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A NEW CLASS OF MEAN CONVERGENCE: STRUCTURED MEAN CONVERGENCE

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ABSTRACT. In this paper, we introduce the concept of *structured mean convergence* (SMC), a generalization of Cesàro convergence that allows averaging over selected index subsets rather than the entire sequence. This approach provides a flexible framework for analyzing the asymptotic behaviour of sequences under customized selection criteria. We explore the fundamental properties of SMC and establish its relationships with classical, Cesàro, and statistical convergence. Additionally, we examine its connection to lacunary mean convergence, proving that lacunary mean convergence is a special case of SMC under certain density conditions. We also present counterexamples demonstrating that SMC does not necessarily imply lacunary mean convergence. These results suggest that SMC provides a broader perspective on summability theory and sequence transformations. Potential applications of SMC include functional analysis, ergodic theory, and number theory, where selective summability plays a crucial role.

1. Introduction

Convergence of sequences is a fundamental concept in mathematical analysis, with significant applications in real analysis, functional analysis, and summability theory. The classical notion of convergence, which requires each term of a sequence to approach a fixed limit, has been extended in various directions to capture more flexible modes of asymptotic behaviour. One of the earliest generalizations is Cesàro convergence, introduced by Ernesto Cesàro in 1890, which allows divergent sequences to be assigned limits through the averaging of their partial sums. Later, statistical convergence was introduced by Fast [3].

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In addition, Šalát [11] contributed to the development of statistical convergence by studying its properties for sequences of real numbers, offering early insights into the behaviour of statistically convergent sequences beyond the foundational work of Fast. This line of research was further developed by Fridy [5].

Lacunary summability was formalized by Freedman, Sember, and Raphael [4]. Building on this framework, Fridy and Orhan [6, 7] systematically studied lacunary statistical convergence and lacunary statistical summability, extending statistical techniques to more irregular index structures. The concept of I--convergence, formally introduced by Kostyrko, Šalát, and Wilczyński [8], serves as a cornerstone in the study of ideal-based convergence. It is worth noting that a concept closely related to I-convergence was introduced independently by Nuray and Ruckle [9] under the name of generalized statistical convergence. This formulation coincides in spirit with the ideal convergence framework later formalized by Kostyrko, Šalát, and Wilczyński [8]. Later, Rakshit, Bal, and Debnath [2] expanded this approach by defining I-statistical Cauchy and I^*-statistical Cauchy sequences. While this paper does not directly address I--convergence or ideal-based methods, the inclusion of such concepts in the literature review serves to emphasize the broader landscape of generalized convergence theories. The possible connections between structured mean convergence and I-convergence remain open and could be explored in future work.

In this paper, we introduce a new type of convergence, termed structured mean convergence (SMC), which generalizes Cesàro convergence by allowing selective averaging over index subsets instead of the entire sequence. Unlike Cesàro convergence, which considers the arithmetic mean of all terms, or statistical convergence, which is based on the natural density of the set of indices, SMC allows for custom selection of index subsets, making it adaptable to different analytical and applied settings. This study is based on the works cited in references [1–11]. For further details, readers are encouraged to consult these sources.

2. Structured mean convergence

To formalize the concept of structured mean convergence (SMC), we define it in terms of averages computed over varying subsets of indices. This approach enables a more flexible treatment of convergence, particularly in cases where the sequence exhibits structured or irregular behaviour.

Definition 2.1. Let $\{x_k\}$ be a sequence in $\mathbb{R}$ and $\{A_n\}$ be a family of index subsets satisfying $A_n \subset \{1, 2, \ldots, n\}$ with $|A_n| \to \infty$ as $n \to \infty$. Then, $\{x_k\}$ is said to be **structured mean convergent (SMC)** to $L \in \mathbb{R}$, denoted as

$SMC - \lim x_k = L$, if

$$\lim_{n\to\infty} \frac{1}{|A_n|} \sum_{k\in A_n} x_k = L.$$

The notion of structured mean convergence naturally extends to sequences in normed linear spaces by taking averages and limits in the norm sense.

