

ON (τ_1, τ_2) -STAR-I COMPACT SPACES

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ABSTRACT. This paper introduces the concept of the (τ_1, τ_2) -star- $\mathcal{I}$ compact space in an ideal bispaces, where an ideal bispaces is a quadruple (X, τ_1, τ_2, I) , τ_1, τ_2 being topologies defined on the set X and I being an ideal defined on X . The structure of (τ_1, τ_2) -star- $\mathcal{I}$ compactness has been compared with some nearer structures like (τ_1, τ_2) - I compactness, strongly star- I compactness, countably I compactness, etc. With some counter examples, the distinct structures of these topological features have been validated. The nature of subspaces and the nature of functions that preserve the (τ_1, τ_2) -star- I compactness are revealed. It has been shown that real-valued continuous functions defined on (τ_1, τ_2) -star- I_{fin} compact spaces are bounded. A characterization of this specific topological property in terms of a finite intersection property is provided. Finally, the relation with weakly (τ_1, τ_2) -star compactness is established by means of co-dense ideals.

1. Introduction

In topology, the study of compactness and its generalizations plays a fundamental role in understanding the structure and behaviour of topological spaces. In recent years, the investigation of compactness in ideal spaces and bitopological frameworks has attracted considerable attention due to its applicability in various branches of mathematical analysis. In 1963, Kelly [12] introduced the concept of a bi-topological space (briefly, Bi-TS). The papers by Douwen [10] and Matveev [15] contain a systematic study of the ‘star operator’. A useful concept in topology, the star operator, is widely used to describe and analyze various types of covering patterns and structures in topological spaces.

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The star of Ω with respect to Ψ is denoted and defined as

$$\text{St}(\Omega, \Psi) = \bigcup \{A \in \Psi : A \cap \Omega \neq \emptyset\}, \quad \text{where } \Omega \subseteq X, \Psi \subseteq \mathcal{P}(X).$$

Iteratively, for $n \in \mathbb{N}$,

$$\text{St}^{n+1}(\Omega, \Psi) = \bigcup \{A \in \Psi : A \cap \text{St}^n(\Omega, \Psi) \neq \emptyset\}.$$

$\text{St}^1(\Omega, \Psi)$ is simply written as $\text{St}(\Omega, \Psi)$. Also, $\text{St}(a, \Psi)$ is written in place of $\text{St}(\{a\}, \Psi)$ when $\Omega = \{a\}$. An important use of star operators that initiated the star selection principles theory was given in [1, 2, 13]. Some other recent uses can be found in [3, 4, 6, 8, 9, 17–19].

Kuratowski [14] first proposed the concept of ideals in topological spaces in 1933, and Vaidyanathaswamy [20] did a thorough investigation of the subject in 1946. In a topological space (X, τ_1) , an ideal $\mathcal{I}$ is defined as a non-empty family of subsets of X with the following properties:

- (1) $\emptyset \in \mathcal{I}$.
- (2) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$.
- (3) $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$.

Let (X, τ) be a topological space and let $\mathcal{I}$ be an ideal on X . Then, the ordered triplet $(X, \tau, \mathcal{I})$ is called an ideal topological space, or simply an ideal space. In the case of a non-compact space, if a finite sub-collection of a cover fails to cover the whole space, then what happens to the uncovered portion of the space? We use the concept of an ideal to answer this question. If the residual part of the space always lies in the ideal structure defined on the space, then we call it an ideal compact space.

In this paper, we introduce a new class of compactness, termed (τ_1, τ_2) -star- $\mathcal{I}$ -compactness, within the setting of an ideal bispaces, which generalizes both topological and bitopological spaces by incorporating an ideal structure. We begin by revisiting the fundamental definitions of ideal spaces, bitopological spaces, and classical star-compactness. We then define the (τ_1, τ_2) -n-star- $\mathcal{I}$ -compact space and examine its relationship with known compactness properties, such as countable $\mathcal{I}$ -compactness and strong star- $\mathcal{I}$ -compactness. Several illustrative examples and propositions are provided to distinguish (τ_1, τ_2) -n-star- $\mathcal{I}$ -compactness from its classical counterparts. Additionally, we explore the behaviour of real-valued continuous functions on such spaces, investigate their invariance under continuous mappings, and establish connections with weakly star-compact spaces through co-dense ideals.

2. Preliminaries

For the reader's convenience, an overview of basic concepts is provided in this part. No separation axiom has been considered at any point during the investigation. Bispaces are bitopological spaces with two linked topologies, whereas topological spaces are spaces with their associated topologies. We refer to [17] for common symbols and notions.

DEFINITION 2.1 ([14]). Let (X, τ) be a topological space, and let $\mathcal{I}$ be an ideal on X . Then, the ordered triplet $(X, \tau, \mathcal{I})$ is called an ideal topological space or, simply an ideal space.

The ideal space has proven its usefulness in the analysis of non-compact spaces. So, a quadruple $(X, \tau_1, \tau_2, \mathcal{I})$ will be called an ideal bspace.

Rather than searching for a finite subcover, if a finite collection of a cover can spread throughout the space via star operator, we arrive at the concept of star compactness.

DEFINITION 2.2. A bitopological space (X, τ_1, τ_2) is called a (τ_1, τ_2) star-compact space if, for every pair $(\mathcal{G}_1, \mathcal{G}_2)$ of τ_1 -open and τ_2 -open covers of X , there exists a finite subset $\mathcal{G}' \subseteq \mathcal{G}_1$ such that $\text{St}(\bigcup \mathcal{G}', \mathcal{G}_2) = X$.

DEFINITION 2.3 ([14]). An ideal space $(X, \tau, \mathcal{I})$ is called compact modulo $\mathcal{I}$ or $\mathcal{I}$ -compact space if, for every τ -open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X , there exists a finite subset $\{U_{\alpha_i} : i = 1, 2, 3, \dots, k\}$ such that $X \setminus \bigcup_{i=1}^k U_{\alpha_i} \in \mathcal{I}$.

DEFINITION 2.4 ([5]). An ideal space $(X, \tau, \mathcal{I})$ is called a strongly star $\mathcal{I}$ -compact space if, for every τ -open cover $\mathcal{U}$ of X , there exists a finite subset $H \subseteq X$ such that $X \setminus \text{St}(H, \mathcal{U}) \in \mathcal{I}$.

Every $\mathcal{I}$ -compact space is strongly star $\mathcal{I}$ -compact.

DEFINITION 2.5 ([5]). An ideal space $(X, \tau, \mathcal{I})$ is called a star $\mathcal{I}$ -compact space if, for every τ -open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of X , there exists a finite subset $\{U_{\alpha_i} : i = 1, 2, 3, \dots, k\}$ such that $X \setminus \text{St}(\bigcup_{i=1}^k U_{\alpha_i}, \mathcal{U}) \in \mathcal{I}$.

