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WEYL'S CLASSIFICATION OF SINGULAR IMPULSIVE q-STURM-LIOUVILLE EQUATIONS

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ABSTRACT. In this work, the singular impulsive q-Sturm-Liouville equations are studied. Weyl's famous classification was made for equations of this type.

1. Introduction

As well-known, Sturm-Liouville problems play an important role in the solution of partial differential equations by the method of separating their variables. Therefore, solutions of such equations under various boundary conditions have been intensively investigated [13, 14]. Impulsive boundary conditions are also one of the investigated conditions [3–6]. Recently, Annaby and Mansour have been studied q-Sturm-Liouville problems [1]. Later, q-Sturm-Liouville problems with impulsive boundary conditions began to attract the attention of mathematicians. In [7], Çetinkaya studied discontinuous q-Sturm-Liouville problems with eigenparameter-dependent boundary conditions. Some spectral properties are investigated and Green's function is constructed. In [10, 11], Karahan and Mamedov investigated a q-Sturm-Liouville problem with discontinuity conditions. In [12], the author studied the existence of spectral functions of singular impulsive q-Sturm-Liouville problems. In [5], Aygar and Bairamov studied the properties of the scattering function of an impulsive q-difference equation. In [6], Bohner and Cebesoy investigated the locations of the eigenvalues and

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spectral singularities of an operator corresponding to impulsive q-difference equations.

H. Weyl, on the other hand, gave his famous classification for the singular Sturm-Liouville problems in 1910 [15]. In this study, Weyl's classification will be made for q-Sturm-Liouville problems under impulsive boundary conditions in the singular case.

2. Notation

In this section, the basic concepts of q-calculus used in this article will be given. For more detailed information, the textbook [9] can be examined.

Let $q \in (0,1)$ and let $A \subset \mathbb{R}$ be a q-geometric set, i.e., if $q\zeta \in A$ for all $\zeta \in A$. We begin by defining the *operator* $\mathcal{D}_q$ by

$$\mathcal{D}_q f(\zeta) = \begin{cases} \frac{f(q\zeta)-f(\zeta)}{(q-1)\zeta}, & \zeta \neq 0, \\ \lim_{n\to\infty} \frac{f(q^n\xi)-f(0)}{q^n\xi}, & \zeta = 0, \end{cases}$$

where $\zeta, \xi \in A$. We define the *Jackson q-integration* by

$$\int_0^\zeta f(\gamma)\, d_q\gamma = \zeta(1-q)\sum_{n=0}^{\infty} q^n f(q^n\zeta) \ \ (\zeta \in A),$$

provided that the series converges, and

$$\int_a^b f(\gamma)\, d_q\gamma = \int_0^b f(\gamma)\, d_q\gamma - \int_0^a f(\gamma)\, d_q\gamma,$$

where $a, b \in A$. From [8], we have

$$\int_0^\infty f(\gamma)\, d_q\gamma = \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

Through the remainder of the paper, we deal only with functions q-regular at zero, i.e., functions satisfying

$$\lim_{n\to\infty} f(\zeta q^n) = f(0),$$

for every $\zeta \in A$.

3. The singular problem

Consider the following problem

$$\Gamma y := \left[-\frac{1}{q}\mathcal{D}_{\frac{1}{q}}\mathcal{D}_q + r(\zeta) \right] y(\zeta) = \lambda y\left(\zeta\right), \tag{1}$$

with boundary conditions

$$y\left(0\right) \cos\beta + \mathcal{D}_q y\left(0\right) \sin\beta = 0, \tag{2}$$

$$y\left(\frac{1}{q^{\varsigma}}\right) \cos\delta + \mathcal{D}_q y\left(\frac{1}{q^{\varsigma}}\right) \sin\delta = 0, \tag{3}$$

and impulsive conditions

$$Y\left(d+\right) = \Lambda Y\left(d-\right), \tag{4}$$

where

$$\zeta \in (0,d) \cup (d,\infty), \quad Y = \begin{pmatrix} y \\ \mathcal{D}_{\frac{1}{q}} y \end{pmatrix}, \quad \beta,\delta \in \mathbb{R}, \quad \lambda \in \mathbb{C}, \quad \varsigma \in \mathbb{N} := \{1,2,\ldots\}, \quad q^{\varsigma} < d;$$

Λ is the 2×2 matrix with entries from $\mathbb{R}$ and $\det \Lambda = \eta > 0$; r is a real-valued, continuous function on $[0,d) \cup (d,\infty)$, one-sided limits $r\left(d\pm\right)$ exist.