Throughout the paper, we refer to the family (A_n) as a *structure* on $\mathbb{N}$. In concrete applications, this structure may encode additional arithmetic or combinatorial information: for instance, A_n may be chosen from arithmetic progressions, prime indices, lacunary intervals, or sets with prescribed density properties. In our basic definition we only require $|A_n| \to \infty$; however, in order to obtain stronger results relating SMC to Cesàro or statistical convergence, we impose further conditions on the "regularity" of the sets A_n, such as positive lower density. In this sense, the term "structured" emphasizes that the averaging sets can be adapted to the specific nature of the problem under consideration.

2.1. Relationship with other types of convergence

Cesàro convergence is defined as

$$C - \lim x_k = L \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = L.$$

Theorem 2.2. *Let $\{x_k\}$ be a real sequence which is Cesàro convergent to L, that is,*

$$C_n := \frac{1}{n} \sum_{k=1}^{n} x_k \longrightarrow L.$$

Assume that the selection sets $A_n \subseteq \{1, \dots, n\}$ satisfy

$$\frac{|A_n|}{n} \longrightarrow 1 \qquad (n \to \infty).$$

Then $\{x_k\}$ is structured mean convergent (SMC) to L with respect to (A_n), i.e.,

$$S_n := \frac{1}{|A_n|} \sum_{k\in A_n} x_k \longrightarrow L.$$

P r o o f. Since $C_n \to L$, the sequence (C_n) is bounded. It is a standard fact that Cesàro convergence implies boundedness of the original sequence; hence there exists a constant $M > 0$ such that $|x_k| \le M$ for all $k \in \mathbb{N}$.

For each n, we decompose the sum as

$$\sum_{k\in A_n} x_k = \sum_{k=1}^{n} x_k - \sum_{k\in A_n^c} x_k,$$

and therefore

$$S_n = \frac{1}{|A_n|} \sum_{k\in A_n} x_k = \frac{n}{|A_n|} C_n - \frac{1}{|A_n|} \sum_{k\in A_n^c} x_k.$$

Hence,

$$|S_n - L| \leq \frac{n}{|A_n|}\,|C_n - L| + \left|\frac{n}{|A_n|} - 1\right|\,|L| + \frac{1}{|A_n|}\left|\sum_{k\in A_n^c} x_k\right|.$$

Using the boundedness of (x_k), the last term satisfies

$$\frac{1}{|A_n|}\left|\sum_{k\in A_n^c} x_k\right| \leq \frac{|A_n^c|}{|A_n|}\,M.$$

Since $|A_n|/n \to 1$, we have

$$n/|A_n| \to 1 \quad \text{and} \quad |A_n^c|/|A_n| \to 0 \quad \text{as} \quad n \to \infty.$$

Together with $C_n \to L$, this implies that all three terms on the right-hand side tend to zero, and hence $S_n \to L$. □

Definition 2.3 ([5]). A sequence $\{x_k\}$ is said to be statistically convergent to L if

$$\lim_{n\to\infty} \frac{1}{n}\big|\{k \leq n : |x_k - L| \geq \varepsilon\}\big| = 0, \quad \forall \varepsilon > 0.$$

SMC can be seen as a refinement of statistical convergence where the selected indices A_n may have different densities rather than relying solely on natural density.

Example. Consider the alternating sequence

$$x_k = (-1)^k.$$

This sequence does not converge classically. However, consider the subset $A_n = \{2, 4, 6, \ldots, 2n\}$, which contains the first n even indices. Then,

$$\frac{1}{|A_n|}\sum_{k\in A_n} x_k = \frac{1}{n}\sum_{j=1}^{n}(-1)^{2j} = \frac{1}{n}\sum_{j=1}^{n} 1 = 1.$$

Therefore, the sequence is SMC-convergent to 1 with respect to the family (A_n).

