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ON HILBERT IDEAL CONVERGENT SEQUENCE SPACES IN NEUTROSOPHIC NORMED SPACES

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ABSTRACT. The neutrosophic set is a generalization of IFNS and was introduced by Smarandache [17]. In this paper, by using the trianglular Hilbert matrix ($\mathscr{H}$) and I-convergence, we define some new sequence spaces $\mathcal{C}^I_{0(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ and $\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ as a domain of trianglular Hilbert matrix $\mathscr{H}$ with respect to neutrosophic norm and study some results about these sequence spaces.

1. Introduction

L. A Zadeh [20] introduced the notion of fuzzy set theory in 1965. Intuitionistic fuzzy set theory was given by Atanassov [1] and studied the concepts of intuitionistic fuzzy sets (IFS). Park [16] studied the notion of intuitionistic fuzzy metric space in 2004, and the concept of intuitionistic fuzzy norm space was further studied by Saadati and Park [18].

T. Bera and N. K. Mahapatra studied neutrosophic soft linear spaces in 2017 [2] and 2018 [3]. Moreover, Murat Kirişci and Necip Şimşek [12] defined neutrosophic normed space.

H. Fast [4] and H. Steinhaus [19] introduced the theory of statistical convergence of sequences of real numbers independently. Then, in 2000, P. Kostyrko, T. Šalát, M. Mačaj [14] introduced I-convergence as a generalization of statistical convergence. Karakus [7] studied statistical convergence on probabilistic normed space. Recently, Khan et al. [8, 9, 11] investigated ideal convergent for single and

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double sequences in intuitionistic fuzzy normed space. Further, Kirişci [12] studied neutrosophic normed spaces and statistical convergence on it. In this paper, we defined two new sequence spaces $\mathcal{C}^I_{0(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ and $\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ and studied the concepts given by M. Kirişci and N. Şimşek into neutrosophic normed space and obtained some interesting results.

Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural and real numbers, respectively. Let ω denote the set of all real/complex sequences, i.e.,

$$\omega := \{\vartheta = (\vartheta_k), k \in \mathbb{N} \mid \vartheta_k \in \mathbb{R}, \text{ or } \mathbb{C}\}, \tag{1}$$

and sequence space ℓ^∞ is as follows

$$\ell^\infty := \left\{\vartheta = (\vartheta_k) \in \omega : \|\vartheta\|_\infty = \sup_{k\in\mathbb{N}} |\vartheta_k| < \infty\right\}. \tag{2}$$

In this paper, we will use the Hilbert triangular matrix $\mathscr{H}$ as a transformation whose domain is the spaces $\mathcal{C}^I_{0(\mathsf{L},\mathsf{M},\mathsf{N})}$ and $\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}$. Let $\mathscr{H}$ be an infinite Hilbert matrix and be defined as

$$\mathscr{H} = (h_{ik}) \quad \text{such that} \quad h_{ik} = \left(\frac{1}{i+k-1}\right) \quad \text{for } i, k \in \mathbb{N}. \tag{3}$$

Hilbert matrix was introduced by David Hilbert in 1894 to study a question in approximation theory. $\mathscr{H}$ is said to be triangular Hilbert matrix if

$$h_{ik} = 0 \quad \text{for } k > i, \quad \text{and} \quad h_{ii} \neq 0 \quad \text{for all } i \in \mathbb{N}.$$

Hilbert matrix was used in classic theory of summability and considered as a bounded linear operator on the spaces of all p-summable sequences spaces (ℓ^p) with norm

$$\|\mathscr{H}\|_p = \frac{\pi}{\sin(\frac{\pi}{p})} \quad \text{for } 1 < p < \infty \qquad [6].$$

Domain of an infinite matrix H in a sequence space $\mathcal{W}$ is defined as

$$H_p = \{\vartheta = (\vartheta_k) \in \omega : H\vartheta \in \mathcal{W}\}.$$

Throughout the paper, $\mathscr{H}$-transform of the sequence $\vartheta = (\vartheta_k) \in \omega$ is defined as follows:

$$\mathscr{H}_i(\vartheta) = \sum_{\alpha=1}^{i} \frac{\vartheta_\alpha}{i+\alpha-1} \quad \text{for } i, \alpha \in \mathbb{N}. \tag{4}$$

2. Preliminaries

Definition 2.1 ([4]). Suppose that $E \subset \mathbb{N}$ and the symbol $|\cdot|$ denotes the cardinality of the set. It is called the asymptotic density of E, denoted $\Delta(E)$, and is defined as

$$\Delta(E) = \lim_{\alpha\to\infty} \frac{1}{\alpha} |\{k \leq \alpha : k \in E\}|. \tag{5}$$

Definition 2.2 ([4]). Let $\vartheta = (\vartheta_k)$ be a number sequence. Then, the sequence is called statistically convergent to the number ℓ if, for each $\epsilon > 0$, the set $E(\epsilon) = \{k \leq p : |\vartheta_k - \ell| > \epsilon\}$ has $\Delta(E) = 0$, i.e.,

$$\lim_{p\to\infty} \frac{1}{p}\big|\{k \leq p : |\vartheta_k - \ell| \geq \epsilon\}\big| = 0. \tag{6}$$

We denote statistical convergence as $\mathrm{st} - \lim = l$.

Definition 2.3 ([4]). A number sequence $\vartheta = (\vartheta_k)$ is said to be a statistically Cauchy sequence if, for every $\epsilon > 0$, $\exists$ is a positive integer (depends on ϵ) $N = N(\epsilon)$ such that

$$\lim_{p\to\infty} \frac{1}{p}\big|\{k \leq p : |\vartheta_k - \vartheta_N| \geq \epsilon\}\big| = 0. \tag{7}$$

Definition 2.4 ([14]). Consider $X \neq \phi$ and $I \subset P(X)$. Then, I is called ideal if it satiesfies the following properties:

(a) $\phi \in I$,

(b) $\sigma_1, \sigma_2 \in I \implies \sigma_1 \cup \sigma_2 \in I$,

(c) for each $\sigma_1 \in I$ and $\sigma_2 \subset \sigma_1$, we have $\sigma_2 \in I$.

If $X \notin I$, then I is called non-trivial ideal.

Definition 2.5 ([14]). A non-trivial ideal I in X is said to be an admissible ideal if it is different from $P(X)$ and contains all singletons, i.e., $\{x\} \in I$ for each $x\in X$.

Definition 2.6 ([14]). Suppose X is a non-empty set. Then, the non-empty family $F \subset P(X)$ is called the filter on X if and only if

(a) $\phi \notin F$,

(b) $\sigma_1, \sigma_2 \in F \implies \sigma_1 \cap \sigma_2 \in F$,

(c) for each $\sigma_1 \in F$ and $\sigma_2 \supset \sigma_1$, we have $\sigma_2 \in F$.

Definition 2.7 ([14]). Suppose $I \subset P(X)$ is a non-trivial ideal. Then, a class

$$F(I) = \{A \subset X : A = X - M, \text{ for some } M \in I\}$$

is a filter on X, called *filter associated* with the ideal I.