The same problem has been studied by Annaby et al. without impulsive boundary conditions [2].

$$H = L_q^2(0,d) \dot{+} L_q^2\left(d,\infty\right)$$

is a Hilbert space endowed with the following inner product

$$\langle y, z \rangle_H := \int_0^d y^{(1)}\overline{z^{(1)}} d_q\zeta + \frac{1}{\eta}\int_d^\infty y^{(2)}\overline{z^{(2)}} d_q\zeta,$$

where

$$y(\zeta) = \begin{cases} y^{(1)}(\zeta), & \zeta \in (0,d), \\ y^{(2)}(\zeta), & \zeta \in (d,\infty), \end{cases} \qquad z(\zeta) = \begin{cases} z^{(1)}(\zeta), & \zeta \in (0,d), \\ z^{(2)}(\zeta), & \zeta \in (d,\infty). \end{cases}$$

Consider the following sets

$$\mathcal{D}_{\max} = \left\{ y \in H : \begin{array}{c} \text{one-sided limits } y\left(d\pm\right) \text{ and} \\ \mathcal{D}_{\frac{1}{q}} y\left(d\pm\right) \text{ exist and finite,} \\ Y(d+) = \Lambda Y(d-) \text{ and} \\ \Gamma y \in H \end{array} \right\}.$$

Then the *maximal operator* $\mathcal{L}_{\max}$ on $\mathcal{D}_{\max}$ is defined by

$$\mathcal{L}_{\max} y = \Gamma y.$$

Let $y, z \in \mathcal{D}_{\max}$. Then the q-Green formula of these functions is given by

$$\int_0^\zeta \left[(\Gamma y)(t)\overline{z(t)} - y(t)\overline{(\Gamma z)(t)}\right] d_q t$$
$$= [y,z](\zeta) - [y,z](d+) + [y,z](d-) - [y,z](0), \quad (5)$$

where

$$\zeta \in (d,\infty) \quad \text{and} \quad [y,z] := y\left(\overline{\mathcal{D}_{\frac{1}{q}} z}\right) - \left(\mathcal{D}_{\frac{1}{q}} y\right)\overline{z}.$$

Consider

$$\phi(\zeta,\lambda) = \begin{cases} \phi^{(1)}(\zeta), & \zeta \in [0,d), \\ \phi^{(2)}(\zeta), & \zeta \in (d,\infty) \end{cases}$$

and

$$\psi(\zeta,\lambda) = \begin{cases} \psi^{(1)}(\zeta), & \zeta \in [0,d), \\ \psi^{(2)}(\zeta), & \zeta \in (d,\infty) \end{cases}$$

as solutions of the problem (1)–(4) satisfying

$$\begin{aligned} \phi^{(1)}(0,\lambda) &= \sin\beta, & \psi^{(1)}(0,\lambda) &= \cos\beta, \\ \mathcal{D}_q\phi^{(1)}(0,\lambda) &= -\cos\beta, & \mathcal{D}_q\psi^{(1)}(0,\lambda) &= \sin\beta, \end{aligned} \quad (6)$$

where $0 \le \beta < \pi$.

Since the q-Wronskian is given as

$$W_q(y,z) := yD_q z - zD_q y, \quad (7)$$

one can define

$$W_q^{(i)} := W_q\left(\psi^{(i)}, \phi^{(i)}\right), \quad i = 1,2.$$

Thus we obtain

$$W_q^{(1)} = \frac{1}{\eta} W_q^{(2)}.$$

Let $W_q := W_q^{(1)} = \frac{1}{\eta} W_q^{(2)}$. Moreover, by (5), we see that

$$\int_0^\zeta \Gamma y\,\overline{z} d_q t - \int_0^\zeta y\,\overline{\Gamma z} d_q t = W_q(y,\overline{z})\left(\frac{1}{q}\zeta\right) - W_q(y,\overline{z})(0), \quad (8)$$

where $\zeta \in (0,d)$.

