \varphi(\nu_B)\big)\Big)\Big)^c \\ &= \Big(\varphi^{-1}\Big(T\big(\varphi(\nu_A), \varphi(\nu_B)\big)\Big), \ \varphi^{-1}\Big(S\big(\varphi(\mu_A), \varphi(\mu_B)\big)\Big)\Big) \\ &= (\nu_A, \mu_A) \odot (\nu_B, \mu_B) \\ &= A^c \odot B^c.\end{aligned}$$

Also, note that,

$$\begin{aligned}\lambda \cdot (A^c) &= \lambda \cdot (\nu_A, \mu_A) \\ &= \Big(\varphi^{-1}\Big(1 - t^{-1}\Big(\lambda t\big(1 - \varphi(\nu_A)\big)\Big)\Big), \ \varphi^{-1}\Big(t^{-1}\Big(\lambda t\big(\varphi(\mu_A)\big)\Big)\Big)\Big) \\ &= \Big(\varphi^{-1}\Big(t^{-1}\Big(\lambda t\big(\varphi(\mu_A)\big)\Big)\Big), \ \varphi^{-1}\Big(1 - t^{-1}\Big(\lambda t\big(1 - \varphi(\nu_A)\big)\Big)\Big)\Big)^c \\ &= (A^\lambda)^c.\end{aligned}$$

□

Theorem 3.10. *Let A, B be two φ-intuitionistic fuzzy sets and let $\lambda > 0$, then*

$$\lambda \cdot (A \cup B) = (\lambda \cdot A) \cup (\lambda \cdot B), \tag{22}$$

$$\lambda \cdot (A \cap B) = (\lambda \cdot A) \cap (\lambda \cdot B), \tag{23}$$

$$(A \cup B)^\lambda = A^\lambda \cup B^\lambda, \tag{24}$$

$$(A \cap B)^\lambda = A^\lambda \cap B^\lambda, \tag{25}$$

$$(A \cup B) \oplus (A \cap B) = A \oplus B, \tag{26}$$

$$(A \cup B) \odot (A \cap B) = A \odot B, \tag{27}$$

$$(A \cup B) \cap B = B, \tag{28}$$

$$(A \cap B) \cup B = B. \tag{29}$$

P r o o f. The proofs of (22) and (26) are the following:

$$
\begin{aligned}
&\mu_{\lambda\cdot(A\cup B)}\\
&=\varphi^{-1}\Big(1-t^{-1}\Big(\lambda t\big(1-\varphi(\max\{\mu_A,\mu_B\})\big)\Big)\Big)\\
&=\varphi^{-1}\Big(1-t^{-1}\Big(\lambda\min\big\{t\big(1-\varphi(\mu_A)\big),\ t\big(1-\varphi(\mu_B)\big)\big\}\Big)\Big)\\
&=\varphi^{-1}\Big(1-t^{-1}\Big(\lambda\max\big\{t\big(1-\varphi(\mu_A)\big),\ t\big(1-\varphi(\mu_B)\big)\big\}\Big)\Big)\\
&=\varphi^{-1}\Big(\max\Big\{1-t^{-1}\Big(\lambda t\big(1-\varphi(\mu_A)\big)\Big),\ 1-t^{-1}\Big(\lambda t\big(1-\varphi(\mu_B)\big)\Big)\Big\}\Big)\\
&=\max\Big\{\varphi^{-1}\Big(1-t^{-1}\Big(\lambda t\big(1-\varphi(\mu_A)\big)\Big)\Big),\ \varphi^{-1}\Big(1-t^{-1}\Big(\lambda t\big(1-\varphi(\mu_B)\big)\Big)\Big)\Big\}\\
&=\mu_{(\lambda\cdot A)\cup(\lambda\cdot B)},
\end{aligned}
$$