Definition 2.8 ([14]). Suppose $I \subset P(\mathbb{X})$ is a non trivial ideal in X. Then, a number sequence $x = (x_k)$ is said to be ideally convergent (I-convergent) to ξ if, for every $\epsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - \xi| \geq \epsilon\} \in I$, and we write it as $I - \lim x = \xi$.

Definition 2.9 ([10, 15]). The binary operation $\star : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-norm if:

(a) $\star$ is commutative and associative,
(b) $\star$ is continuous,
(c) $s \star 1 = s \ \forall \ s \in [0,1]$,
(d) $s \leq u$ and $p \leq q \Longrightarrow s \star p \leq u \star q$, for each s, p, u, $q \in [0,1]$.

Definition 2.10 ([15]). The binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-conorm if:

(a) $\diamond$ is commutative and associative,
(b) $\diamond$ is continuous,
(c) $s \diamond 0 = s \ \forall \ s \in [0,1]$,
(d) $s \leq u$ and $p \leq q \Longrightarrow s \diamond p \leq u \diamond q$ for each s, p, u, $q \in [0,1]$.

Definition 2.11 ([12]). Let X as a vector space and $\mathcal{N} = \{\langle x, \mathsf{L}(\vartheta), \mathsf{M}(\vartheta), \mathsf{N}(\vartheta)\rangle : \vartheta \in X\}$ be a normed space such that $\mathcal{N} : X \times (0, \infty) \to [0,1]$. Assume that $\star$ and $\diamond$ are a continuous t-norm and a continuous t-conorm, respectively. Then, the four-tuple $(X, \mathcal{N}, \star, \diamond)$ is called a neutrosophic normed space (NNS) if the following conditions are satisfied; for all $\vartheta, \vartheta_0 \in X$ and $\jmath, s > 0$, and for each $\alpha \neq 0$,

(i) $0 \leq \mathsf{L}(\vartheta, \jmath) \leq 1$, $0 \leq \mathsf{M}(\vartheta, \jmath) \leq 1$, $0 \leq \mathsf{N}(\vartheta, \jmath) \leq 1$, $\forall \ \jmath \in R^+$,
(ii) $\mathsf{L}(\vartheta, \jmath) + \mathsf{M}(\vartheta, \jmath) + \mathsf{N}(\vartheta, \jmath) \leq 3$, for $\jmath \in R^+$,
(iii) $\mathsf{L}(\vartheta, \jmath) = 1$ (for $\jmath > 0$) iff $\vartheta = 0$,
(iv) $\mathsf{L}(\alpha\vartheta, \jmath) = \mathsf{L}(\vartheta, \frac{\jmath}{|\alpha|})$,
(v) $\mathsf{L}(\vartheta, \jmath) \star \mathsf{L}(\vartheta_0, s) \leq \mathsf{L}(\vartheta + \vartheta_0, \jmath + s)$,
(vi) $\mathsf{L}(\vartheta, .)$ is a continuous non-decreasing function,
(vii) $\lim_{\jmath \to \infty} \mathsf{L}(\vartheta, \jmath) = 1$,
(viii) $\mathsf{M}(\vartheta, \jmath) = 0$ (for $\jmath > 0$) iff $\vartheta = 0$,
(ix) $\mathsf{M}(\alpha\vartheta, \jmath) = \mathsf{M}(\vartheta, \frac{\jmath}{|\alpha|})$,
(x) $\mathsf{M}(\vartheta, \jmath) \diamond \mathsf{M}(\vartheta_0, s) \geq \mathsf{M}(\vartheta + \vartheta_0, \jmath + s)$,
(xi) $\mathsf{M}(\vartheta, \diamond)$ is a continuous non-increasing function,
(xii) $\lim_{\jmath \to \infty} \mathsf{M}(\vartheta, \jmath) = 0$,
(xiii) $\mathsf{N}(\vartheta, \jmath) = 0$ (for $\jmath > 0$) iff $\vartheta = 0$,
(xiv) $\mathsf{N}(\alpha\vartheta, \jmath) = \mathsf{N}(\vartheta, \frac{\jmath}{|\alpha|})$,
(xv) $\mathsf{N}(\vartheta, \jmath) \diamond \mathsf{N}(\vartheta_), s) \geq \mathsf{N}(\vartheta + \vartheta_), \jmath + s)$,
(xvi) $\mathsf{N}(\vartheta, .)$ is a continuous non-increasing function,
(xvii) $\lim_{\jmath \to \infty} \mathsf{N}(\vartheta, \jmath) = 0$,

(xviii) if $\jmath \leq 0$, then $\mathsf{L}(\vartheta, \jmath) = 0$, $\mathsf{M}(\vartheta, \jmath) = 1$, $\mathsf{N}(\vartheta, \jmath) = 1$.

These three tupples, $\mathcal{N} = \langle \mathsf{L}, \mathsf{M}, \mathsf{N} \rangle$, are called the neutrosophic norm (NN).

Example ([12]). Suppose that $(X, \| \ . \ \|)$ is a NNS. Give the operations as $u_1 \star u_2 = u_1 u_2$ and $u_1 \diamond u_2 = u_1 + u_2 - u_1 u_2$. For $\jmath > ||u_1||$,

$$\mathsf{L}(u_1, \jmath) = \frac{\jmath}{\jmath + ||u_1||}, \quad \mathsf{M}(u_1, \jmath) = \frac{||u_1||}{\jmath + ||u_1||}, \quad (u_1, \jmath) = \frac{||u_1||}{\jmath} \tag{8}$$

for all $u_1, u_2 \in X$, and $\jmath > 0$. If we take $\jmath \leq ||u_1||$, then

$$\mathsf{L}(u_1, \jmath) = 0, \quad \mathsf{M}(u_1, \jmath) = 1 \quad \text{and} \quad \mathsf{N}(u_1, \jmath) = 1. \tag{9}$$

Hence, $(X, \mathcal{N}, \star, \diamond)$ is NNS such that $\mathcal{N} : X \times (0, \infty) \to [0, 1]$.

Definition 2.12. [12] Let $(X, \mathcal{N}, \star, \diamond)$ be a neutrosophic normed space. A sequence $\vartheta = (\vartheta_k) \in X$ is said to be convergent to ξ with respect to the neutrosophic norm $\mathcal{N} = \langle \mathsf{L}, \mathsf{M}, \mathsf{N} \rangle$ if, for every $0 < \epsilon < 1$ and $\jmath > 0$, there exists $K \in \mathbb{N}$ such that $\mathsf{L}(\vartheta_k - \xi, \jmath) > 1 - \epsilon$, $\mathsf{M}(\vartheta_k - \xi, \jmath) < \epsilon$ and $\mathsf{N}(\vartheta_k - \xi, \jmath) < \epsilon$, i.e., for all $\jmath > 0$, we have

$$\lim_{k \to \infty} \mathsf{L}(\vartheta_k - \xi, \jmath) = 1, \quad \lim_{k \to \infty} \mathsf{M}(\vartheta_k - \xi, \jmath) = 0 \quad \text{and} \quad \lim_{k \to \infty} \mathsf{N}(\vartheta_k - \xi, \jmath) = 0. \tag{10}$$

The convergence of a sequence in $(X, \mathcal{N}, \star, \circ)$ is denoted by $\mathcal{N} - \lim \vartheta_k = \xi$.