Example. Consider again the sequence $x_k = (-1)^k$, and let A_n be the set of prime indices

$$A_n = \{p \leq n : p \ \text{is prime}\}.$$

Define $\pi(n) = |A_n|$, the number of primes less than or equal to n. Then,

$$\frac{1}{\pi(n)}\sum_{p\leq n,\, p\,\text{prime}} (-1)^p$$

is a sequence of averages taken over the signs corresponding to prime indices. Although $(-1)^k$ is Cesàro convergent to 0 over the full index set, its SMC-average over primes may exhibit different behaviour due to the irregular distribution of primes.

This illustrates that (SMC)-convergence may detect alternative limiting behaviours depending on the chosen structure (A_n), even when Cesàro convergence exists.

Example. Consider the sequence

$$x_k = \begin{cases} 1, & \text{if } k \in A, \\ 2, & \text{otherwise,} \end{cases}$$

where $A \subseteq \mathbb{N}$ is a subset of natural numbers with zero natural density, i.e.,

$$\lim_{n\to\infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} = 0.$$

Then, the sequence (x_k) is statistically convergent to 2, since the proportion of terms differing from 2 vanishes

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : x_k \neq 2\}| = 0.$$

Now, consider the structured subsets

$$A_n = A \cap \{1, 2, \ldots, n\}.$$

Then the structured mean becomes

$$\frac{1}{|A_n|} \sum_{k\in A_n} x_k = \frac{1}{|A_n|} \sum_{k\in A_n} 1 = 1,$$

and hence,

$$\lim_{n\to\infty} \frac{1}{|A_n|} \sum_{k\in A_n} x_k = 1.$$

Thus, while the sequence is statistically convergent to 2, its structured mean convergence with respect to the sparse index family (A_n) leads to the different limit 1.

Theorem 2.4. *If a sequence $\{x_k\}$ is classically convergent to L, then it is structured mean convergent (SMC) to L for every selection set $A_n \subseteq \{1, 2, \ldots, n\}$ satisfying $|A_n| \to \infty$ as $n \to \infty$.*

Proof. Since $x_k \to L$ classically, for every $\varepsilon > 0$ there exists an integer N such that for all $k > N$ we have $|x_k - L| < \varepsilon$.

Let $A_n \subseteq \{1, 2, \ldots, n\}$ be such that $|A_n| \to \infty$. We split the structured average as follows

$$\left| \frac{1}{|A_n|} \sum_{k\in A_n} x_k - L \right| \leq \frac{1}{|A_n|} \sum_{k\in A_n \cap \{k \leq N\}} |x_k - L| + \frac{1}{|A_n|} \sum_{k\in A_n \cap \{k > N\}} |x_k - L|.$$

For the first term, note that the set $\{k \le N\}$ contains at most N indices, and hence the numerator is bounded while the denominator satisfies $|A_n| \to \infty$. Therefore,

$$\frac{1}{|A_n|} \sum_{k \in A_n \cap \{k \le N\}} |x_k - L| \longrightarrow 0.$$

For the second term, since $|x_k - L| < \varepsilon$ for all $k > N$,

$$\frac{1}{|A_n|} \sum_{k \in A_n \cap \{k > N\}} |x_k - L| \le \frac{|A_n \cap \{k > N\}|}{|A_n|} \cdot \varepsilon.$$

We now show that

$$\frac{|A_n \cap \{k > N\}|}{|A_n|} \longrightarrow 1.$$

Indeed, because at most N elements of A_n can lie in $\{k \le N\}$, we have

$$|A_n \cap \{k \le N\}| \le N \qquad \text{and therefore} \qquad \frac{|A_n \cap \{k \le N\}|}{|A_n|} \le \frac{N}{|A_n|} \longrightarrow 0.$$

Hence,

$$\frac{|A_n \cap \{k > N\}|}{|A_n|} = 1 - \frac{|A_n \cap \{k \le N\}|}{|A_n|} \longrightarrow 1\,.$$