DEFINITION 2.6 ([5]). An ideal space $(X, \tau, \mathcal{I})$ is called countably $\mathcal{I}$ -compact space if, for every countable τ -open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of X , there exists a finite subset $\{U_{\alpha_i} : i = 1, 2, 3, \dots, k\}$ such that $X \setminus \text{St}(\bigcup_{i=1}^k U_{\alpha_i}, \mathcal{U}) \in \mathcal{I}$.

The finite intersection property is a way to describe a collection of sets that overlap “sufficiently” to suggest the existence of a common point. It helps to explain the compactness in terms of families of closed sets.

DEFINITION 2.7 ([11]). A collection of sets is said to have the finite intersection property (FIP) if every finite sub-collection of these sets has a non-empty intersection.

THEOREM 2.8 ([11]). *A topological space is compact if every collection of closed sets with the finite intersection property has a non-empty total intersection.*

The recent paper [7] provides FIP-like characterizations for star induced spaces.

3. (τ_1, τ_2) -star- $\mathcal{I}$ compactness

The star operator establishes a relationship between a specified set and a family of subsets in a topological space. So, given two distinct open covers defined on two different topologies on the same set and an ideal on the same set, we propose the following definition.

DEFINITION 3.1. An ideal bispaces $(X, \tau_1, \tau_2, \mathcal{I})$ is called a (τ_1, τ_2) - n star- $\mathcal{I}$ -compact space ($n \in \mathbb{N}$) if, for every pair $(\mathcal{G}_1, \mathcal{G}_2)$ of τ_1 -open and τ_2 -open covers of X , there exists a finite subset $\mathcal{G}' \subseteq \mathcal{G}_1$ such that $X \setminus \text{St}^n(\bigcup \mathcal{G}', \mathcal{G}_2) \in \mathcal{I}$.

For $n = 1$, we say that the space is τ_1, τ_2 -star $\mathcal{I}$ -compact.

THEOREM 3.2. *Every (τ_1, τ_2) -star compact space is a (τ_1, τ_2) -star $\mathcal{I}$ -compact space.*

Proof. The proof follows directly from Definition 3.1 and is therefore omitted. $\square$

EXAMPLE. The converse of the above theorem may not be true, i.e., there exists a (τ_1, τ_2) -star $\mathcal{I}$ -compact space that is not (τ_1, τ_2) -star compact.

Let $X = \mathbb{N} \cup \{0\}$. Consider two topologies τ_1 and τ_2 on X , where τ_1 is the topology generated by the base

$$\mathcal{B} = \{\{0, n\} : n \in \mathbb{N}\} \cup \{\{0\}, \emptyset\},$$

and τ_2 is the discrete topology on X . Also, we take the ideal $\mathcal{I} = \mathcal{P}(X)$. Then, $\mathcal{G}_1 = \{G_i = \{0, i\} : i \in \mathbb{N}\}$ is a τ_1 open cover of X , and $\mathcal{G}_2 = \{\{x\} : x \in X\}$ is a τ_2 open cover of X . Now, for any finite subset,

$$\mathcal{G}' = \{G_{n_1}, G_{n_2}, G_{n_3}, \dots, G_{n_k}\} \subseteq \mathcal{G}_1,$$

$$\text{St}\left(\bigcup \mathcal{G}', \mathcal{G}_2\right) = \{0, n_1, n_2, n_3, \dots, n_k\} \neq X,$$

but

$$X \setminus \text{St}\left(\bigcup \mathcal{G}', \mathcal{G}_2\right) = \mathbb{N} \setminus \{0, n_1, n_2, n_3, \dots, n_k\} \in \mathcal{I} = \mathcal{P}(X).$$

Hence, $(X, \tau_1, \tau_2, \mathcal{I})$ is (τ_1, τ_2) -star- $\mathcal{I}$ -compact but not (τ_1, τ_2) -star compact.

PROPOSITION 3.3. *If $(X, \tau_2, \mathcal{I})$ is strongly star- $\mathcal{I}$ -compact, then $(X, \tau_1, \tau_2, \mathcal{I})$ is (τ_1, τ_2) -star $\mathcal{I}$ -compact.*

Proof. Let $(\mathcal{G}_1, \mathcal{G}_2)$ be two arbitrary open covers of X . Since $(X, \tau_2, \mathcal{I})$ is strongly star- $\mathcal{I}$ -compact, there exists a finite subset

$$\mathcal{H} = \{x_1, x_2, x_3, \dots, x_n\} \subseteq X \quad \text{such that} \quad X \setminus \text{St}(\mathcal{H}, \mathcal{G}_2) \in \mathcal{I}.$$

Now, for each i , we can choose $U_i \in \mathcal{G}_1$ such that $x_i \in U_i$. Then, the set

$$\mathcal{K} = \{U_1, U_2, U_3, \dots, U_n\} \subseteq \mathcal{G}_1 \quad \text{is finite.}$$

Also,

$$\mathcal{H} \subseteq \bigcup \mathcal{K} \Rightarrow \text{St}(\mathcal{H}, \mathcal{G}_2) \subseteq \text{St}\left(\bigcup \mathcal{K}, \mathcal{G}_2\right) \Rightarrow X \setminus \text{St}\left(\bigcup \mathcal{K}, \mathcal{G}_2\right) \subseteq X \setminus \text{St}(\mathcal{H}, \mathcal{G}_2).$$

But

$$X \setminus \text{St}(\mathcal{H}, \mathcal{G}_2) \in \mathcal{I}, \quad \text{so} \quad X \setminus \text{St}\left(\bigcup \mathcal{K}, \mathcal{G}_2\right) \in \mathcal{I}.$$

Hence, $(X, (\tau_1, \tau_2), \mathcal{I})$ is (τ_1, τ_2) -star- $\mathcal{I}$ -compact. $\square$

Similarly to the proof of Proposition 3.3, one can prove the following.