Theorem 3.1. *Let ϕ and ψ be two solutions of the boundary-value problem (BVP)* (1), (4)*. Then, $W_q(\phi,\psi)$ is independent of ζ.*

P r o o f. Let $\lambda \in \mathbb{C}$ and $\zeta \in (0,d)$. From (5), we get

$$\int_0^\zeta \Gamma\phi\psi d_q t - \int_0^\zeta \phi\Gamma\psi d_q t = (\lambda-\lambda)\int_0^\zeta \phi(t,\lambda)\psi(t,\lambda) d_q t$$
$$= W_q(\phi,\psi)\left(\frac{1}{q}\zeta,\lambda\right) - W_q(\phi,\psi)(0,\lambda). \tag{9}$$

due to

$$\Gamma\phi = \lambda\phi \quad \text{and} \quad \Gamma\psi = \lambda\psi.$$

Consequently, we obtain

$$W_q(\phi,\psi)\left(\frac{1}{q}\zeta,\lambda\right) = W_q(\phi,\psi)(0,\lambda).$$

Likewise, for $\zeta \in (d,\infty)$, we get

$$W_q(\phi,\psi)\left(\frac{1}{q}\zeta,\lambda\right) = \eta W_q(\phi,\psi)(0,\lambda).$$ □

Lemma 3.2. *Let $\phi(\zeta,\lambda)$ be a solution of the BVP* (1), (4) *and let $\sigma = \operatorname{Im}\lambda$, where $\lambda \in \mathbb{C}$. Then the following relations hold.*

1)
$$2i\sigma \int_0^b |\phi(\zeta,\lambda)|^2 d_q\zeta$$
$$= W_q\left(\phi,\overline{\phi}\right)\left(\frac{1}{q}b,\lambda\right) - W_q\left(\phi,\overline{\phi}\right)(0,\lambda), \tag{10}$$

where $b \in (0,d)$.

2)
$$2i\sigma\left[\int_0^d |\phi(\zeta,\lambda)|^2 d_q\zeta + \frac{1}{\eta}\int_d^b |\phi(\zeta,\lambda)|^2 d_q\zeta\right]$$
$$= W_q\left(\phi,\overline{\phi}\right)\left(\frac{1}{q}b,\lambda\right) - W_q\left(\phi,\overline{\phi}\right)(0,\lambda), \tag{11}$$

where $b \in (d,\infty)$.

P r o o f. The proof is straightforward. □

It is obvious that every solution of (1) can be written as

$$\vartheta(\zeta,\lambda) := \phi(\zeta,\lambda) + m\psi(\zeta,\lambda),$$

where $\zeta \in [0,d) \cup (d,\infty)$ and m is a constant. Then ϑ satisfies (3) if and only if

$$m = -\frac{\Xi\cos\delta + \Omega\sin\delta}{\Phi\cos\delta + \Upsilon\sin\delta}, \tag{12}$$

where

$$\begin{aligned} \Omega &= \mathcal{D}_q\phi\left(\tfrac{1}{q^\varsigma},\lambda\right), & \Upsilon &= \mathcal{D}_q\psi\left(\tfrac{1}{q^\varsigma},\lambda\right), \\ \Xi &= \phi\left(\tfrac{1}{q^\varsigma},\lambda\right), & \Phi &= \psi\left(\tfrac{1}{q^\varsigma},\lambda\right). \end{aligned}$$

From (12), we obtain

$$m = -\frac{\Xi z + \Omega}{\Phi z + \Upsilon}, \tag{13}$$

where $\cot\delta = z$. Thus, m describes a circle

$$C_{\frac{1}{q^\varsigma}} \ (\varsigma \in \mathbb{N}).$$

Now let's calculate the radius and center of this circle.

Theorem 3.3. *Let $\varsigma \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, where $\sigma = \operatorname{Im}\lambda \neq 0$. Then, we have*

$$\Theta_\varsigma(\lambda) = -\frac{W_q\left(\phi,\overline{\psi}\right)\left(\frac{1}{q^\varsigma},\lambda\right)}{W_q\left(\psi,\overline{\psi}\right)\left(\frac{1}{q^\varsigma},\lambda\right)}, \tag{14}$$

$$r_\varsigma(\lambda) = \left(2\sigma\int_0^d |\psi(\zeta,\lambda)|^2\, d_q\zeta + \frac{2\sigma}{\eta}\int_d^{\frac{1}{q^\varsigma}} |\psi(\zeta,\lambda)|^2\, d_q\zeta\right)^{-1}, \tag{15}$$

where Θ_ς is the center of the circle $C_{\frac{1}{q^\varsigma}}$ and r_ς is its radius.