Definition 2.13 ([12])**.** Let $(X, \mathcal{N}, \star, \diamond)$ be a NNS. A sequence $\vartheta = (\vartheta_k) \in X$ is said to be *Cauchy* sequence with respect to the neutrosophic norm $\mathcal{N} = \langle \mathsf{L}, \mathsf{M}, \mathsf{N} \rangle$ if, for every $0 < \epsilon < 1$ and $\jmath > 0$, there exists $N \in \mathbb{N}$ such that $\mathsf{L}(\vartheta_k - \vartheta_m, \jmath) > 1 - \epsilon$, $\mathsf{M}(\vartheta_k - \vartheta_m, \jmath) < \epsilon$ and $\mathsf{N}(\vartheta_k - \vartheta_m, \jmath) < \epsilon$, for $k, m \geq N$.

Definition 2.14 ([12])**.** Let $(X, \mathcal{N}, \star, \diamond)$ be a neutrosophic normed space. Then, the open ball with center $v \in X$ and radius r is defined as, for $0 < r < 1$, $v \in X$ and $\jmath > 0$,

$$\mathscr{B}_v(r, \jmath) = \{ y \in X : \mathsf{L}(v - y, \jmath) > 1 - r, \quad \mathsf{M}(v - y, \jmath) < r, \quad \mathsf{N}(v - y, \jmath) < r \}. \tag{11}$$

Definition 2.15 ([12])**.** Let $(X, \mathcal{N}, \star, \diamond)$ be a NNS and $Y \subseteq X$. Then, Y is said to be open if, for each $y \in Y$, there exist $t > 0$, $0 < r < 1$ such that $B_y(r, t) \subseteq Y$.

Definition 2.16 ([12])**.** Let $(X, \mathcal{N}, \star, \diamond)$ be an NNS, a sequence $\vartheta = (\vartheta_k) \in X$, $k \in \mathbb{N}$ is said to be *statistically convergent* with respect to the neutrosophic norm $\mathcal{N} = \langle \mathsf{L}, \mathsf{M}, \mathsf{N} \rangle$ if there exists $\xi \in X$ such that the set

$$\mathbb{P}_\epsilon := \{ k \leq n \in \mathbb{N} : \mathsf{L}(\vartheta_k - \xi, \jmath) \leq 1 - \epsilon \ \ or \ \ \mathsf{M}(\vartheta_k - \xi, \jmath) \geq \epsilon, \ \mathsf{N}(\vartheta_k - \xi, \jmath) \geq \epsilon \} \tag{12}$$

or equivalently,

$$\mathbb{P}_\epsilon := \{k \leq n \in \mathbb{N} : \mathsf{L}(\vartheta_k - \xi, \jmath) > 1 - \epsilon \text{ or } \mathsf{M}(\vartheta_k - \xi, \jmath) < \epsilon, \mathsf{N}(\vartheta_k - \xi, \jmath) < \epsilon\}, \tag{13}$$

has $\Delta(\mathbb{P}_\epsilon) = 0$ for every $\epsilon > 0$ and $\jmath > 0$. We write it as

$$\mathbb{S}_{\mathcal{N}} - \lim \vartheta_k = \xi, \quad \text{or} \quad \vartheta_k \to \xi(\mathbb{S}_{\mathcal{N}}).$$

Definition 2.17 ([13]). Let $(X, \mathcal{N}, \star, \diamond)$ be an NNS, a sequence $\vartheta = (\vartheta_k) \in X$, $k \in \mathbb{N}$ is said to be I-convergent to ξ, with respect to the neutrosophic norm

$$\mathcal{N} = \langle \mathsf{L}, \mathsf{M}, \mathsf{N} \rangle$$

if, for every $\epsilon > 0$ and $\jmath > 0$,

$$\{k \in \mathcal{N} : \mathsf{L}(\vartheta_k - \xi, \jmath) \leq 1 - \epsilon \text{ or } \mathsf{M}(\vartheta_k - \xi, \jmath) \geq \epsilon, \ \mathsf{N}(\vartheta_k - \xi, \jmath) \geq \epsilon\} \in I. \tag{14}$$

Definition 2.18 ([8]). A sequence $\vartheta = (\vartheta_n) \in \omega$ is said to be Hilbert I-convergent to a number $\xi \in \mathbb{R}$ if, for every $\epsilon > 0$, the set

$$\{n \in \mathbb{N} : |\mathscr{H}_n(\vartheta) - \xi| \geq \epsilon\} \in I.$$

3. Main Results

Definition 3.1. Let $(X, \mathcal{N}, \star, \diamond)$ be a neutrosophic normed space. A sequence $\vartheta = (\vartheta_k) \in X$ is said to be neutrosophic Hilbert I-convergent to ξ with respect to the neutrosophic norm $\mathcal{N} = \big\langle \mathsf{L}, \mathsf{M}, \mathsf{N} \big\rangle$ if for every $\epsilon \in (0, 1)$ and $\jmath > 0$ we have the set

$$\mathcal{M}_1 = \{\, n \in \mathbb{N} : \mathsf{L}\big(\mathscr{H}_n(\vartheta) - \xi, \jmath\big) \leq 1 - \epsilon \quad \text{or} \quad \mathsf{M}\big(\mathscr{H}_n(\vartheta) - \xi, \jmath\big) \geq \epsilon, \quad \mathsf{N}\big(\mathscr{H}_n(\vartheta) - \xi, \jmath\big) \geq \epsilon \,\} \in I$$

and we write $I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})\text{–}\lim(\vartheta_k) = \xi$.

Definition 3.2. Let $(X, \mathcal{N}, \star, \diamond)$ be a neutrosophic normed space. Then, a sequence $\vartheta = (\vartheta_k) \in X$ is said to be neutrosophic Hilbert I-Cauchy with respect to the neutrosophic norm $\mathcal{N} = \langle \mathsf{L}, \mathsf{M}, \mathsf{N} \rangle$ if, for every $\epsilon \in (0, 1)$ and $\jmath > 0$, there exists a $N \in \mathbb{N}$ such that the set

$$\mathcal{M}_2 = \{\, k \in \mathbb{N} : \mathsf{L}\big(\mathscr{H}_k(\vartheta) - \mathscr{H}_N(\vartheta), \jmath\big) \leq 1 - \epsilon \quad \text{or} \quad \mathsf{M}\big(\mathscr{H}_k(\vartheta) - \mathscr{H}_N(\vartheta), \jmath\big) \geq \epsilon, \quad \mathsf{N}\big(\mathscr{H}_k(\vartheta) - \mathscr{H}_N(\vartheta), \jmath\big) \geq \epsilon \,\} \in I.$$