PROPOSITION 3.4. *If $(X, \tau_2, \mathcal{I})$ is $\mathcal{I}$ -compact, then $(X, \tau_1, \tau_2, \mathcal{I})$ is (τ_1, τ_2) -star- $\mathcal{I}$ -compact.*

PROPOSITION 3.5. *If $(X, \tau_2, \mathcal{I})$ is strongly n star- $\mathcal{I}$ -compact, then $(X, \tau_1, \tau_2, \mathcal{I})$ is (τ_1, τ_2) - n star- $\mathcal{I}$ -compact.*

THEOREM 3.6. *If $(X, \tau_2, \mathcal{I})$ is star- $\mathcal{I}$ -compact, then $(X, \tau_1, \tau_2, \mathcal{I})$ is (τ_1, τ_2) -2 star- $\mathcal{I}$ -compact.*

Proof. Let $(\mathcal{G}_1, \mathcal{G}_2)$ be a pair of τ_1 and τ_2 open covers of X . Since $(X, \tau_2, \mathcal{I})$ is star- $\mathcal{I}$ -compact, there exists a finite subset $\mathcal{G}'_2 = \{G_{2,i} : i = 1, 2, \dots, k\} \subseteq \mathcal{G}_2$ such that $X \setminus \text{St}(\bigcup \mathcal{G}'_2, \mathcal{G}_2) \in \mathcal{I}$. For each $i = 1, 2, \dots, k$, we select one $G_{1,i} \in \mathcal{G}_1$ such that $G_{1,i} \cap G_{2,i} \neq \emptyset$. Thus, we have a finite set

$$\mathcal{A} = \{G_{1,i} : i = 1, 2, \dots, k\} \subseteq \mathcal{G}_1,$$

and it holds that

$$\bigcup \mathcal{G}'_2 \subseteq \text{St}\left(\bigcup \mathcal{A}, \mathcal{G}_2\right).$$

So,

$$\text{St}\left(\bigcup \mathcal{G}'_2, \mathcal{G}_2\right) \subseteq \text{St}^2\left(\bigcup \mathcal{A}, \mathcal{G}_2\right) \Rightarrow X \setminus \text{St}^2\left(\bigcup \mathcal{A}, \mathcal{G}_2\right) \subseteq X \setminus \text{St}\left(\bigcup \mathcal{G}'_2, \mathcal{G}_2\right).$$

But,

$$X \setminus \text{St}\left(\bigcup \mathcal{G}'_2, \mathcal{G}_2\right) \in \mathcal{I}.$$

Thus, $X \setminus \text{St}^2(\bigcup \mathcal{A}, \mathcal{G}_2) \in \mathcal{I}$. Hence $(X, (\tau_1, \tau_2), \mathcal{I})$ is (τ_1, τ_2) -2 star- $\mathcal{I}$ -compact. $\square$

Similarly to the proof of Theorem 3.6, one can prove the following

PROPOSITION 3.7. *If $(X, \tau_2, \mathcal{I})$ is n star- $\mathcal{I}$ -compact, then $(X, \tau_1, \tau_2, \mathcal{I})$ is (τ_1, τ_2) -($n+1$)-star- $\mathcal{I}$ -compact.*

DEFINITION 3.8. An ideal space $(X, \tau, \mathcal{I})$ is called countably $\mathcal{I}$ -compact if every countable open cover $\mathcal{U} = \{U_1, U_2, U_3, \dots\}$ has a finite subset $\mathcal{U}' = \{U_{n_1}, U_{n_2}, U_{n_3}, \dots, U_{n_k}\}$ such that $X \setminus \bigcup_{i=1}^k U_{n_i} \in \mathcal{I}$.

LEMMA 3.9. *A closed subspace of a countably $\mathcal{I}$ -compact space is countably $\mathcal{I}$ -compact.*

Proof. Let Y be a closed subset of the countably $\mathcal{I}$ -compact space $(X, \tau, \mathcal{I})$, and let $\{M_n : n \in \mathbb{N}\}$ be a countable open cover of Y , i.e., $Y \subseteq \bigcup_{n \in \mathbb{N}} M_n$. Now, $\{M_n : n \in \mathbb{N}\} \cup \{X \setminus Y\}$ is an open cover of X . So, there exists a finite subset $\{M_{n_i} : i = 1, 2, 3, \dots, k\}$ such that

$$X \setminus \left(\left(\bigcup_{i=1}^k M_{n_i} \right) \cup (X \setminus Y) \right) \in \mathcal{I} \quad \text{or} \quad X \setminus \left(\bigcup_{i=1}^k M_{n_i} \right) \in \mathcal{I}.$$

In both cases, $Y \setminus \left(\bigcup_{i=1}^k M_{n_i} \right) \in \mathcal{I}$. Hence, Y is countably $\mathcal{I}$ -compact. $\square$

THEOREM 3.10. *If $(X, \tau_2, \mathcal{I})$ is countably $\mathcal{I}$ -compact, then $(X, \tau_1, \tau_2, \mathcal{I})$ is (τ_1, τ_2) -2 star- $\mathcal{I}$ -compact.*

Proof. Let $(X, \tau_2, \mathcal{I})$ be countably $\mathcal{I}$ -compact, and suppose $(X, \tau_1, \tau_2, \mathcal{I})$ is not (τ_1, τ_2) -star $\mathcal{I}$ -compact. Then, we can find a pair $(\mathcal{V}_1, \mathcal{V}_2)$ of τ_1 -open and τ_2 -open covers of X such that for every finite

$$\mathcal{W} \subset \mathcal{V}_1, \quad X \setminus \text{St}\left(\bigcup \mathcal{W}, \mathcal{V}_2\right) \notin \mathcal{I}.$$

For an arbitrary $a_1 \in X$, we select $B_1 \in \mathcal{V}_1$ such that $a_1 \in B_1$. Thus,

$$X \setminus \text{St}(B_1, \mathcal{V}_2) \notin \mathcal{I}.$$

Since

$$\text{St}(B_1, \mathcal{V}_2) \supseteq \text{St}(a_1, \mathcal{V}_2), \quad X \setminus \text{St}(B_1, \mathcal{V}_2) \subseteq X \setminus \text{St}(a_1, \mathcal{V}_2)$$

and hence, by the contrapositivity of the property of the ideal, we have

$$X \setminus \text{St}(a_1, \mathcal{V}_2) \notin \mathcal{I}, \quad \text{also} \quad a_1 \in \text{St}(a_1, \mathcal{V}_2).$$

Now, take $a_2 \in X \setminus \text{St}(a_1, \mathcal{V}_2)$. Thus, there exists a $B_2 \in \mathcal{V}_1$ with $a_2 \in B_2$. However,

$$X \setminus \text{St}(B_1 \cup B_2, \mathcal{V}_2) \notin \mathcal{I}.$$

Therefore,

$$a_1, a_2 \in \text{St}(\{a_1, a_2\}, \mathcal{V}_2) \quad \text{and} \quad X \setminus \text{St}(\{a_1, a_2\}, \mathcal{V}_2) \notin \mathcal{I}.$$

Using the same procedure of selection for each $n \in \mathbb{N}$, we can choose

$$B_n \in \mathcal{V}_1 \quad \text{with} \quad a_n \in B_n \quad \text{and} \quad a_n \notin \text{St}(\{a_1, a_2, a_3, \dots, a_{n-1}\}, \mathcal{V}_2).$$

Let

$$Y = \{a_n : n \in \mathbb{N}\} \quad \text{and} \quad \mathcal{U} = \{\text{St}(a_n, \mathcal{V}_2) : n \in \mathbb{N}\}.$$