Thus the entire second term is asymptotically bounded by ε. Combining both estimates, we obtain

$$\left| \frac{1}{|A_n|} \sum_{k \in A_n} x_k - L \right| \le \varepsilon + o(1).$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\lim_{n \to \infty} \frac{1}{|A_n|} \sum_{k \in A_n} x_k = L.$$

Therefore $\{x_k\}$ is (SMC)-convergent to L. □

Theorem 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and bounded. If $\{x_k\}$ converges classically to L, then for every family of selection sets $A_n \subseteq \{1, 2, \ldots, n\}$ with $|A_n| \to \infty$, the sequence $\{f(x_k)\}$ is structured mean convergent (SMC) to $f(L)$, i.e.,*

$$\lim_{n \to \infty} \frac{1}{|A_n|} \sum_{k \in A_n} f(x_k) = f(L).$$

Proof. Since $x_k \to L$ classically and f is continuous at L, we have $f(x_k) \to f(L)$ in the usual sense. Fix $\varepsilon > 0$. By continuity of f at L, there exists $\delta > 0$ such that

$$|x - L| < \delta \quad \Rightarrow \quad |f(x) - f(L)| < \varepsilon.$$

Because $x_k \to L$, there exists an integer N such that

$$k > N \quad \Rightarrow \quad |x_k - L| < \delta \quad \Rightarrow \quad |f(x_k) - f(L)| < \varepsilon.$$

Let $A_n \subseteq \{1, \ldots, n\}$ with $|A_n| \to \infty$. Then

$$\frac{1}{|A_n|} \sum_{k \in A_n} f(x_k) - f(L) = \frac{1}{|A_n|} \sum_{k \in A_n} \big(f(x_k) - f(L)\big).$$

Split the sum into "initial" and "tail" parts

$$A_n = A_n^{(1)} \cup A_n^{(2)}, \qquad A_n^{(1)} = A_n \cap \{k \le N\}, \qquad A_n^{(2)} = A_n \cap \{k > N\}.$$

Then

$$\begin{aligned} &\left| \frac{1}{|A_n|} \sum_{k \in A_n} \big(f(x_k) - f(L)\big) \right| \\ &\le \left| \frac{1}{|A_n|} \sum_{k \in A_n^{(1)}} \big(f(x_k) - f(L)\big) \right| + \left| \frac{1}{|A_n|} \sum_{k \in A_n^{(2)}} \big(f(x_k) - f(L)\big) \right|. \end{aligned}$$

For the first term, note that $A_n^{(1)} \subseteq \{1, \ldots, N\}$, so

$$\left| \frac{1}{|A_n|} \sum_{k \in A_n^{(1)}} \big(f(x_k) - f(L)\big) \right| \le \frac{|A_n^{(1)}|}{|A_n|} \cdot \max_{1 \le k \le N} |f(x_k) - f(L)|.$$

Since $|A_n| \to \infty$, we have $|A_n^{(1)}| \le N$ and hence

$$\frac{|A_n^{(1)}|}{|A_n|} \le \frac{N}{|A_n|} \longrightarrow 0,$$

so the first term tends to 0.

For the second term, we use the estimate $|f(x_k) - f(L)| < \varepsilon$ for all $k > N$, valid in particular for all $k \in A_n^{(2)}$. Thus

$$\left| \frac{1}{|A_n|} \sum_{k \in A_n^{(2)}} \big(f(x_k) - f(L)\big) \right| \le \frac{|A_n^{(2)}|}{|A_n|} \varepsilon \le \varepsilon.$$

Combining these, we obtain

$$\left| \frac{1}{|A_n|} \sum_{k \in A_n} f(x_k) - f(L) \right| \le \varepsilon + o(1).$$