In this section, we introduce the following sequence spaces:

$$\begin{aligned}\mathcal{C}^{I}_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H}) = \Big\{\vartheta = (\vartheta_k) \in \omega : \big\{i \in \mathbb{N} :\ &\text{for some } \ \xi \in \mathbb{C},\\ &\mathsf{L}\big(\mathscr{H}_i(\vartheta) - \xi, \jmath\big) \leq 1 - \epsilon \quad \text{or}\\ &\mathsf{M}\big(\mathscr{H}_i(\vartheta) - \xi, \jmath\big) \geq \epsilon,\\ &\mathsf{N}\big(\mathscr{H}_i(\vartheta) - \xi, \jmath\big) \geq \epsilon\big\} \in I\Big\}, \quad (15)\end{aligned}$$

$$\begin{aligned}\mathcal{C}^{I}_{0(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H}) = \Big\{\vartheta = (\vartheta_k) \in \omega : \big\{i \in \mathbb{N} :\ &\mathsf{L}\big(\mathscr{H}_i(\vartheta), \jmath\big) \leq 1 - \epsilon \quad \text{or}\\ &\mathsf{M}\big(\mathscr{H}_i(\vartheta), \jmath\big) \geq \epsilon,\\ &\mathsf{N}\big(\mathscr{H}_i(\vartheta), \jmath\big) \geq \epsilon\big\} \in I\Big\}. \quad (16)\end{aligned}$$

We define an open ball with respect to the Hilbert triangular matrix $\mathscr{H}$ with center at ϑ and radius $r > 0$ with respect to the neutrosophic norm and $\epsilon \in (0, 1)$ as follows:

$$\begin{aligned}\mathscr{B}_{\vartheta}(r, \epsilon)(\mathscr{H}) = \Big\{y = (y_k) \in \omega : \big\{i \in \mathbb{N} :\ &\mathsf{L}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(y),\, r\big) \leq 1 - \epsilon \quad \text{or}\\ &\mathsf{M}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(y),\, r\big) \geq \epsilon,\\ &\mathsf{N}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(y),\, r\big) \geq \epsilon\big\} \in I\Big\}. \quad (17)\end{aligned}$$

Theorem 3.3. *The spaces*

$$\mathcal{C}^{I}_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H}) \quad \textit{and} \quad \mathcal{C}^{I}_{0(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H}) \quad \textit{are linear spaces over } \mathbb{R}.$$

Proof. We will prove the result for $\mathcal{C}^{I}_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$. The proof of linearity of the space $\mathcal{C}^{I}_{0(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ follows similarly. Let

$$\vartheta = (\vartheta_k), y = (y_k) \in \mathcal{C}^{I}_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H}).$$

Then, there exist $\xi_1, \xi_2 \in \mathbb{C}$, such that $\vartheta = (\vartheta_k)$ and $y = (y_k)$ Hilbert I–converge to ξ_1 and ξ_2, respectively. We will show that for any scalars ζ_1 and ζ_2 the sequence $\zeta_1\vartheta_k + \zeta_2 y_k$ Hilbert I–converges to $\zeta_1\xi_1 + \zeta_2\xi_2$. For $\jmath > 0$ and $0 < \epsilon < 1$, consider the following sets:

$$\begin{aligned}A = \Bigg\{i \in \mathbb{N} : \mathsf{L}&\left(\mathscr{H}_i(\vartheta) - \xi_1, \frac{\jmath}{2|\zeta_1|}\right) \leq 1 - \epsilon \quad \text{or}\\ \mathsf{M}&\left(\mathscr{H}_i(\vartheta) - \xi_1, \frac{\jmath}{2|\zeta_1|}\right) \geq \epsilon,\\ \mathsf{N}&\left(\mathscr{H}_i(\vartheta) - \xi_1, \frac{\jmath}{2|\zeta_1|}\right) \geq \epsilon\Bigg\} \in I.\end{aligned}$$

$$
\begin{aligned}
A^c = \Big\{ i \in \mathbb{N} : \mathsf{L}\left(\mathscr{H}_i(\vartheta) - \xi_1, \frac{\jmath}{2|\zeta_1|}\right) &> 1-\epsilon \quad \text{or} \\
\mathsf{M}\left(\mathscr{H}_i(\vartheta) - \xi_1, \frac{\jmath}{2|\zeta_1|}\right) &< \epsilon, \\
\mathsf{N}\left(\mathscr{H}_i(\vartheta) - \xi_1, \frac{\jmath}{2|\zeta_1|}\right) &< \epsilon \Big\} \in F(I).
\end{aligned}
$$

$$
\begin{aligned}
B = \Big\{ i \in \mathbb{N} : \mathsf{L}\left(\mathscr{H}_i(y) - \xi_2, \frac{\jmath}{2|\zeta_2|}\right) &\leq 1-\epsilon \quad \text{or} \\
\mathsf{M}\left(\mathscr{H}_i(y) - \xi_2, \frac{\jmath}{2|\zeta_2|}\right) &\geq \epsilon, \\
\mathsf{N}\left(\mathscr{H}_i(y) - \xi_2, \frac{\jmath}{2|\zeta_2|}\right) &\geq \epsilon \Big\} \in I.
\end{aligned}
$$

$$
\begin{aligned}
B^c = \Big\{ i \in \mathbb{N} : \mathsf{L}\left(\mathscr{H}_i(y) - \xi_2, \frac{t}{2|\zeta_2|}\right) &> 1-\epsilon \quad \text{or} \\
\mathsf{M}\left(\mathscr{H}_i(y) - \xi_2, \frac{\jmath}{2|\zeta_2|}\right) &< \epsilon, \\
\mathsf{N}\left(\mathscr{H}_i(y) - \xi_2, \frac{\jmath}{2|\zeta_2|}\right) &< \epsilon \Big\} \in F(I).
\end{aligned}
$$

Define the set $D = A \cup B$ so that $D \in I$. It follows that $\phi \neq D^c \in F(I)$.

Let $q \in D^c$. In this case,

$$
\begin{aligned}
\mathsf{L}\left(\mathscr{H}_q(\vartheta) - \xi_1, \frac{\jmath}{2|\zeta_1|}\right) &> 1-\epsilon \quad \text{or} \\
\mathsf{M}\left(\mathscr{H}_q(\vartheta) - \xi_1, \frac{\jmath}{2|\zeta_1|}\right) &< \epsilon, \\
\mathsf{N}\left(\mathscr{H}_q(\vartheta) - \xi_1, \frac{\jmath}{2|\zeta_1|}\right) &< \epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\mathsf{L}\left(\mathscr{H}_q(y) - \xi_2, \frac{\jmath}{2|\zeta_2|}\right) &> 1-\epsilon \quad \text{or} \\
\mathsf{M}\left(\mathscr{H}_q(y) - \xi_2, \frac{\jmath}{2|\zeta_2|}\right) &< \epsilon, \\
\mathsf{N}\left(\mathscr{H}_q(y) - \xi_2, \frac{\jmath}{2|\zeta_2|}\right) &< \epsilon.
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{L}\Big(\big(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_2\mathscr{H}_q(y)\big)-\big(\zeta_1\xi_1-\zeta_2\xi_2\big),\jmath\Big)\\
&\geq \mathsf{L}\left(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_1\xi_1,\frac{\jmath}{2}\right)\star \mathsf{L}\left(\zeta_2\mathscr{H}_q(y)-\zeta_2\xi_2,\frac{\jmath}{2}\right)\\
&= \mathsf{L}\left(\mathscr{H}_q(\vartheta)-\xi_1,\frac{\jmath}{2|\zeta_1|}\right)\star \mathsf{L}\left(\mathscr{H}_q(y)-\xi_2,\frac{\jmath}{2|\zeta_2|}\right)\\
&> (1-\epsilon)\star(1-\epsilon)=1-\epsilon.
\end{aligned}
$$