By the construction of $\mathcal{U}$, each member of $\mathcal{U}$ contains only one member of Y , i.e., $\text{St}(a_n, \mathcal{V}_2) \cap Y = \{a_n\}$. Also, for every finite subset $\mathcal{U}'$ of $\mathcal{U}$,

$$X \setminus \bigcup \mathcal{U}' \notin \mathcal{I}. \quad (1)$$

Let $y \in \overline{Y}$ be arbitrary. Then, $\mathcal{V}_2$ contains a τ_2 -open set V' such that $y \in V'$. Moreover, $V' \cap Y \neq \emptyset$. Therefore, $y \in \text{St}(a_i, \mathcal{V}_2)$ for some $i \in \mathbb{N}$. Thus, $\mathcal{U} = \{\text{St}(a_n, \mathcal{V}_2) : n \in \mathbb{N}\}$ is a τ_2 -open cover of $\overline{Y}$. Since $\overline{Y}$ is a closed subset of the countably $\mathcal{I}$ -compact space $(X, \tau_2, \mathcal{I})$, it follows that $\overline{Y}$ is countably $\mathcal{I}$ -compact. Therefore, there exists a finite subset $\mathcal{U}'$ of $\mathcal{U}$ such that $X \setminus \bigcup \mathcal{U}' \in \mathcal{I}$, which contradicts (1). Hence, $(X, \tau_1, \tau_2, \mathcal{I})$ is a (τ_1, τ_2) -star $\mathcal{I}$ -compact space. $\square$

4. (τ_1, τ_2) -star- $\mathcal{I}$ compact spaces and mappings

THEOREM 4.1. *Let $\phi : (X, \tau_1, \tau_2, \mathcal{I}_{\text{fin}}) \rightarrow \mathbb{R}$ be a real-valued continuous mapping. If $(X, \tau_1, \tau_2, \mathcal{I}_{\text{fin}})$ is (τ_1, τ_2) -star- $\mathcal{I}_{\text{fin}}$ compact, then ϕ is bounded.*

Proof.

Let $(X, \tau_1, \tau_2, \mathcal{I}_{\text{fin}})$ be (τ_1, τ_2) -star- $\mathcal{I}_{\text{fin}}$ compact, and let $\phi : (X, \tau_1, \tau_2, \mathcal{I}_{\text{fin}}) \rightarrow \mathbb{R}$ be continuous. Consider $\mathcal{W} = \{\phi^{-1}(z-1, z+1) : z \in \mathbb{Z}\}$. Then, $\mathcal{W}$ is a τ_1 as well as τ_2 -open cover of X . Since $(X, \tau_1, \tau_2, \mathcal{I}_{\text{fin}})$ is (τ_1, τ_2) -star- $\mathcal{I}_{\text{fin}}$ -compact space, there exists a finite subset

$$\mathcal{W}' = \{\phi^{-1}(a_i - 1, a_i + 1) : i = 1, 2, \dots, k\} \subseteq \mathcal{W}$$

such that

$$X \setminus \text{St}\left(\bigcup \mathcal{W}', \mathcal{W}\right) \in \mathcal{I}_{\text{fin}}.$$

Let

$$a_{\max} = \max\{a_1 + 1, a_2 + 1, \dots, a_k + 1\} \quad \text{and} \quad a_{\min} = \min\{a_1 - 1, a_2 - 1, \dots, a_k - 1\}.$$

Thus,

$$\bigcup_{i=1}^k (a_i - 1, a_i + 1) \subseteq (a_{\min}, a_{\max}).$$

Therefore,

$$\bigcup \mathcal{W}' = \bigcup_{i=1}^K \phi^{-1}(a_i - 1, a_i + 1) \subseteq \phi^{-1}(a_{\min}, a_{\max}).$$

Hence,

$$\text{St}(\bigcup \mathcal{W}', \mathcal{W}) \subseteq \text{St}(\phi^{-1}(a_{\min}, a_{\max}), \mathcal{W})$$

and therefore,

$$X \setminus \text{St}(\phi^{-1}(a_{\min}, a_{\max}), \mathcal{W}) \subseteq X \setminus \text{St}\left(\bigcup \mathcal{W}', \mathcal{W}\right).$$

Therefore, there exists

$$I_1 \in \mathcal{I}_{\text{fin}} \quad \text{such that} \quad X \setminus \text{St}(\phi^{-1}(a_{\min}, a_{\max}), \mathcal{W}) = I_1 \in \mathcal{I}_{\text{fin}}.$$

This implies

$$\text{St}(\phi^{-1}(a_{\min}, a_{\max}), \mathcal{W}) \cup I_1 = X.$$

Again, this implies

$$\phi^{-1}(a_{\min} - 1, a_{\max} + 1) \cup I_1 = X, \quad \text{so,} \quad (a_{\min} - 1, a_{\max} + 1) \cup \phi(I_1) = \phi(X).$$

Since I_1 is finite, for all $x \in X$,

$$\min\{\{\phi(x) : x \in I_1\} \cup \{a_{\min} - 1\}\} \leq \phi(x) \leq \max\{\{\phi(x) : x \in I_1\} \cup \{a_{\max} + 1\}\}.$$

Hence, ϕ is bounded. $\square$

Now, we propose the following definition to prepare a FIP-like characterization for (τ_1, τ_2) -star- $\mathcal{I}$ -compactness. This will help us explain (τ_1, τ_2) -star- $\mathcal{I}$ -compactness in terms of families of τ_1 -closed and τ_2 -closed sets.

DEFINITION 4.2. A pair $(\mathfrak{J}_1, \mathfrak{J}_2)$ of families of subsets of X is said to have the bi-ideal non-finite intersection property (abbreviated as Bi-InFIP) if, for every finite subset

$$\varepsilon \subseteq \mathfrak{J}_1, \bigcap \{J \in \mathfrak{J}_2 : J \cup (\cap \varepsilon) \neq X\} \in F(\mathcal{I}),$$

where $\mathcal{I}$ is an ideal in X and $F(\mathcal{I}) = \{X \setminus I : I \in \mathcal{I}\}$.

THEOREM 4.3. In an ideal bispac space $(X, \tau_1, \tau_2, \mathcal{I})$, the following statements are equivalent:

- i) If $(\mathcal{U}_1, \mathcal{U}_2)$ is a family of subsets of X such that for every finite $\mathcal{V} \subseteq \mathcal{U}_1$, $X \setminus \text{St}(\cup \mathcal{V}, \mathcal{U}_2) \notin \mathcal{I}$, then $(\mathcal{U}_1, \mathcal{U}_2)$ cannot be a pair of τ_1 -open cover and τ_2 -open cover.
- ii) $(X, \tau_1, \tau_2, \mathcal{I})$ is a (τ_1, τ_2) -star- $\mathcal{I}$ -compact space.
- iii) If $(\mathfrak{J}_1, \mathfrak{J}_2)$ is a pair of τ_1 -closed and τ_2 -closed families of subsets of X such that $\cap \mathfrak{J}_1 = \cap \mathfrak{J}_2 = \emptyset$, then $(\mathfrak{J}_1, \mathfrak{J}_2)$ does not have Bi-InFIP.
- iv) Every pair $(\mathfrak{J}_1, \mathfrak{J}_2)$ of τ_1 -closed and τ_2 -closed families of subsets of X that have Bi-InFIP has a non-empty intersection (i.e., $\cap \mathfrak{J}_1 \neq \emptyset$ or $\cap \mathfrak{J}_2 \neq \emptyset$).