Letting $n \to \infty$ and then $\varepsilon \to 0$, we conclude that

$$\lim_{n \to \infty} \frac{1}{|A_n|} \sum_{k \in A_n} f(x_k) = f(L),$$

which shows that $\{f(x_k)\}$ is (SMC)-convergent to $f(L)$. $\square$

Theorem 2.6. *Let $T : X \to Y$ be a bounded (not necessarily compact) linear operator between Banach spaces. If $\{x_k\} \subset X$ is structured mean convergent to L, then $\{T(x_k)\} \subset Y$ is structured mean convergent to $T(L)$.*

Proof. By assumption, $\{x_k\}$ is (SMC) to L, that is,

$$\lim_{n\to\infty} \frac{1}{|A_n|} \sum_{k\in A_n} x_k = L$$

in the norm of X. For each n, linearity of T gives

$$\frac{1}{|A_n|} \sum_{k\in A_n} T(x_k) = T\left(\frac{1}{|A_n|} \sum_{k\in A_n} x_k\right).$$

Since T is continuous, we may pass to the limit

$$\lim_{n\to\infty} \frac{1}{|A_n|} \sum_{k\in A_n} T(x_k) = T\left(\lim_{n\to\infty} \frac{1}{|A_n|} \sum_{k\in A_n} x_k\right) = T(L).$$

Hence $\{T(x_k)\}$ is (SMC)-convergent to $T(L)$. □

Theorem 2.7. *Let $\{x_k\} \subset \mathbb{R}$ be a bounded sequence. If $\{x_k\}$ is statistically convergent to L, then it is (SMC)-convergent to L with respect to any family $A_n \subseteq \{1, \dots, n\}$ satisfying*

$$\liminf_{n\to\infty} \frac{|A_n|}{n} > 0.$$

Proof. Since $x_k \to L$ statistically, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |x_k - L| \ge \varepsilon\}| = 0.$$

Hence, the set

$$B_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \ge \varepsilon\}$$

has natural density zero.

Assume that $\{x_k\}$ is bounded. Then $\{x_k - L\}$ is also bounded, i.e., there exists $M > 0$ such that $|x_k - L| \le M$ for all k.

Let $A_n \subseteq \{1, \dots, n\}$ satisfy

$$\liminf_{n\to\infty} \frac{|A_n|}{n} \ge c > 0.$$

Define the "bad set"

$$E_n = A_n \cap B_\varepsilon.$$

Since B_ε has density zero, we have

$$\frac{|E_n|}{n} \longrightarrow 0.$$

Dividing by $|A_n|/n \geq c > 0$, we obtain

$$\frac{|E_n|}{|A_n|} \longrightarrow 0.$$

Now estimate the structured mean

$$\left| \frac{1}{|A_n|} \sum_{k \in A_n} x_k - L \right| \leq \left| \frac{1}{|A_n|} \sum_{k \in A_n \setminus E_n} (x_k - L) \right| + \left| \frac{1}{|A_n|} \sum_{k \in E_n} (x_k - L) \right|.$$

For $k \in A_n \setminus E_n$ we have $|x_k - L| < \varepsilon$, so

$$\left| \frac{1}{|A_n|} \sum_{k \in A_n \setminus E_n} (x_k - L) \right| \leq \varepsilon.$$

For $k \in E_n$, we have $|x_k - L| \leq M$, hence

$$\left| \frac{1}{|A_n|} \sum_{k \in E_n} (x_k - L) \right| \leq \frac{|E_n|}{|A_n|} M.$$

Since $|E_n|/|A_n| \to 0$, this term vanishes. Therefore,

$$\left| \frac{1}{|A_n|} \sum_{k \in A_n} x_k - L \right| \leq \varepsilon + o(1).$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\frac{1}{|A_n|} \sum_{k \in A_n} x_k \longrightarrow L.$$

Thus the sequence is (SMC)-convergent to L. □

Example. Let $A \subset \mathbb{N}$ be a set with natural density zero, and define the sequence (x_k) by

$$x_k = \begin{cases} 1, & \text{if } k \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then (x_k) is statistically convergent to 0, since

$$\frac{1}{n} \left| \{k \leq n : |x_k - 0| \geq \varepsilon\} \right| = \frac{|A \cap \{1, \ldots, n\}|}{n} \to 0 \quad \text{as } n \to \infty.$$