$$
\begin{aligned}
&\nu_{\lambda\cdot(A\cup B)}\\
&=\varphi^{-1}\Big(t^{-1}\Big(\lambda t\big(\varphi(\min\{\nu_A,\ \nu_B\})\big)\Big)\Big)\\
&=\varphi^{-1}\Big(t^{-1}\Big(\lambda\min\big\{t\big(\varphi(\nu_A)\big),\ t\big(\varphi(\nu_B)\big)\big\}\Big)\Big)\\
&=\varphi^{-1}\left(t^{-1}\left(\lambda\min\big\{t\big(\varphi(\nu_A)\big),\ t\big(\varphi(\nu_B)\big)\big\}\right)\right)\\
&=\varphi^{-1}\left(\min\Big\{t^{-1}\Big(\lambda t\big(\varphi(\nu_A)\big)\Big),\ \varphi^{-1}\Big(t^{-1}\Big(\lambda t\big(\varphi(\nu_B)\big)\Big)\Big)\Big\}\right)\\
&=\min\Big\{\varphi^{-1}\Big(t^{-1}\Big(\lambda t\big(\varphi(\nu_A)\big)\Big)\Big),\ \varphi^{-1}\Big(t^{-1}\Big(\lambda t\big(\varphi(\nu_B)\big)\Big)\Big)\Big\}\\
&=\nu_{(\lambda\cdot A)\cup(\lambda\cdot B)}
\end{aligned}
$$

thanks to the monotonicity of φ and t.

$$
\begin{aligned}
\mu_{(A\cup B)\oplus(A\cap B)}&=\varphi^{-1}\Big(S\big(\varphi(\max\{\mu_A,\mu_B\}),\ \varphi(\min\{\mu_A,\mu_B\})\big)\Big)\\
&=\varphi^{-1}\Big(S\big(\varphi(\mu_A),\ \varphi(\mu_B)\big)\Big)=\mu_{A\oplus B},\\
\nu_{(A\cup B)\oplus(A\cap B)}&=\varphi^{-1}\Big(T\big(\varphi(\min\{\nu_A,\ \nu_B\}),\ \varphi(\max\{\nu_A,\nu_B\})\big)\Big)\\
&=\varphi^{-1}\Big(T\big(\varphi(\nu_A),\ \varphi(\nu_B)\big)\Big)=\nu_{A\oplus B}
\end{aligned}
$$

thanks to the commutativity of S and T.

The proofs for (23), (24), and (25) are analogous to (22), the proof for (27) is analogous to (26) and the proofs for (28) and (29) can be found in the mentioned literature (see, e.g., [14] or [18]). □

Theorem 3.11. *Let A, B, C be three φ-intuitionistic fuzzy sets. Then*

$$(A \cup B) \cup C = A \cup (B \cup C), \tag{30}$$

$$(A \cap B) \cap C = A \cap (B \cap C), \tag{31}$$

$$(A \oplus B) \oplus C = A \oplus (B \oplus C), \tag{32}$$

$$(A \odot B) \odot C = A \odot (B \odot C). \tag{33}$$

Proof. The proof of equation (32) is as follows:

$$\begin{aligned}
\mu_{(A \oplus B) \oplus C} &= \varphi^{-1}\left(S\Big(\varphi\Big(\varphi^{-1}\Big(S\big(\varphi(\mu_A),\ \varphi(\mu_B)\big)\Big)\Big),\ \varphi(\mu_C)\Big)\right) \\
&= \varphi^{-1}\Big(S\Big(S\big(\varphi(\mu_A),\ \varphi(\mu_B)\big)\Big),\ \varphi(\mu_C)\Big) \\
&= \varphi^{-1}\Big(S\Big(S\big(\varphi(\mu_B),\ \varphi(\mu_C)\big)\Big),\ \varphi(\mu_A)\Big) \\
&= \varphi^{-1}\left(S\Big(\varphi\Big(\varphi^{-1}\Big(S\big(\varphi(\mu_B),\ \varphi(\mu_C)\big)\Big)\Big),\ \varphi(\mu_A)\Big)\right) \\
&= \mu_{A \oplus (B \oplus C)}, \\
\nu_{(A \oplus B) \oplus C} &= \varphi^{-1}\left(T\Big(\varphi\Big(\varphi^{-1}\Big(T\big(\varphi(\nu_A),\ \varphi(\nu_B)\big)\Big)\Big),\ \varphi(\nu_C)\Big)\right) \\
&= \varphi^{-1}\Big(T\Big(T\big(\varphi(\nu_A),\ \varphi(\nu_B)\big)\Big),\ \varphi(\nu_C)\Big) \\
&= \varphi^{-1}\Big(T\Big(T\big(\varphi(\nu_B),\ \varphi(\nu_C)\big)\Big),\ \varphi(\nu_A)\Big) \\
&= \nu_{A \oplus (B \oplus C)}
\end{aligned}$$