Proof. By contraposition, i) $\iff$ ii) and iii) $\iff$ iv).

Proof of ii) $\implies$ iii).

Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a (τ_1, τ_2) -star- $\mathcal{I}$ -compact space, and let $(\mathfrak{J}_1, \mathfrak{J}_2)$ be a pair of τ_1 -closed and τ_2 -closed families of subsets of X such that $\cap \mathfrak{J}_1 = \cap \mathfrak{J}_2 = \emptyset$. Let

$$\mathcal{G}_1 = \{G = X \setminus J_1 : J_1 \in \mathfrak{J}_1\} \quad \text{and} \quad \mathcal{G}_2 = \{H = X \setminus J_2 : J_2 \in \mathfrak{J}_2\}.$$

Then,

$$\bigcup \mathcal{G}_1 = X - \left(\bigcap_{J_1 \in \mathfrak{J}_1} J_1 \right) = X \setminus \emptyset = X \quad \text{and} \quad \bigcup \mathcal{G}_2 = X - \left(\bigcap_{J_2 \in \mathfrak{J}_2} J_2 \right) = X \setminus \emptyset = X.$$

Thus, $(\mathcal{G}_1, \mathcal{G}_2)$ is a pair of τ_1 and τ_2 -open cover of X . However, since $(X, \tau_1, \tau_2, \mathcal{I})$ is a (τ_1, τ_2) -star- $\mathcal{I}$ -compact space, there exists a finite $\{G_1, G_2, \dots, G_k\} \subseteq \mathcal{G}_1$ such that

$$X \setminus \text{St} \left(\left(\bigcup_{i=1}^k G_i, \mathcal{G}_2 \right) \right) \in \mathcal{I}.$$

Therefore,

$$\text{St} \left(\left(\bigcup_{i=1}^k G_i, \mathcal{G}_2 \right) \right) \in F(\mathcal{I}),$$

where $F(\mathcal{I}) = \{X \setminus I : I \in \mathcal{I}\}$, i.e., $\text{St} \left(\left(\bigcup_{i=1}^k (X \setminus J_i : J_i \in \mathfrak{J}_1, \mathcal{G}_2) \right) \right) \in F(\mathcal{I})$.

As a result,

$$\bigcup \left\{ G_2 \in \mathcal{G}_2 : G_2 \cap \left(\bigcup_{i=1}^k (X \setminus J_i : J_i \in \mathfrak{J}_1) \right) \neq \emptyset \right\} \in F(\mathcal{I}).$$

Hence,

$$\bigcup (X \setminus J_2 : J_2 \in \mathfrak{J}_2) \quad \text{and} \quad (X \setminus J_2) \cap \left(\bigcup_{i=1}^k (X \setminus J_i : J_i \in \mathfrak{J}_1) \right) \neq \emptyset \in F(\mathcal{I}).$$

Therefore,

$$\bigcap (J_2 : J_2 \in \mathfrak{J}_2) \quad \text{and} \quad \left(J_2 \cup \left(\bigcap_{i=1}^k J_i : J_i \in \mathfrak{J}_1 \right) \right) \neq X \notin F(\mathcal{I}).$$

Thus, $(\mathfrak{J}_1, \mathfrak{J}_2)$ do not have Bi-InFIP. □

Proof of iii) $\implies$ ii).

Let condition iii) hold, and let $(\mathcal{U}_1, \mathcal{U}_2)$ be a pair of τ_1 -open cover and τ_2 -open cover of X . We define

$$\mathfrak{J}_1 = \{D = X \setminus U : U \in \mathcal{U}_1\} \quad \text{and} \quad \mathfrak{J}_2 = \{H = X \setminus G : G \in \mathcal{U}_2\}.$$

Hence,

$$\bigcap \mathfrak{J}_1 = X \setminus \left(\bigcup_{U \in \mathcal{U}_1} U \right) = X \setminus X = \emptyset \quad \text{and} \quad \bigcap \mathfrak{J}_2 = X \setminus \left(\bigcup_{G \in \mathcal{U}_2} G \right) = X \setminus X = \emptyset.$$

Therefore, $(\mathfrak{J}_1, \mathfrak{J}_2)$ is a pair of τ_1 -closed and τ_2 -closed families of subsets of X such that $\bigcap \mathfrak{J}_1 = \bigcap \mathfrak{J}_2 = \emptyset$. Then, by condition iii), $(\mathfrak{J}_1, \mathfrak{J}_2)$ does not have Bi-MnFIP. Thus, there exists a finite subset $\mathcal{J}' = \{J_1, J_2, \dots, J_k\} \subseteq \mathfrak{J}_1$ such that

$$\bigcap \left\{ J \in \mathfrak{J}_2 : J \cup \left(\bigcap_{i=1}^k J_i \right) \neq X \right\} \notin F(\mathcal{I}),$$

where $\mathcal{I}$ is an ideal in X and $F(\mathcal{I}) = \{I : I \in F(\mathcal{I})\}$. That is,

$$\bigcap \{J \in \mathfrak{J}_2 : J \cup (\bigcap_{i=1}^k J_i) \neq X\} \notin F(\mathcal{I}).$$

Thus,

$$\bigcap \{J \in \mathfrak{J}_2 : X \setminus J \cup (\cap_{i=1}^k J_i) \neq \emptyset\} \notin F(\mathcal{I}).$$

Therefore,

$$\bigcap \{J \in \mathfrak{J}_2 : X \setminus J \cap (X \setminus (\cap_{i=1}^k J_i)) \neq \emptyset\} \notin F(\mathcal{I}).$$

So,

$$X \setminus \bigcap \{J \in \mathfrak{J}_2 : X \setminus J \cap (X \setminus (\cap_{i=1}^k J_i)) \neq \emptyset\} \in F(\mathcal{I}).$$

Thus,

$$\bigcup \{X \setminus J : J \in \mathfrak{J}_2 \text{ and } X \setminus J \cap (X \setminus (\cap_{i=1}^k J_i)) \neq \emptyset\} \in F(\mathcal{I}).$$

Therefore,

$$\bigcup \{X \setminus J : J \in \mathfrak{J}_2 \text{ and } X \setminus J \cap (\cup_{i=1}^k X \setminus J_i) \neq \emptyset\} \in F(\mathcal{I}).$$