This implies that

$$
\mathsf{L}\Big(\big(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_2\mathscr{H}_q(y)br\big)-\big(\zeta_1\xi_1-\zeta_2\xi_2\big),\jmath\Big)>1-\epsilon.
$$

A similar way,

$$
\begin{aligned}
&\mathsf{M}\Big(\big(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_2\mathscr{H}_q(y)\big)-\big(\zeta_1\xi_1-\zeta_2\xi_2\big),\jmath\Big)\\
&\leq \mathsf{M}\left(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_1\xi_1,\frac{\jmath}{2}\right)\diamond \mathsf{M}\left(\zeta_2\mathscr{H}_q(y)-\zeta_2\xi_2,\frac{\jmath}{2}\right)\\
&= \mathsf{M}\left(\mathscr{H}_q(\vartheta)-\xi_1,\frac{\jmath}{2|\zeta_1|}\right)\diamond \mathsf{M}\left(\mathscr{H}_q(y)-\xi_2,\frac{\jmath}{2|\zeta_2|}\right)<\epsilon\diamond\epsilon=\epsilon.
\end{aligned}
$$

$$
\Longrightarrow \mathsf{M}\Big(\big(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_2\mathscr{H}_q(y)\big)-\big(\zeta_1\xi_1-\zeta_2\xi_2\big),\jmath\Big)<\epsilon.
$$

And,

$$
\begin{aligned}
&\mathsf{N}\Big(\big(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_2\mathscr{H}_q(y)\big)-\big(\zeta_1\xi_1-\zeta_2\xi_2\big),\jmath\Big)\\
&\leq \mathsf{N}\left(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_1\xi_1,\frac{\jmath}{2}\right)\diamond \mathsf{N}\left(\zeta_2\mathscr{H}_q(y)-\zeta_2\xi_2,\frac{\jmath}{2}\right)\\
&= \mathsf{N}\left(\mathscr{H}_q(\vartheta)-\xi_1,\frac{\jmath}{2|\zeta_1|}\right)\diamond \mathsf{N}\left(\mathscr{H}_q(y)-\xi_2,\frac{\jmath}{2|\zeta_2|}\right)<\epsilon\diamond\epsilon=\epsilon.
\end{aligned}
$$

$$
\Longrightarrow \mathsf{N}\Big(\big(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_2\mathscr{H}_q(y)\big)-\big(\zeta_1\xi_1-\zeta_2\xi_2\big),\jmath\Big)<\epsilon.
$$

Therefore, we have

$$
\begin{aligned}
D^c\subset\Big\{q\in\mathbb{N}: &\,\mathsf{L}\Big(\big(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_2\mathscr{H}_q(y)\big)-\big(\zeta_1\xi_1-\zeta_2\xi_2\big),\jmath\Big)>1-\epsilon \quad \text{or}\\
&\mathsf{M}\Big(\big(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_2\mathscr{H}_q(y)\big)-\big(\zeta_1\xi_1-\zeta_2\xi_2\big),\jmath\Big)<\epsilon,\\
&\mathsf{N}\Big(\big(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_2\mathscr{H}_q(y)\big)-\big(\zeta_1\xi_1-\zeta_2\xi_2\big),\jmath\Big)<\epsilon\Big\}.
\end{aligned}
$$

By the definition of the *filter associated* with the ideal I, we have

$$
\begin{aligned}
\Big\{q\in\mathbb{N}: &\,\mathsf{L}\Big(\big(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_2\mathscr{H}_q(y)\big)-\big(\zeta_1\xi_1-\zeta_2\xi_2\big),\jmath\Big)>1-\epsilon \quad \text{or}\\
&\mathsf{M}\Big(\big(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_2\mathscr{H}_q(y)\big)-\big(\zeta_1\xi_1-\zeta_2\xi_2\big),\jmath\Big)<\epsilon,\\
&\mathsf{N}\Big(\big(\zeta_1\mathscr{H}_q(\vartheta)-\zeta_2\mathscr{H}_q(y)\big)-\big(\zeta_1\xi_1-\zeta_2\xi_2\big),\jmath\Big)<\epsilon\Big\}\in F(I).
\end{aligned}
$$

Hence, the sequence $(\zeta_1\vartheta_k + \zeta_2 y_k)$ Hilbert I–converges to $\zeta_1\xi_1 + \zeta_2\xi_2$. Therefore, $(\zeta_1\vartheta_i+\zeta_2 y_i) \in \mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$. Hence, $\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ is a linear space. □

Theorem 3.4. *Every open ball $\mathscr{B}_\vartheta(r,\epsilon)(\mathscr{H})$ with center at ϑ and radius $r>0$ with respect to the parameter of the neutrosophic norm and $0<\epsilon<1$ is an open set in $\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$.*

Proof. Consider an open ball with centre at ϑ and radius $r>0$ with neutrosophic parameter $0<\epsilon<1$,

$$\begin{aligned}\mathscr{B}_\vartheta(r,\epsilon)(\mathscr{H}) = \Big\{y=(y_i)\in\omega : \big\{i\in\mathbb{N} : \mathsf{L}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r\big) &\le 1-\epsilon \quad \text{or}\\ \mathsf{M}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r\big) &\ge \epsilon,\\ \mathsf{N}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r\big) &\ge \epsilon\big\} \in I\Big\}.\end{aligned}$$

Then,

$$\begin{aligned}\mathscr{B}^c_\vartheta(r,\epsilon)(\mathscr{H}) = \Big\{y=(y_i)\in\omega : \big\{i\in\mathbb{N} : \mathsf{L}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r\big) &> 1-\epsilon \quad \text{or}\\ \mathsf{M}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r\big) &< \epsilon,\\ \mathsf{N}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r\big) &< \epsilon\big\}\in F(I)\Big\}.\end{aligned}$$

Let $y=(y_k)\in\mathscr{B}^c_\vartheta\big(r,\epsilon\big)(\mathscr{H})$. Then, we have the set such that

$$\begin{aligned}\big\{i\in\mathbb{N} : \mathsf{L}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r\big) &> 1-\epsilon \quad \text{or}\\ \mathsf{M}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_j(y),r\big) &< \epsilon,\\ \mathsf{N}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r\big) &< \epsilon\big\}.\end{aligned}$$

$$\begin{aligned}\mathsf{L}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r\big) &> 1-\epsilon,\\ \mathsf{M}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r\big) &< \epsilon\\ \text{and}\qquad \mathsf{N}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r\big) &< \epsilon\end{aligned}$$

there exists $r_0\in(0,r)$ such that,

$$\begin{aligned}\mathsf{L}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r_0\big) &> 1-\epsilon,\\ \mathsf{M}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r_0\big) &< \epsilon\\ \text{and}\qquad \mathsf{N}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r_0\big) &< \epsilon.\end{aligned}$$

Put

$$\epsilon_0 = \mathsf{L}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_i(y),r_0\big), \implies \epsilon_0 > 1-\epsilon.$$

Then,

$$\exists\ s\in(0,1) \quad \text{such that} \quad \epsilon_0 > 1-s > 1-\epsilon.$$

For $\epsilon_0 > 1-s$, we can have $\epsilon_1, \epsilon_2, \epsilon_3 \in (0,1)$ such that

$$\epsilon_0 \star \epsilon_1 > 1-s, \quad (1-\epsilon_0) \diamond (1-\epsilon_2) < s \quad \text{and} \quad (1-\epsilon_0) \diamond (1-\epsilon_3) < s.$$

Let $\epsilon_4 = \max\{\epsilon_1, \epsilon_2, \epsilon_3\}$.