Now, consider the selection set

$$A_n = \{1 \le k \le n : k \equiv 0 \pmod 2\},$$

i.e., the set of even indices up to n. Clearly,

$$A_n \subseteq \{1, \dots, n\} \quad \text{and} \quad \frac{|A_n|}{n} \to \frac{1}{2},$$

so A_n has positive density. By the previous theorem, since (x_k) is statistically convergent to 0 and the selection sets A_n have positive density, it follows that (x_k) is also (SMC)-convergent to 0 with respect to A_n.

3. Relationship between lacunary mean convergence and structured mean convergence

A sequence $\{x_k\}$ is said to be lacunary mean convergent (or θ-mean convergent) to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} x_k = L,$$

where $I_r = (k_{r-1}, k_r]$ are intervals defined by a lacunary sequence

$$\theta = \{k_r\} \quad \text{and} \quad h_r = k_r - k_{r-1} \qquad [4].$$

A particular choice of the selection set A_r that directly leads to lacunary mean convergence is

$$A_r = \{k_{r-1} + 1, k_{r-1} + 2, \dots, k_r\}.$$

In this case, the selected mean simplifies as follows

$$\frac{1}{|A_r|} \sum_{k\in A_r} x_k = \frac{1}{h_r} \sum_{k\in I_r} x_k.$$

Since lacunary mean convergence is defined as

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} x_k = L,$$

it follows that

$$\lim_{r\to\infty} \frac{1}{|A_r|} \sum_{k\in A_r} x_k = L.$$

Thus, under this specific choice, the structured mean convergence (SMC) coincides exactly with lacunary mean convergence.

This observation highlights that (SMC) provides a more general framework that can encompass lacunary mean convergence when the selection set is taken as lacunary intervals. However, for a truly distinct generalization, the selection set A_n should be chosen independently of fixed lacunary intervals, allowing for more flexible averaging mechanisms.

Theorem 3.1. *Let $\{x_k\}$ be lacunary mean convergent to L with respect to a lacunary sequence $\theta = \{k_r\}$, where*

$$I_r = (k_{r-1}, k_r], \qquad h_r = k_r - k_{r-1}.$$

Fix a subset $A \subseteq \mathbb{N}$, and for each r define

$$B_r = A \cap I_r.$$

Assume that

$$\liminf_{r\to\infty} \frac{|B_r|}{h_r} \geq c > 0.$$

Define the structured index sets

$$A_n = \bigcup_{r: I_r \subseteq \{1,\ldots,n\}} B_r.$$

Then the sequence $\{x_k\}$ is structured mean convergent (SMC) to L with respect to the family (A_n).

Proof. Since $\{x_k\}$ is lacunary mean convergent to L, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} x_k = L.$$

For each r, the subset $B_r \subseteq I_r$ satisfies $|B_r| \geq c h_r$ asymptotically. Hence,

$$\frac{1}{|B_r|} \sum_{k\in B_r} x_k = \frac{1}{|B_r|} \left(\sum_{k\in I_r} x_k - \sum_{k\in I_r \setminus B_r} x_k \right).$$

Because the lacunary mean over I_r converges to L, and the proportion of omitted indices

$$|I_r \setminus B_r|/h_r \leq 1 - c < 1,$$

a standard averaging estimate shows that

$$\frac{1}{|B_r|} \sum_{k\in B_r} x_k \longrightarrow L.$$

Now observe that every A_n is a disjoint union of finitely many such blocks B_r corresponding to those intervals $I_r \subseteq \{1, \ldots, n\}$. Since each block average tends to L, and each block contains at least a proportion c of its parent interval, the structured means satisfy

$$\frac{1}{|A_n|} \sum_{k\in A_n} x_k \longrightarrow L.$$

Thus $\{x_k\}$ is (SMC)-convergent to L with respect to (A_n). □

Example. Let $\theta = \{k_r\} = \{2^r\}$, and define the lacunary intervals $I_r = (2^{r-1}, 2^r]$, so that $h_r = 2^{r-1}$. Consider the sequence $\{x_k\}$ defined by

$$x_k = \begin{cases} 2 & \text{if } k \text{ is even,} \\ -2 & \text{if } k \text{ is odd.} \end{cases}$$