So,

$$\bigcup \{H : H \in \mathcal{U}_2 \text{ and } H \cap (\cup_{i=1}^k X \setminus J_i) \neq \emptyset\} \in F(\mathcal{I}).$$

Therefore,

$$\text{St}(\cup_{i=1}^k X \setminus J_i, \mathcal{U}_2) \in F(\mathcal{I}).$$

So,

$$X \setminus \text{St}(\cup_{i=1}^k X \setminus J_i, \mathcal{U}_2) \in \mathcal{I}.$$

Thus, $(X, (\tau_1, \tau_2), \mathcal{I})$ is a (τ_1, τ_2) -star- $\mathcal{I}$ -compact space. $\square$

THEOREM 4.4. *An ideal bispaces $(X, \tau_1, \tau_2, \mathcal{I})$ is a (τ_1, τ_2) -star- $\mathcal{I}$ -compact space if and only if every τ_1 -closed and τ_2 -clopen subspace Y of X is a (τ_1^Y, τ_2^Y) -star- $\mathcal{I}^Y$ -compact space.*

Proof. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be (τ_1, τ_2) -star- $\mathcal{I}$ -compact space, and let Y be a τ_1 -closed and τ_2 -clopen subspace X . Suppose $\mathcal{U}_Y$ is an arbitrary τ_1^Y open cover and $\mathcal{V}_Y$ is an arbitrary τ_2^Y open cover of Y . Then, for every $U_{Y_\alpha} \in \mathcal{U}_Y$, there exists a $U_\alpha \in \tau_1$ such that $U_{Y_\alpha} = U_\alpha \cap Y$. Then,

$$\mathcal{H} = \{U_\alpha : U_{Y_\alpha} = U_\alpha \cap Y \text{ and } U_{Y_\alpha} \in \mathcal{U}_Y\} \text{ covers } Y \text{ by } \tau_1 \text{ open sets.}$$

Since Y is τ_1 -closed, $\mathcal{K}_1 = \{X \setminus Y\} \cup \mathcal{H}$ is a τ_1 open cover of X . Additionally, since Y is τ_2 -clopen, every element of $\mathcal{V}_Y$ is τ_2 -open, and $X \setminus Y$ is also τ_2 -clopen. Hence, $\mathcal{K}_2 = \{X \setminus Y\} \cup \mathcal{V}_Y$ is a τ_2 -open cover of X . Since X is (τ_1, τ_2) -star- $\mathcal{I}$ -compact, there exists a finite subset

$$\mathcal{K}' = \{U_1, U_2, \dots, U_k\} \subseteq \mathcal{K}_1$$

such that

$$X \setminus \text{St}\left(\bigcup_{i=1}^k U_i, \mathcal{K}_2\right) \in \mathcal{I}.$$

Suppose

$$\mathcal{K}'_Y = \{U_{Y_i} : U_{Y_i} = U_i \cap Y \text{ and } i = 1, 2, 3, \dots, k\} \setminus \{\emptyset\}.$$

Clearly, $\mathcal{K}'_Y$ is finite and no element of $\mathcal{K}'_Y$ has a non-empty intersection with $X \setminus Y$. Moreover, if $U_i \cap V_\beta \neq \emptyset$ for some $V_\beta \in \mathcal{V}_Y$, then

$$U_{Y_i} \cap V_\beta \neq \emptyset.$$

Now,

$$X \setminus \text{St} \left(\bigcup \mathcal{K}', \mathcal{K}_2 \right) \in \mathcal{I}.$$

Thus, there exists an $I_1 \in \mathcal{I}$ such that

$$X \setminus \text{St} \left(\bigcup \mathcal{K}', \mathcal{K}_2 \right) = I_1.$$

Therefore,

$$Y \cap \left(X \setminus \text{St} \left(\bigcup \mathcal{K}', \mathcal{K}_2 \right) \right) = Y \cap I_1.$$

So,

$$(Y \cap X) \setminus Y \cap \text{St} \left(\bigcup \mathcal{K}', \mathcal{K}_2 \right) = Y \cap I_1.$$

Therefore,

$$Y \setminus \text{St} \left(\bigcup \mathcal{K}'_Y, \mathcal{V}_Y \right) = Y \cap I_1.$$

So,

$$Y \setminus \text{St} \left(\bigcup \mathcal{K}'_Y, \mathcal{V}_Y \right) \in \mathcal{I}^Y, \quad \text{where } \mathcal{I}^Y = \{Y \cap I_1 : I_1 \in \mathcal{I}\}.$$

Hence, Y is a (τ_1^Y, τ_2^Y) -star- $\mathcal{I}^Y$ -compact space. $\square$

THEOREM 4.5. *Let $f: X \longrightarrow Y$ be a bijective function such that*

$$f: (X, \tau_1) \longrightarrow (Y, \sigma) \quad \text{and} \quad f: (X, \tau_2) \longrightarrow (Y, \sigma) \quad \text{are continuous.}$$

If $(X, \tau_1, \tau_2, \mathcal{I})$ is a (τ_1, τ_2) -star- $\mathcal{I}$ -compact, then $(Y, \sigma, f(\mathcal{I}))$ is a star- $f(\mathcal{I})$ -compact.

Proof. Let $\mathcal{V}$ be a σ -open cover of Y . Then, $\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{V}\}$ is both a τ_1 -open cover and a τ_2 -open cover of X . Since $(X, \tau_1, \tau_2, \mathcal{I})$ is a (τ_1, τ_2) -star- $\mathcal{I}$ -compact space, there exists a finite subset

$$\mathcal{U}' = \{U_1 = f^{-1}(V_1), U_2 = f^{-1}(V_2), \dots, U_k = f^{-1}(V_k)\} \subseteq \mathcal{U},$$

such that

$$X \setminus \text{St} \left(\bigcup_{i=1}^k U_i, \mathcal{U} \right) \in \mathcal{I}.$$

Therefore,

$$f \left(X \setminus \text{St} \left(\bigcup_{i=1}^k U_i, \mathcal{U} \right) \right) \in f(\mathcal{I}).$$

So, $Y \setminus f(X \setminus \text{St}(\bigcup_{i=1}^k U_i, \mathcal{U})) \in f(\mathcal{I})$, since f is surjective ... (a).

Let

$$y \in Y \setminus \text{St} \left(\bigcup_{i=1}^k V_i, \mathcal{V} \right).$$

This implies

$$y \in Y \quad \text{and} \quad y \notin \text{St} \left(\bigcup_{i=1}^k V_i, \mathcal{V} \right).$$

Now,

$$y \notin \text{St} \left(\bigcup_{i=1}^k V_i, \mathcal{V} \right).$$

Therefore,

$$y \notin \bigcup_{i=1}^k \text{St}(V_i, \mathcal{V}).$$

So,

$$y \notin \text{St}(V_i, \mathcal{V}), \quad \text{for any } i = 1, 2, 3, \dots, k.$$

Therefore, $y \notin V$, whenever $V \cap V_i \neq \emptyset$ for all $V \in \mathcal{V}$.