thanks to the associativity of S and T.

The proof for equation (33) is analogous, and the proofs for equations (30) and (31) can be found in the mentioned literature (see, e.g., [14] or [18]). □

Theorem 3.12. *Let A, B, C be three φ-intuitionistic fuzzy sets. Then*

$$\begin{aligned}(A \cup B) \cap C &= (A \cap C) \cup (B \cap C),\\ (A \cap B) \cup C &= (A \cup C) \cap (B \cup C),\\ (A \cup B) \oplus C &= (A \oplus C) \cup (B \oplus C),\\ (A \cap B) \oplus C &= (A \oplus C) \cap (B \oplus C),\\ (A \cup B) \odot C &= (A \odot C) \cup (B \odot C),\\ (A \cap B) \odot C &= (A \odot C) \cap (B \odot C).\end{aligned} \tag{34}$$

Proof. The proof of equation (34) is the following:

$$\begin{aligned}\mu_{(A\cup B)\oplus C} &= S(\max\{\mu_A, \mu_B\},\ \mu_C)\\ &= \varphi^{-1}\Big(S\big(\varphi(\max\{\mu_A, \mu_B\}), \varphi(\mu_C)\big)\Big)\\ &= \max\Big\{\varphi^{-1}\Big(S\big(\varphi(\mu_A),\ \varphi(\mu_C)\big)\Big),\ \varphi^{-1}\Big(S\big(\varphi(\mu_B),\ \varphi(\mu_C)\big)\Big)\Big\}\\ &= \mu_{(A\oplus C)\cup(B\oplus C)},\\ \nu_{(A\cup B)\oplus C} &= T(\min\{\nu_A, \nu_B\},\ \nu_C)\\ &= \varphi^{-1}\Big(T\big(\varphi(\min\{\nu_A, \nu_B\}),\ \varphi(\nu_C)\big)\Big)\\ &= \min\Big\{\varphi^{-1}\Big(T\big(\varphi(\nu_A),\ \varphi(\nu_C)\big)\Big),\ \varphi^{-1}\Big(T\big(\varphi(\nu_B),\ \varphi(\nu_C)\big)\Big)\Big\}\\ &= \nu_{(A\oplus C)\cup(B\oplus C)}\end{aligned}$$

thanks to the monotonicity of φ, S, and T. The rest of the proofs of Theorem 3.12 can be proven analogously. □

4. Concluding remarks

Our exploration of intuitionistic values and fuzzy sets through automorphisms represents a significant step forward in the mathematical modelling of uncertainty. By extending the foundational concepts of intuitionistic fuzzy sets with strong negations, we have introduced a versatile and powerful framework that offers new insights and tools to deal with vagueness and imprecision. This work not only enriches the theoretical landscape of fuzzy logic, but also creates new tools for applications in various fields.

Future work will focus on further exploring the implications of this framework, developing more complex operations, finding the principle of choosing the optimal instance of the framework, i.e., the most suitable negation for a specific application, and applying these concepts to real-world problems.

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Received Jun 23, 2025
Revised November 27, 2025
Accepted January 5, 2026
Publ. online April 30, 2026

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