Now, consider the open ball $\mathscr{B}^c_y(r-r_0, 1-\epsilon_4)(\mathscr{H})$. We will show that

$$\mathscr{B}^c_y(r-r_0, 1-\epsilon_4)(\mathscr{H}) \subset \mathscr{B}^c_\vartheta(r,\epsilon)(\mathscr{H}).$$

Let $z=(z_k) \in \mathscr{B}^c_y(r-r_0, 1-\epsilon_4)(\mathscr{H})$, then we have the set

$$\begin{aligned} \big\{i \in \mathbb{N} : \mathsf{L}\big(\mathscr{H}_i(y) - \mathscr{H}_i(z), r-r_0\big) &> \epsilon_4 \quad \text{or} \\ \mathsf{M}\big(\mathscr{H}_i(y) - \mathscr{H}_i(z), r-r_0\big) &< 1-\epsilon_4, \\ \mathsf{N}\big(\mathscr{H}_i(y) - \mathscr{H}_i(z), r-r_0\big) &< 1-\epsilon_4\big\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathsf{L}\big(\mathscr{H}i(\vartheta) - \mathscr{H}_i(z), r\big) &\geq \mathsf{L}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(y), r_0\big) \star \mathsf{L}\big(\mathscr{H}_i(y) - \mathscr{H}_i(z), r-r_0\big) \\ &\geq \epsilon_0 \star \epsilon_4 \geq \epsilon_0 \star \epsilon_1 \\ &> (1-s) > (1-\epsilon). \end{aligned}$$

This implies that $\mathsf{L}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(z), r\big) > 1-\epsilon$,

$$\begin{aligned} \mathsf{M}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(z), r\big) &\leq \mathsf{M}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(y), r_0\big) \diamond \mathsf{M}\big(\mathscr{H}_i(y) - \mathscr{H}_i(z), r-r_0\big) \\ &\leq (1-\epsilon_0) \diamond (1-\epsilon_4) \leq (1-\epsilon_0) \diamond (1-\epsilon_2) \\ &\leq (s) < \epsilon. \end{aligned}$$

Hence, we get

$$\mathsf{M}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(z), r\big) < \epsilon$$

and

$$\begin{aligned} \mathsf{N}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(z), r\big) &\leq \mathsf{N}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(y), r_0\big) \diamond \mathsf{N}\big(\mathscr{H}_i(y) - \mathscr{H}_i(z), r-r_0\big) \\ &\leq (1-\epsilon_0) \diamond (1-\epsilon_4) \leq (1-\epsilon_0) \diamond (1-\epsilon_3) \\ &\leq (s) < \epsilon. \end{aligned}$$

This implies that

$$\mathsf{N}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(z), r\big) < \epsilon.$$

Therefore, we have a set such that

$$\begin{aligned} \big\{i \in \mathbb{N} : \mathsf{L}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(z), r\big) &> 1-\epsilon \quad \text{or} \\ \mathsf{M}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(z), r\big) &< \epsilon, \\ \mathsf{N}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(z), r\big) &< \epsilon\big\} \in F(I). \end{aligned}$$

Hence, we get $z = (z_k) \in \mathscr{B}^c_\vartheta(r, \epsilon)(\mathscr{H})$. This implies that

$$\mathscr{B}^c_y(r - r_0, 1 - \epsilon_4)(\mathscr{H}) \subset \mathscr{B}^c_\vartheta(r, \epsilon)(\mathscr{H}). \qquad \square$$

Remark 1. The spaces

$$\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H}) \quad \text{and} \quad \mathcal{C}^I_{0(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$$

are NNS with respect to the neutrosophic norm $\mathcal{N} = \langle \mathsf{L}, \mathsf{M}, \mathsf{N} \rangle$.

Now, define

$$\tau_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H}) = \big\{ W \subset \mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H}) : \text{for each } \vartheta = (\vartheta_k) \in W,\ \exists\, r > 0 \text{ and } \epsilon \in (0,1) \text{ such that } \mathscr{B}_\vartheta(r, \epsilon)(\mathscr{H}) \subset W \big\}.$$

Then, $\tau_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ defines a topology on the sequence space $\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$. The collection defined by

$$\mathcal{B} = \Big\{ \mathscr{B}_\vartheta(r, \epsilon) : \vartheta = (\vartheta_k) \in \mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H}), r > 0 \text{ and } \epsilon \in (0,1) \Big\}$$

is a base for the topology $\tau_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ on the space $\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$.

Theorem 3.5. *The topology $\tau_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ on the space $\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ is first countable.*

Proof. For each $\vartheta = (\vartheta_i) \in \mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$, consider the set

$$\mathcal{B} = \left\{ \mathscr{B}_\vartheta\left(\frac{1}{n}, \frac{1}{n}\right)(\mathscr{H}) : n = 1, 2, 3, 4, \ldots \right\},$$

which is a countable local base at ϑ. Therefore, the topology $\tau_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ on the space $\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ is first countable. $\square$

Theorem 3.6. *The spaces $\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ and $\mathcal{C}^I_{0(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ are Hausdorff spaces.*

Proof. We shall prove the result only for $\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ and another one follows similarly. Let $\vartheta = (\vartheta_i)$ and $y = (y_i) \in \mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ such that $\vartheta \neq y$. Then for each $i \in \mathbb{N}$ and $r > 0$, we have,

$$0 < \mathsf{L}\left(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(y), r\right) < 1,$$

$$0 < \mathsf{M}\left(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(y), r\right) < 1,$$

and

$$0 < \mathsf{N}\left(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(y), r\right) < 1. \tag{18}$$

Put

$$\begin{aligned} &\epsilon_1 = \mathsf{L}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(y), r\big), && \epsilon_2 = \mathsf{M}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(y), r\big), \\ &\epsilon_3 = \mathsf{N}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_i(y), r\big) \quad \textit{and} && \epsilon \ = \max\{\epsilon_1, 1-\epsilon_2, 1-\epsilon_3\}. \end{aligned} \tag{19}$$

Then, for each $\epsilon_0 > (\epsilon, 1)$ there exist $\epsilon_4, \epsilon_5, \epsilon_6 \in (0,1)$ such that

$$\epsilon_4 \star \epsilon_4 \geq \epsilon_0, (1-\epsilon_5) \diamond (1-\epsilon_5) \leq (1-\epsilon_0) \quad \text{and} \quad (1-\epsilon_6) \diamond (1-\epsilon_6) \leq (1-\epsilon_0).$$