Proof.
It is clear that the center of the circle $C_{\frac{1}{q^\varsigma}}$ is the symmetric point at ∞. Let

$$m(\lambda,z') = \infty, \quad m(\lambda,z'') = \Theta_\varsigma(\lambda).$$

Then we see that

$$z' = \overline{z''}.$$

On the other hand,

$$m(\lambda, z') = \infty \Leftrightarrow z' = -\frac{\mathcal{D}_q \psi\left(\frac{1}{q^\varsigma}, \lambda\right)}{\psi\left(\frac{1}{q^\varsigma}, \lambda\right)}.$$

Hence

$$\begin{aligned}
\Theta_\varsigma(\lambda) &= m\left(\lambda, -\frac{\mathcal{D}_q \psi\left(\frac{1}{q^\varsigma}, \lambda\right)}{\psi\left(\frac{1}{q^\varsigma}, \lambda\right)}\right) \\
&= -\frac{\phi\left(\frac{1}{q^\varsigma}, \lambda\right)\left(-\overline{\frac{\mathcal{D}_q \psi\left(\frac{1}{q^\varsigma}, \lambda\right)}{\psi\left(\frac{1}{q^\varsigma}, \lambda\right)}}\right) + \mathcal{D}_q \phi\left(\frac{1}{q^\varsigma}, \lambda\right)}{\psi\left(\frac{1}{q^\varsigma}, \lambda\right)\left(-\overline{\frac{\mathcal{D}_q \psi\left(\frac{1}{q^\varsigma}, \lambda\right)}{\psi\left(\frac{1}{q^\varsigma}, \lambda\right)}}\right) + \mathcal{D}_q \psi\left(\frac{1}{q^\varsigma}, \lambda\right)} \\
&= -\frac{\phi\left(\frac{1}{q^\varsigma}, \lambda\right)\mathcal{D}_q \psi\left(\frac{1}{q^\varsigma}, \overline{\lambda}\right) - \psi\left(\frac{1}{q^\varsigma}, \overline{\lambda}\right)\mathcal{D}_q \phi\left(\frac{1}{q^\varsigma}, \lambda\right)}{\psi\left(\frac{1}{q^\varsigma}, \lambda\right)\mathcal{D}_q \psi\left(\frac{1}{q^\varsigma}, \overline{\lambda}\right) - \psi\left(\frac{1}{q^\varsigma}, \overline{\lambda}\right)\mathcal{D}_q \psi\left(\frac{1}{q^\varsigma}, \lambda\right)} \\
&= -\frac{W_q(\phi, \overline{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)}{W_q(\psi, \overline{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)},
\end{aligned}$$

where $\varsigma \in \mathbb{N}$. Since $W_q(\phi, \psi)\left(\frac{1}{q^\varsigma}\right) = \eta(\varsigma \in \mathbb{N})$ we conclude that

$$\begin{aligned}
r_n(\lambda) &= \left|\frac{\mathcal{D}_q \phi\left(\frac{1}{q^\varsigma}, \lambda\right)}{\mathcal{D}_q \psi\left(\frac{1}{q^\varsigma}, \lambda\right)} - \frac{W_q(\phi, \overline{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)}{W_q(\psi, \overline{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)}\right| \\
&= \left|\frac{\mathcal{D}_q \psi\left(\frac{1}{q^\varsigma}, \overline{\lambda}\right) W_q(\phi, \psi)\left(\frac{1}{q^\varsigma}, \lambda\right)}{\mathcal{D}_q \psi\left(\frac{1}{q^\varsigma}, \lambda\right) W_q(\psi, \overline{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)}\right| \\
&= \left|\frac{W_q(\phi, \psi)\left(\frac{1}{q^\varsigma}, \lambda\right)}{W_q(\psi, \overline{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)}\right| = \frac{\eta}{\left|W_q(\psi, \overline{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)\right|}.
\end{aligned}$$

By Lemma 3.2, we find

$$W_q(\psi, \overline{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right) = 2i\sigma \int_0^d |\psi(\zeta, \lambda)|^2 \, d_q\zeta + \frac{2i\sigma}{\eta} \int_d^{\frac{1}{q^\varsigma}} |\psi(\zeta, \lambda)|^2 \, d_q\zeta.$$