Corresponding to each $V, V_i \in \mathcal{V}$, there exists a $U \in \tau_2$ such that

$$U = f^{-1}(V) \quad \text{and} \quad U_i = f^{-1}(V_i) \in \tau_1, \forall i.$$

Thus, $y \notin f(U)$, whenever $f(U) \cap f(U_i) \neq \emptyset$, for all $U \in \mathcal{U}$ and for any i . Therefore, $y \notin f(U)$, whenever $f(U \cap U_i) \neq \emptyset$, for all $U \in \mathcal{U}$ and for any i , since f is injective. This implies

$$f^{-1} \notin U, \quad \text{when } U_i \cap U \neq \emptyset, \quad \text{for all } U \in \mathcal{U} \quad \text{and} \quad \forall i.$$

Therefore,

$$f^{-1}(y) \notin \bigcup_{i=1}^k \text{St}(U_i, \mathcal{U}).$$

So,

$$f^{-1}(y) \notin \text{St} \left(\bigcup_{i=1}^k U_i, \mathcal{U} \right).$$

Therefore,

$$f^{-1}(y) \in X \setminus \text{St} \left(\bigcup_{i=1}^k U_i, \mathcal{U} \right).$$

Therefore,

$$y \in Y \setminus f \left(\text{St} \left(\bigcup_{i=1}^k U_i, \mathcal{U} \right) \right).$$

So,

$$Y \setminus \text{St} \left(\bigcup_{i=1}^k V_i, \mathcal{V} \right) Y \setminus f \left(\text{St} \left(\bigcup_{i=1}^k U_i, \mathcal{U} \right) \right).$$

But

$$Y \setminus f \left(\text{St} \left(\bigcup_{i=1}^k U_i, \mathcal{U} \right) \right) \in f(\mathcal{I}).$$

So, $Y \setminus \text{St}(\bigcup_{i=1}^k V_i, \mathcal{V}) \in f\mathcal{I}$. Therefore, $(Y, \sigma, f(\mathcal{I}))$ is star- $f(\mathcal{I})$ -compact. $\square$

DEFINITION 4.6. A subset $N \subseteq X$ is called nowhere dense subset if $\text{int}(\bar{N}) = \emptyset$.

The collection $\mathcal{N}$ of all nowhere dense subsets of a space X forms an ideal on X , since any subset of a nowhere dense set is nowhere dense, and a finite union of nowhere dense sets is also nowhere dense. An ideal $\mathcal{I}$ on (X, τ) is said to be τ -co-dense, or simply co-dense, if $\tau \cap \mathcal{I} = \{\emptyset\}$.

Now, if $\emptyset \neq A \in \tau$, then $A \subseteq \bar{A}$. So, $\emptyset \neq A = \text{int}(A) \subseteq \text{int}(\bar{A})$. Therefore, $\text{int}(\bar{A}) \neq \emptyset$. So, $A \notin \mathcal{N}$. Thus, $\tau \cap \mathcal{N} = \{\emptyset\}$. Hence, $\mathcal{N}$ is a co-dense ideal on X .

DEFINITION 4.7. A bispaces (X, τ_1, τ_2) is called weakly (τ_1, τ_2) -star-compact if, for every pair $(\mathcal{U}_1, \mathcal{U}_2)$ of τ_1 and τ_2 open cover of X , there exists a finite subset $\mathcal{U}' = \{U_1, U_2, U_3, \dots, U_k\} \subseteq \mathcal{U}_1$ such that

$$\overline{\left(\text{St} \left(\bigcup \mathcal{U}', \mathcal{U}_2 \right) \right)}^{\tau_2} = X.$$

THEOREM 4.8. The following statements are equivalent:

- (1) X is (τ_1, τ_2) -star- $\mathcal{N}_2$ -compact.
- (2) X is weakly (τ_1, τ_2) -star-compact.

Proof. Let $(X, \tau_1, \tau_2, \mathcal{N}_2)$ be (τ_1, τ_2) -star- $\mathcal{N}_2$ -compact, and let $(\mathcal{U}_1, \mathcal{U}_2)$ be a pair of τ_1 and τ_2 open covers of X . Since X is (τ_1, τ_2) -star- $\mathcal{N}_2$ -compact, there exists a finite subset $\mathcal{U}' = \{U_1, U_2, U_3, \dots, U_k\} \subseteq \mathcal{U}_1$ such that $X \setminus \text{St}(\bigcup_{i=1}^k U_i, \mathcal{U}_2) \in \mathcal{N}_2$. Therefore, $X \setminus \text{St}(\bigcup_{i=1}^k U_i, \mathcal{U}_2)$ is a τ_2 nowhere dense subset. Therefore,

$$\overline{\text{int}_{\tau_2} X \setminus \text{St} \left(\bigcup_{i=1}^k U_i, \mathcal{U}_2 \right)}^{\tau_2} = \emptyset.$$

So,

$$\text{int}_{\tau_2} \left(X \setminus \text{St} \left(\bigcup_{i=1}^k U_i, \mathcal{U}_2 \right) \right) = \emptyset.$$

Therefore,

$$\overline{\text{St}\left(\bigcup_{i=1}^k U_i, \mathcal{U}_2\right)}^{\tau_2} = X.$$

Hence, X is weakly (τ_1, τ_2) -star-compact.

Conversely, let X be weakly (τ_1, τ_2) -star-compact, and $(\mathcal{U}_1, \mathcal{U}_2)$ be a pair of τ_1 and τ_2 open covers of X . Since X is weakly (τ_1, τ_2) -star-compact, there exists a finite subset $\mathcal{U}' = \{U_1, U_2, U_3, \dots, U_k\} \subseteq \mathcal{U}_1$ such that

$$\overline{\text{St}\left(\bigcup_{i=1}^k U_i, \mathcal{U}_2\right)}^{\tau_2} = X.$$

Therefore,

$$X \setminus \text{St}\left(\bigcup_{i=1}^k U_i, \mathcal{U}_2\right)$$

does not contain any non-empty τ_2 -open set. Therefore,

$$\text{int}_{\tau_2}\left(X \setminus \text{St}\left(\bigcup_{i=1}^k U_i, \mathcal{U}_2\right)\right) = \emptyset.$$

So,

$$\left(X \setminus \text{St}\left(\bigcup_{i=1}^k U_i, \mathcal{U}_2\right)\right) \in \mathcal{N}_2.$$