Again, putting $\epsilon_7 = \max\{\epsilon_4, 1-\epsilon_5, 1-\epsilon_6,\}$, consider the open balls

$$\mathscr{B}_\vartheta\Big(1-\epsilon_7, \frac{r}{2}\Big)(\mathscr{H}) \quad \text{and} \quad \mathscr{B}_y\Big(1-\epsilon_7, \frac{r}{2}\Big)(\mathscr{H})$$

centered at ϑ and y, respectively. We show that

$$\mathscr{B}^c_\vartheta\Big(1-\epsilon_7, \frac{r}{2}\Big)(\mathscr{H}) \cap \mathscr{B}^c_y\Big(1-\epsilon_7, \frac{r}{2}\Big)(\mathscr{H}) = \phi.$$

If possible, let

$$\vartheta = (\vartheta_i) \in \mathscr{B}^c_\vartheta\Big(1-\epsilon_7, \frac{r}{2}\Big)(\mathscr{H}) \cap \mathscr{B}^c_y\Big(1-\epsilon_7, \frac{r}{2}\Big)(\mathscr{H}).$$

Then for the set $\{k \in \mathbb{N}\}$, we have

$$\begin{aligned} \epsilon_1 &= \mathsf{L}\big(\mathscr{H}_k(\vartheta) - \mathscr{H}_k(y), r\big) \\ &\geq \mathsf{L}\left(\mathscr{H}_k(\vartheta) - \mathscr{H}_k(z), \frac{r}{2}\right) \star \mathsf{L}\left(\mathscr{H}_k(z) - \mathscr{H}_k(y), \frac{r}{2}\right) \\ &> \epsilon_7 \star \epsilon_7 \geq \epsilon_4 \star \epsilon_4 \geq \epsilon_0 > \epsilon_1 \end{aligned} \tag{20}$$

$$\begin{aligned} \epsilon_2 &= \mathsf{M}\big(\mathscr{H}_k(\vartheta) - \mathscr{H}_k(y), r\big) \\ &\leq \mathsf{M}\left(\mathscr{H}_k(\vartheta) - \mathscr{H}_k(z), \frac{r}{2}\right) \diamond \mathsf{M}\left(\mathscr{H}_k(z) - \mathscr{H}_k(y), \frac{r}{2}\right) \\ &< (1-\epsilon_7) \diamond (1-\epsilon_7) \leq (1-\epsilon_5) \diamond (1-\epsilon_5) \\ &< (1-\epsilon_0) < \epsilon_2. \end{aligned} \tag{21}$$

And

$$\begin{aligned} \epsilon_3 &= \mathsf{N}\big(\mathscr{H}_k(\vartheta) - \mathscr{H}_k(y), r\big) \\ &\leq \mathsf{N}\left(\mathscr{H}_k(\vartheta) - \mathscr{H}_j(z), \frac{r}{2}\right) \diamond \mathsf{N}\left(\mathscr{H}_k(z) - \mathscr{H}_k(y), \frac{r}{2}\right) \\ &< (1-\epsilon_7) \diamond (1-\epsilon_7) \leq (1-\epsilon_6) \diamond (1-\epsilon_6) \\ &< (1-\epsilon_0) < \epsilon_3. \end{aligned} \tag{22}$$

Equations (20), (21) and (22) imply a contradiction. Therefore,

$$\mathscr{B}^c_\vartheta\Big(1-\epsilon_7, \frac{r}{2}\Big)(\mathscr{H}) \cap \mathscr{B}^c_y\Big(1-\epsilon_7, \frac{r}{2}\Big)(\mathscr{H}) = \phi.$$

Hence, the space $\mathcal{C}^I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H})$ is a Hausdorff space. □

Theorem 3.7. *A sequence $\vartheta = (\vartheta_k) \in \omega$ is a neutrosophic Hilbert I-convergent with respect to the neutrosophic norms $\langle \mathsf{L}, \mathsf{M}, \mathsf{N} \rangle$ if and only if it is a neutrosophic Hilbert I-Cauchy with respect to the same norms.*

Proof. Suppose $\vartheta = (\vartheta_k) \in \omega$ is neutrosophic Hilbert I-convergent with respect to neutrosophic norms $\langle \mathsf{L}, \mathsf{M}, \mathsf{N} \rangle$ such that

$$I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H}) - \lim(\vartheta_k) = \xi.$$

For a given $\epsilon \in (0,1)$, there exists $\epsilon_1 \in (0,1)$ such that

$$(1-\epsilon_1) \star (1-\epsilon_1) > 1-\epsilon \quad \text{and} \quad \epsilon_1 \diamond \epsilon_1 < \epsilon.$$

Since $I_{(\mathsf{L},\mathsf{M},\mathsf{N})}(\mathscr{H}) - \lim(\vartheta_k) = \xi$, therefore for all $\jmath > 0$

$$\begin{aligned} D = \Big\{ i \in \mathbb{N} : \ & \mathsf{L}\left(\mathscr{H}_i(\vartheta) - \xi, \jmath\right) \le 1 - \epsilon_1 \quad \text{or} \\ & \mathsf{M}\big(\mathscr{H}_i(\vartheta) - \xi, \jmath\big) \ge \epsilon_1, \\ & \mathsf{N}\big(\mathscr{H}_i(\vartheta) - \xi, \jmath\big) \ge \epsilon_1 \Big\} \in I \end{aligned}$$

that implies

$$\begin{aligned} D^c = \Big\{ i \in \mathbb{N} : & \mathsf{L}\big(\mathscr{H}_i(\vartheta) - \xi, \jmath\big) > 1 - \epsilon_1 \quad \text{or} \\ & \mathsf{M}\big(\mathscr{H}_i(\vartheta) - \xi, \jmath\big) < \epsilon_1, \\ & \mathsf{N}\big(\mathscr{H}_i(\vartheta) - \xi, \jmath\big) < \epsilon_1 \Big\} \in F(I). \end{aligned}$$

For $i \in D^c$, we have

$$\begin{aligned} & \mathsf{L}\big(\mathscr{H}_i(\vartheta) - \xi, \jmath\big) > 1 - \epsilon_1 \quad \text{or} \\ & \mathsf{M}\big(\mathscr{H}_i(\vartheta) - \xi, \jmath\big) < \epsilon_1, \\ & \mathsf{N}\big(\mathscr{H}_j(y) - \xi, \jmath\big) < \epsilon_1. \end{aligned}$$

For fix $k \in D^c$, let

$$\begin{aligned} Y = \Big\{ i \in \mathbb{N} : & \mathsf{L}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\big) \le 1 - \epsilon \quad \text{or} \\ & \mathsf{M}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\big) \ge \epsilon, \\ & \mathsf{N}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\big) \ge \epsilon \Big\} \in I. \end{aligned}$$

We show that $Y \subset D$. Let $i \in Y$, we have

$$\begin{aligned} & \mathsf{L}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\big) \le 1 - \epsilon \quad \text{or} \\ & \mathsf{M}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\big) \ge \epsilon, \\ & \mathsf{N}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\big) \ge \epsilon. \end{aligned}$$

We have two possible cases, first consider $\mathsf{L}\big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\big) \le 1 - \epsilon$. Then, $\mathsf{L}\big(\mathscr{H}_i(\vartheta) - \xi, \frac{\jmath}{2}\big) \le 1 - \epsilon_1$.