Hence

$$\left|W_q\left(\psi,\overline{\psi}\left(\frac{1}{q^\varsigma},\lambda\right)\right)\right| = 2\left|\sigma\right|\int_0^d \left|\psi(\zeta,\lambda)\right|^2 d_q\zeta + \frac{2\left|\sigma\right|}{\eta}\int_d^{\frac{1}{q^\varsigma}} \left|\psi(\zeta,\lambda)\right|^2 d_q\zeta,$$

which proves the theorem. □

Theorem 3.4. *Let $\lambda=\rho+i\sigma$, where $\operatorname{Im}\lambda=\sigma>0$. Then the exterior of the circle $C_{\frac{1}{q^\varsigma}}$ $(\varsigma\in\mathbb{N})$ is mapped onto the upper half-plane of the z-plane.*

P r o o f. From (11), we find

$$\begin{aligned}\operatorname{Im}\left\{\frac{\mathcal{D}_q\psi\left(\frac{1}{q^\varsigma},\lambda\right)}{\psi\left(\frac{1}{q^\varsigma},\lambda\right)}\right\} &= \frac{1}{2}i\left\{-\frac{\mathcal{D}_q\psi\left(\frac{1}{q^\varsigma},\lambda\right)}{\psi\left(\frac{1}{q^\varsigma},\lambda\right)}+\frac{\mathcal{D}_q\psi\left(\frac{1}{q^\varsigma},\overline{\lambda}\right)}{\psi\left(\frac{1}{q^\varsigma},\overline{\lambda}\right)}\right\}\\ &= \frac{1}{2}i\frac{W_q\left(\overline{\psi},\psi\right)\left(\frac{1}{q^\varsigma},\lambda\right)}{\left|\psi\left(\frac{1}{q^\varsigma},\lambda\right)\right|^2}\\ &= -\frac{1}{2}i\frac{W_q\left(\psi,\overline{\psi}\right)\left(\frac{1}{q^\varsigma},\lambda\right)}{\left|\psi\left(\frac{1}{q^\varsigma},\lambda\right)\right|^2}\\ &= \frac{\sigma}{\left|\psi\left(\frac{1}{q^\varsigma},\lambda\right)\right|^2}\left(\int_0^d\left|\psi(\zeta,\lambda)\right|^2 d_q\zeta+\frac{1}{\eta}\int_d^{\frac{1}{q^\varsigma}}\left|\psi(\zeta,\lambda)\right|^2 d_q\zeta\right)\\ &> 0\ ,\end{aligned}$$

where $\varsigma\in\mathbb{N}$. □

Theorem 3.5. *Let $\varsigma\in\mathbb{N}$ and $\operatorname{Im}\lambda=\sigma>0$. Then m lies on the circles $C_{\frac{1}{q^\varsigma}}$ if and only if*

$$\int_0^d\left|\phi(\zeta,\lambda)+m\psi(\zeta,\lambda)\right|^2 d_q\zeta+\frac{1}{\eta}\int_d^{\frac{1}{q^\varsigma}}\left|\phi(\zeta,\lambda)+m\psi(\zeta,\lambda)\right|^2 d_q\zeta=\frac{\operatorname{Im} m}{\sigma}, \tag{16}$$

and m is inside the circle $C_{\frac{1}{q^\varsigma}}$ if and only if

$$\int_0^d\left|\phi(\zeta,\lambda)+m\psi(\zeta,\lambda)\right|^2 d_q\zeta+\frac{1}{\eta}\int_d^{\frac{1}{q^\varsigma}}\left|\phi(\zeta,\lambda)+m\psi(\zeta,\lambda)\right|^2 d_q\zeta<\frac{\operatorname{Im} m}{\sigma}.$$

P r o o f. Let $m \in \mathbb{C}$. Then, we obtain

$$\begin{aligned}
&W_q(\phi + m\psi, \overline{\phi + m\psi})(0, \lambda) \\
&= W_q\left(\phi, \overline{\phi}\right)(0, \lambda) + mW_q\left(\psi, \overline{\phi}\right)(0, \lambda) \\
&\quad + \overline{m}W_q\left(\phi, \overline{\psi}\right)(0, \lambda) + |m|^2 W_q\left(\psi, \overline{\psi}\right)(0, \lambda) \\
&= -m + \overline{m} = -2i \operatorname{Im} m.
\end{aligned}$$