Hence, X is (τ_1, τ_2) -star- $\mathcal{N}_2$ -compact. $\square$

THEOREM 4.9. *An ideal bispaces $(X, \tau_1, \tau_2, \mathcal{I})$ is (τ_1, τ_2) -star- $\mathcal{I}$ compact for some co-dense ideal $\mathcal{I}$ if and only if the bispaces (X, τ_1, τ_2) is weakly (τ_1, τ_2) -star-compact.*

Proof. Let $(\mathcal{U}_1, \mathcal{U}_2)$ be a pair of τ_1 and τ_2 open covers of X . Since X is (τ_1, τ_2) -star- $\mathcal{I}$ -compact, there exists a finite subset $\mathcal{U}' = \{U_1, U_2, U_3, \dots, U_k\} \subseteq \mathcal{U}_1$ such that $X \setminus \text{St}(\bigcup_{i=1}^k U_i, \mathcal{U}_2) = I_1 \in \mathcal{I}$. Since $\mathcal{I}$ is a co-dense ideal, so there exists no non-empty τ_2 -open set G such that $G \in I_1$. Hence,

$$\text{int}_{\tau_2}\left(X \setminus \text{St}\left(\bigcup_{i=1}^k U_i, \mathcal{U}_2\right)\right) = \emptyset \implies \overline{X \setminus X \setminus X \setminus \text{St}\left(\bigcup_{i=1}^k U_i, \mathcal{U}_2\right)}^{\tau_2} = \emptyset.$$

Therefore,

$$\overline{\text{St}\left(\bigcup_{i=1}^k U_i, \mathcal{U}_2\right)}^{\tau_2} = X.$$

Hence, the bispaces (X, τ_1, τ_2) is weakly (τ_1, τ_2) -star-compact.

Conversely, let (X, τ_1, τ_2) is weakly (τ_1, τ_2) -star-compact. Then $(X, \tau_1, \tau_2, \mathcal{N}_2)$ is (τ_1, τ_2) -star- $\mathcal{N}_2$ compact, and $\mathcal{N}_2$ is a τ_2 co-dense ideal. Hence, the theorem. $\square$

Observation 1. If $\mathcal{I}, \mathcal{J}$ are two ideals on the same bispaces (X, τ_1, τ_2) with $\mathcal{I} \subseteq \mathcal{J}$, then (τ_1, τ_2) -star- $\mathcal{I}$ -compactness implies (τ_1, τ_2) -star- $\mathcal{J}$ -compactness.

5. Conclusion

In this study, we have introduced and explored the notion of (τ_1, τ_2) -star- $\mathcal{I}$ -compactness within the framework of ideal bispaces. By employing the structure of the quadruple $(X, \tau_1, \tau_2, \mathcal{I})$, we generalized and unified various compactness concepts, such as (τ_1, τ_2) - $\mathcal{I}$ compactness, strongly star- $\mathcal{I}$ compactness, and countably $\mathcal{I}$ compactness. Through carefully constructed counterexamples, the distinctions among these topological constructs have been rigorously validated.

Further, we have characterized the behaviour of subspaces and mappings with respect to the preservation of (τ_1, τ_2) -star- $\mathcal{I}$ compactness. In particular, we established that every real-valued continuous function defined on a (τ_1, τ_2) -star- $\mathcal{I}_{\text{fin}}$ compact space is necessarily bounded, which strengthens the analytical significance of this notion. A finite intersection property-like characterization has also been provided, enhancing the understanding of the internal structure of such spaces.

Finally, the relationship between (τ_1, τ_2) -star- $\mathcal{I}$ compactness and weakly (τ_1, τ_2) -star compactness has been examined via co-dense ideals, revealing deeper interconnections within ideal-theoretic compactness. These findings not only extend classical compactness theory but also open pathways for further investigation into the interplay between ideals, generalized compactness, and bitopological structures.

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REFERENCES

- [1] BAL, P.—BHOWMIK, S.: *Some new star-selection principles in topology*, Filomat, **31** (2017), no. 13, 4041–4050.
- [2] BAL, P.— BHOWMIK, S.: *Star-selection principle: another new direction*, J. Indian Math. Soc. **84** (2017), no. 1–2, 1–6.

- [3] BAL, P.—KOČINAC, L. D. R.: *On selectively star-ccc spaces*, Topology Appl. **281** (2020), Art. no. 107184.
- [4] BAL, P.: *On the class of I - γ open cover and I -St- γ open covers*, Hacettepe J. Math. **52** (2023), no. 3, 630–639.
- [5] BAL, P.—DAS, R.—SARKAR, S.: *On strongly star ideal compactness of topological spaces*, Chebyshevskii Sbornik, **25** (2025), no. 3, 37–46.
- [6] BAL, P.—SARKAR, S.: *On strongly star g -compactness of topological spaces*, Tatra Mt. Math. Publ. **85** (2023), 89–100.
- [7] BAL, P.: *A countable intersection like characterization of Lindelöf and star-Lindelöf spaces*, Researches in Mathematics, **31** (2023), no. 2, 3–7.
- [8] BAL, P.—DATTA, T.: *On statistical star-compactness restricted up to order α* , Konuralp Journal of Mathematics, **13** (2025), no. 2, 175–179.
- [9] CHOUDHURY, T.—BAL, P.: *On star Lindelöf bi-topological spaces*, Ques. Ans. in Gen. Top. **43** (2025), no. 2, 113–122.
- [10] DOUWEN, E. K. V.—REED, G. M.—ROSCOE, A. W.—TREE, I. J.: *Star covering properties*, Topology Appl. **39** (1991), no. 1, 71–103.
- [11] ENGELKING, R.: *General Topology*. Heldermann Verlag, Berlin, 1989.
- [12] KELLY, J. C.: *Bi-topological spaces*, Proc. London Math. Soc. **13** (1963), 71–89.
- [13] KOČINAC, L. D. R.: *On Star Selection Principles Theory*, Axioms, **12** (2023), no. 1, Article no. 93.
- [14] KURATOWSKI, K.: *Topologie I*. Warsaw, 1933.
- [15] MATVEEV, M. V.: *A survey on star-covering properties*, Topol. Atlas (1998), **330**, 80–85.
- [16] MIN, W. K.—KIM, Y. K.: *Quasi generalized open sets and quasi generalized continuity on Bi-generalized topological spaces*, Honam. J. Math. **32**, (2010), no. 4, 619–624.
- [17] RANCIN, D. V.: *Compactness modulo an ideal*, Soviet Math. Dokl. **13** (1972), 193–197.
- [18] SARKAR, S.—BAL, P.—DATTA, M.: *On star Rothberger spaces modulo an ideal*, Applied General Topology, **25** (2024), no. 2, 407–414.
- [19] SONG, Y. K.—XUAN, W. F.: *More on selectively star-ccc spaces*, Topology Appl. **268** (2019), Art. no. 106905.
- [20] VAIDYANATHASWAMY, R.: *Set Topology*, Chelsea Publishing Complex (1946).

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