If possible, let $\mathsf{L}\big(\mathscr{H}_i(\vartheta)-\xi,\frac{\jmath}{2}\big)>1-\epsilon_1$. Then,

$$\begin{aligned}1-\epsilon &\geq \mathsf{L}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_k(\vartheta),\jmath\big)\\ &\geq \mathsf{L}\left(\mathscr{H}_i(\vartheta)-\xi,\frac{\jmath}{2}\right)\star\mathsf{L}\left(\mathscr{H}_k(\vartheta)-\xi,\frac{\jmath}{2}\right)\\ &>(1-\epsilon_1)\star(1-\epsilon_1)\\ &>(1-\epsilon),\end{aligned}$$

which is a contradiction.

Similarly, consider

$$\mathsf{M}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_k(\vartheta),\jmath\big)\geq\epsilon,\quad\text{then}\quad\mathsf{M}\left(\mathscr{H}_i(\vartheta)-\xi,\frac{\jmath}{2}\right)\geq\epsilon_1.$$

If possible, suppose

$$\mathsf{M}\Big(\mathscr{H}_i(\vartheta)-\xi,\frac{\jmath}{2}\Big)<\epsilon_1.$$

Hence,

$$\begin{aligned}\epsilon &\leq \mathsf{M}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_k(\vartheta),\jmath\big)\\ &\leq \mathsf{M}\left(\mathscr{H}_i(\vartheta)-\xi,\frac{\jmath}{2}\right)\diamond\mathsf{M}\left(\mathscr{H}_k(\vartheta)-\xi,\frac{\jmath}{2}\right)\\ &<\epsilon_1\diamond\epsilon_1<\epsilon,\end{aligned}$$

which is again a contradiction.

$$\mathsf{N}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_k(\vartheta),\jmath\big)\geq\epsilon,\quad\text{then}\quad\mathsf{N}\left(\mathscr{H}_i(\vartheta)-\xi,\frac{\jmath}{2}\right)\geq\epsilon_1.$$

If possible, suppose

$$\mathsf{N}\left(\mathscr{H}_i(\vartheta)-\xi,\frac{\jmath}{2}\right)<\epsilon_1.$$

Hence,

$$\begin{aligned}\epsilon &\leq \mathsf{N}\big(\mathscr{H}_i(\vartheta)-\mathscr{H}_k(\vartheta),\jmath\big)\\ &\leq \mathsf{N}\left(\mathscr{H}_i(\vartheta)-\xi,\frac{\jmath}{2}\right)\diamond\ \mathsf{N}\left(\mathscr{H}_k(\vartheta)-\xi,\frac{\jmath}{2}\right)\\ &<\epsilon_1\diamond\epsilon_1<\epsilon,\end{aligned}$$

which is again a contradiction. Therefore for $i\in Y$, we have

$$\begin{aligned}\Big\{&\mathsf{L}\left(\mathscr{H}_i(\vartheta)-\xi,\frac{\jmath}{2}\right)\leq(1-\epsilon_1),\\ &\mathsf{M}\left(\mathscr{H}_i(\vartheta)-\xi,\frac{\jmath}{2}\right)\geq\epsilon_1\quad\text{and}\\ &\mathsf{N}\left(\mathscr{H}_i(\vartheta)-\xi,\frac{\jmath}{2}\right)\geq\epsilon_1\Big\}\in I.\end{aligned}$$

This implies that $i\in D$. Hence, $Y\subset D$. Since $D\in I$, then consequently, the sequence $\vartheta=(\vartheta_i)$ is a neutrosophic Hilbert I-Cauchy with respect to the norms $\langle\mathsf{L},\mathsf{M},\mathsf{N}\rangle$.

Conversely, suppose that the sequence $\vartheta = (\vartheta_i) \in \omega$ is neutrosophic Hilbert I-Cauchy with respect to the norms $\langle \mathsf{L}, \mathsf{M}, \mathsf{N} \rangle$ denoted by E. Let, on the contrary, the sequence $\vartheta = (\vartheta_i) \in \omega$ be not neutrosophic Hilbert I-convergent; we denote it by T. Then, there exists $i \in \mathbb{N}$ such that

$$\begin{aligned} E = \Big\{ i \in \mathbb{N} : \mathsf{L}\Big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\Big) \leq 1 - \epsilon \quad \text{or} \\ \mathsf{M}\Big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\Big) \geq \epsilon, \\ \mathsf{N}\Big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\Big) \geq \epsilon \Big\} \in I. \end{aligned}$$

Let, on the contrary,

$$\begin{aligned} T = \Bigg\{ i \in \mathbb{N} : \mathsf{L}\left(\mathscr{H}_i(\vartheta) - \xi, \frac{\jmath}{2}\right) > 1 - \epsilon_1 \quad \text{or} \\ \mathsf{M}\left(\mathscr{H}_i(\vartheta) - \xi, \frac{\jmath}{2}\right) < \epsilon_1, \\ \mathsf{N}\left(\mathscr{H}_j(y) - \xi, \frac{t}{2}\right) < \epsilon_1 \Bigg\} \notin I. \end{aligned}$$

This implies that,

$$\begin{aligned} 1 - \epsilon &\geq \mathsf{L}\Big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\Big) \\ &\geq \mathsf{L}\Big(\mathscr{H}_i(\vartheta) - \xi, \frac{\jmath}{2}\Big) \star \mathsf{L}\Big(\mathscr{H}_k(\vartheta) - \xi, \frac{\jmath}{2}\Big) \\ &> (1 - \epsilon_1) \star (1 - \epsilon_1) \\ &> 1 - \epsilon, \end{aligned}$$

which is a contradiction. Now,

$$\begin{aligned} \epsilon &\leq \mathsf{M}\Big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\Big) \\ &\leq \mathsf{M}\Big(\mathscr{H}_i(\vartheta) - \xi, \frac{\jmath}{2}\Big) \diamond \mathsf{M}\Big(\mathscr{H}_k(\vartheta) - \xi, \frac{\jmath}{2}\Big) \\ &< \epsilon_1 \diamond \epsilon_1 \\ &< \epsilon, \end{aligned}$$

which is again a contradiction. And

$$\begin{aligned} \epsilon &\leq \mathsf{N}\Big(\mathscr{H}_i(\vartheta) - \mathscr{H}_k(\vartheta), \jmath\Big) \\ &\leq \mathsf{N}\Big(\mathscr{H}_i(\vartheta) - \xi, \frac{\jmath}{2}\Big) \diamond \mathsf{N}\Big(\mathscr{H}_k(\vartheta) - \xi, \frac{\jmath}{2}\Big) \\ &< \epsilon_1 \diamond \epsilon_1 \\ &< \epsilon, \end{aligned}$$

which is again a contradiction. Therefore, $T \in I$, and hence, $\vartheta = (\vartheta_i) \in \omega$ is neutrosophic Hilbert I-convergent. $\square$

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