From Lemma 3.2, we conclude that

$$\begin{aligned}
&2\sigma \int_0^d |\phi(\zeta, \lambda) + m\psi(\zeta, \lambda)|^2 \, d_q\zeta + \frac{2\sigma}{\eta} \int_d^{\frac{1}{q^\varsigma}} |\phi(\zeta, \lambda) + m\psi(\zeta, \lambda)|^2 \, d_q\zeta \\
&\qquad = \frac{1}{i}\left(W_q(\phi + m\psi, \phi + m\psi)\left(\frac{1}{q^\varsigma}, \lambda\right) + 2i \operatorname{Im} m\right), \quad (17)
\end{aligned}$$

where $\varsigma \in \mathbb{N}$. By Theorem 3.4, for $\sigma > 0$, m is inside $C_{\frac{1}{q^\varsigma}}$ if $\operatorname{Im} z < 0$. By (13), we deduce that

$$m = -\frac{\Xi z + \Omega}{\Phi z + \Upsilon} \;\Leftrightarrow\; z = -\frac{\Omega + m\Upsilon}{\Xi + m\Phi}.$$

Thus

$$\begin{aligned}
i\left(z - \overline{z}\right) &= i\left\{-\frac{\Omega + m\Upsilon}{\Xi + m\Phi} + \frac{\overline{\Omega} + \overline{m\Upsilon}}{\overline{\Xi} + \overline{m\Phi}}\right\} \\
&= i\frac{W_q\left(\phi + m\psi, \overline{\phi + m\psi}\right)\left(\frac{1}{q^\varsigma}, \lambda\right)}{|\Xi + m\Phi|^2},
\end{aligned}$$

where $\varsigma \in \mathbb{N}$. Then, we see that

$$\operatorname{Im} z < 0 \;\Leftrightarrow\; iW_q\left(\phi + m\psi, \overline{\phi + m\psi}\right)\left(\frac{1}{q^\varsigma}, \lambda\right) > 0, \quad (18)$$

where $\varsigma \in \mathbb{N}$. From (17) and (18), we find

$$\int_0^d |\phi(\zeta, \lambda) + m\psi(\zeta, \lambda)|^2 \, d_q\zeta + \frac{1}{\eta}\int_d^{\frac{1}{q^\varsigma}} |\phi(\zeta, \lambda) + m\psi(\zeta, \lambda)|^2 \, d_q\zeta < \frac{\operatorname{Im} m}{\sigma}, \quad (19)$$

where $\varsigma \in \mathbb{N}$. On the other hand, m is on the circle $C_{\frac{1}{q^\varsigma}}$ if and only if $\operatorname{Im} z = 0$. Hence, we obtain

$$W_q\left(\phi + m\psi, \overline{\phi + m\psi}\right)\left(\frac{1}{q^\varsigma}, \lambda\right) = 0, \quad (20)$$

where $\varsigma \in \mathbb{N}$. By virtue of (20) and (17), we get the desired result. $\square$

Theorem 3.6. *Let* $\operatorname{Im}\lambda = \sigma > 0$ *and let* k *be a point such that* $d < \frac{1}{q^k} < \frac{1}{q^\varsigma}$*. Then* $C_{\frac{1}{q^k}} \subset C_{\frac{1}{q^\varsigma}}$*.*

Proof. By (17), we find

$$\int_0^d |\phi(\zeta,\lambda)+m\psi(\zeta,\lambda)|^2\, d_q\zeta + \frac{1}{\eta}\int_d^{\frac{1}{q^k}} |\phi(\zeta,\lambda)+m\psi(\zeta,\lambda)|^2\, d_q\zeta$$

$$< \int_0^d |\phi(\zeta,\lambda)+m\psi(\zeta,\lambda)|^2\, d_q\zeta + \frac{1}{\eta}\int_d^{\frac{1}{q^\varsigma}} |\phi(\zeta,\lambda)+m\psi(\zeta,\lambda)|^2\, d_q\zeta$$

$$< \frac{\operatorname{Im} m}{\sigma},$$

which proves the theorem. □

Corollary 3.7. *As* $\varsigma \to \infty$*, the circles* $C_{\frac{1}{q^\varsigma}}$ *contract either to a circle or a point. These represent limit-circle and limit-point cases, respectively.*