In each lacunary interval I_r, approximately half of the terms are even and half are odd. Thus, the lacunary mean becomes

$$\frac{1}{h_r} \sum_{k \in I_r} x_k = \frac{1}{h_r} \left(\frac{h_r}{2} \cdot 2 + \frac{h_r}{2} \cdot (-2) \right) = 0.$$

Hence, the sequence $\{x_k\}$ is lacunary mean convergent to 0.

Now define the selection set

$$A_n = \{k \leq n : k \text{ is even}\}.$$

Since for every r, the intersection $A_n \cap I_r$ contains approximately half of the elements of I_r, we have

$$\frac{|A_n \cap I_r|}{h_r} \approx \frac{1}{2} > 0.$$

Thus, the density condition in the previous theorem is satisfied, and we conclude that the sequence $\{x_k\}$ is also structured mean convergent (SMC) to 0 with respect to the selection sets A_n.

Example. Let $\theta = \{2^r\}$, and consider the sequence $\{x_k\}$ given by

$$x_k = \begin{cases} 1, & \text{if } k = 2^r \text{ for some } r, \\ 0, & \text{otherwise.} \end{cases}$$

Then the sequence is lacunary mean convergent to 0, because in each lacunary interval $I_r = (2^{r-1}, 2^r]$, only the last element $k = 2^r$ is nonzero. Hence,

$$\frac{1}{h_r} \sum_{k \in I_r} x_k = \frac{1}{2^{r-1}} \to 0.$$

Now define the selection set

$$A_n = \{2^r : 2^r \leq n\}.$$

Clearly, each $A_n \cap I_r$ contains exactly one element, while $h_r = 2^{r-1}$, so

$$\frac{|A_n \cap I_r|}{h_r} = \frac{1}{2^{r-1}} \to 0.$$

Thus, the density condition of the previous theorem is violated. Moreover,

$$\frac{1}{|A_n|} \sum_{k \in A_n} x_k = 1,$$

showing that the (SMC)-limit is 1, not 0. Hence, the converse implication does not hold in general.

Conclusion

In this paper, we introduced the concept of structured mean convergence (SMC) as a generalization of classical and statistical convergence methods. By defining convergence with respect to customizable index subsets, (SMC) provides a flexible framework that captures various forms of asymptotic behaviour. We established basic properties of (SMC), explored its relationship with classical, Cesàro, and statistical convergence, and provided illustrative examples to support our theoretical findings. We also showed that lacunary mean convergence can be viewed as a special case of (SMC) under appropriate density assumptions on the selected index sets, while suitable counterexamples demonstrate that the converse implication fails in general.

Although the paper does not focus on ideal-based methods such as I-convergence, the general structure of (SMC) suggests potential connections to these frameworks. Investigating such connections, for instance by characterizing (SMC)-limits via suitable ideals or filters on $\mathbb{N}$, could be a promising direction for future research.

From an applications point of view, (SMC) offers a natural tool for studying averages along structured or sparse subsequences. For instance, in ergodic theory, (SMC) could be used to analyze averages of the form

$$\frac{1}{|A_n|}\sum_{k\in A_n} f(T^k x)$$

along specially chosen index sets A_n, going beyond the classical Cesàro averages. In number theory, (SMC) may provide a framework for understanding the mean behaviour of arithmetic functions over selected subsets of the integers, such as primes, polynomial sequences, or lacunary sets. Future work may also focus on developing (SMC)-Cauchy criteria, studying completeness properties of spaces of (SMC)-convergent sequences under additional restrictions on (A_n), and exploring further interactions with summability theory and operator theory.

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