Theorem 3.8. *Let* m *be a point lying on or inside the limiting circle* C_∞*, and*

$$\Psi(\zeta,\lambda) := \phi(\zeta,\lambda) + m\psi(\zeta,\lambda), \quad \operatorname{Im}\lambda = \sigma > 0,$$

be the solution of (1)*. Then we have*

$$\int_0^d |\Psi(\zeta,\lambda)|^2\, d_q\zeta + \frac{1}{\eta}\int_d^\infty |\Psi(\zeta,\lambda)|^2\, d_q\zeta < \infty,$$

i.e., $\Psi(.,\lambda) \in H$*.*

Proof. Let $\varsigma \in \mathbb{N}$ $(q^\varsigma < d)$. Since

$$\int_0^d |\Psi(\zeta,\lambda))|^2\, d_q\zeta + \frac{1}{\eta}\int_d^{\frac{1}{q^\varsigma}} |\Psi(\zeta,\lambda)|^2\, d_q\zeta < \frac{\operatorname{Im} m}{\sigma}, \tag{21}$$

we arrive at

$$\int_0^d |\Psi(\zeta,\lambda)|^2\, d_q\zeta + \frac{1}{\eta}\int_d^\infty |\Psi(\zeta,\lambda)|^2\, d_q\zeta < \frac{\operatorname{Im} m}{\sigma},$$

i.e., $\Psi(.,\lambda) \in H$. □

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REFERENCES

[1] ANNABY, M. H.—MANSOUR Z. S.: *Basic Sturm-Liouville problems*, J. Phys. A **38** (2005), no. 17, 3775–3797.

[2] ANNABY, M. H.—MANSOUR Z. S.—SOLIMAN I. A.: *q-Titchmarsh-Weyl theory: series expansion*, Nagoya Math. J. **205** (2012), 67–118.

[3] AYDEMIR, K.—OLĞAR, H.—MUKHTAROV, O.: *The principal eigenvalue and the principal eigenfunction of a boundary-value-transmission problem.* Turkish J. Math. Comput. Sci. **11** (2019), no. 2, 97–100.

[4] AYDEMIR, K.—OLĞAR, H.—MUKHTAROV, O. SH.—MUHTAROV, F.: *Differential operator equations with interface conditions in modified direct sum spaces*, Filomat **32** (2018), no. 3, 921–931.

[5] AYGAR, Y.—BAIRAMOV, E.: *Scattering theory of impulsive sturm-liouville equation in quantum calculus.* Bulletin of the Malaysian Mathematical Sciences Society, **42** (2019), 3247–3259.

[6] BOHNER, M.—CEBESOY, S.: *Spectral analysis of an impulsive quantum difference operato.* Mathematical Methods in the Applied Sciences, **42** (2019), no. 16, 5331–5339.

[7] ÇETINKAYA, F. A.: *A discontinuous q-fractional boundary value problem with eigenparameter dependent boundary conditions*, Miskolc Mathematical Notes **20** (2019), no. 2, 795–806.

[8] HAHN, W.: *Beiträge zur theorie der heineschen reihen. die 24 integrale der hypergeometrischen q-differenzengleichung. das q-analogon der laplace-transformation.* Mathematische Nachrichten **2** (1949), no. 6, 340–379.

[9] KAC, V. G.—CHEUNG P.: *Quantum Calculus*, Vol. 113. Springer-Verlag, Berlin, 2002.

[10] KARAHAN, D.: . *On a q-analogue of the Sturm-Liouville operator with discontinuity conditions*, Bulletin of Samara State Technical University, Ser. Phys. Math. Sci. Vol. 26 (2022), no. 3, pp. 407–418,

[11] KARAHAN, D.—MAMEDOV, K. R.: *On a q-boundary value problem with discontinuity conditions*, Bulletin of the South Ural State University, Ser. Mathematics, Mechanics, Physics, Vol. 13 (2021), no. (4) pp. 5–12, DOI:10.14529/mmph210401

[12] PALAMUT KOŞAR, N.: *The Parseval identity for q-Sturm-Liouville problems with transmission conditions*, Adv. Difference Equ. **2021** (2021), Paper no. 251, pp. 12.

[13] SARGSJAN, I. S. ET AL.: *Sturm-Liouville and Dirac Operators*. Vol. 59, Springer Science & Business Media, Dordrecht, 2012.

[14] TITCHMARSH, E. C.—WEISS, G.: . *Eigenfunction expansions associated with second--order differential equations*, Part 1. Physics Today, **15** (1962) no. 8, p. 52; https://doi.org/10.1063/1.3058324

[15] WEYL, H.: *Über gewöhnliche differentialgleichungen mit singularitäten und die zugehörigen entwicklungen willkürlicher funktionen*, Math. Annal. textbf68 (1910), no. 2, 220–269.

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