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CESÀRO CONVERGENCE OF DOUBLE SEQUENCES OF COMPLEX UNCERTAIN VARIABLES

Debasish Datta* — Binod Chandra Tripathy

Department of Mathematics, Tripura University, Agartala, West Tripura, INDIA

ABSTRACT. In this article, we introduce the concept of Cesàro summable double sequences of complex uncertain variable. We define the notion for almost sure, mean, measure and distribution. The deviation from the case of single sequences has been discussed providing suitable examples. We prove some basic results and properties.

1. Introduction

Convergence of sequence, as a fundamental theory of mathematics, plays a very important role in probability theory. There are many convergence concepts of random sequences such as convergence in probability, convergence in mean, convergence almost surely, convergence in distribution, etc.

Definition 1.1 ([7])**.** Let L be a σ algebra on a nonempty set Γ. A set function M is called an uncertain measure if it satisfies the following axioms:

Axiom 1: (Normality Axiom). $M\{\Gamma\} = 1$;

Axiom 2: (Duality Axiom). $M\{\Lambda\} + M\{\Lambda^c\} = 1$, for any $\Lambda \in L$;

Axiom 3: (Subadditivity Axiom). For every countable sequence $\{\Lambda_j\} \in L$, we have

$$M\{\cup_{j=1}^{\infty}\Lambda_j\} \leq \Sigma_{j=1}^{\infty} M\{\Lambda_j\}.$$

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*Corresponding author.

The triplet (Γ, L, M) is called an uncertainty space, and each element Λ in L is called an event. To obtain the uncertain measure of a compound event, the product uncertain measure is defined in [7] as follows:

Axiom 4: (Product Axiom.) Let $(\Gamma_k,\ L_k,\ M_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ The product uncertain measure M is an uncertain measure satisfying

$$M\{\Pi_{k=1}^{\infty}\Lambda_k\} = \Lambda_{k=1}^{\infty}M_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1, 2, \ldots,$ respectively.

Definition 1.2 ([7])**.** An uncertain variable ξ is a measurable function from an uncertainty space (Γ, L, M) to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma : \xi(\gamma) \in B\}$ is an event.

Definition 1.3 ([7])**.** The uncertainty distribution ϕ of an uncertain variable ξ is defined by

$$\phi(x) = M\{\xi \le x\}, \quad \text{for all} \quad x \in R.$$

Definition 1.4 ([7])**.** A complex uncertain double sequence space E is said to be *solid* if $\{\eta_{n,k}(\gamma)\zeta_{n,k}(\gamma)\} \in E$, whenever $\{\eta_{n,k}(\gamma)\} \in E$ for all complex numbers $\eta_{n,k}(\gamma)$ with $0 \le |\eta_{n,k}(\gamma)| \le 1$, for all $n, k \in N$.

Definition 1.5. A complex uncertain double sequence space E is said to be *symmetric* if $\{\zeta_{n,k}(\gamma)\} \in E$ implies $\{\xi_{\pi(n,k)}(\gamma)\} \in E$, where π is a permutation of $N \times N$.

Definition 1.6. A complex uncertain double sequence space E is said to be *monotone* if it contains the canonical preimages of all its step spaces.

Definition 1.7. A complex uncertain double sequence space E is said to be *convergence free* if $\{\zeta_{n,k}(\gamma)\} \in E$, whenever $\{\eta_{n,k}(\gamma)\} \in E$, and $\zeta_{n,k}(\gamma) = \theta$, whenever $\eta_{n,k}(\gamma) = \theta$.

2. Mathematical Preliminary

Definition 2.1. A double sequence $\{a_{n,k}\}$ is said to be *Cesàro summable* to L if

$$\lim_{n,k\to\infty} \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} a_{i,j} = L.$$

This implies

$$\left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} a_{i,j} - L \right\| < \varepsilon, \quad \text{for all} \quad n, k \in N.$$

Remark 1. In this case of single sequences, a sequence is convergent means that it is Cesàro summable, but this fails in the case of double sequences. The following double sequence is a null sequence, but it is not Cesàro summable.

$$\begin{bmatrix} 1 & 2^3 & 3^3 & \cdots & n^3 & \cdots \\ 2^3 & \frac{1}{1.2} & \frac{1}{1.3} & \cdots & \frac{1}{1.n} & \cdots \\ 3^3 & \frac{1}{2.1} & \frac{1}{2.2} & \cdots & \frac{1}{2.n} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ n^3 & \frac{1}{n.1} & \frac{1}{n.2} & \cdots & \frac{1}{n.n} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{bmatrix}.$$

Definition 2.2. The complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ is said to be *Cesàro convergent almost sure* in Pringshim's sense to $\zeta(\gamma)$ if there exists an event Λ with $M(\Lambda) = 1$ and for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \zeta_{i,j}(\gamma) - \zeta(\gamma) \right\| < \varepsilon, \quad \text{for all } n, k \geq n_0 \text{ and } \gamma \in \Lambda.$$

Definition 2.3. The complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ is said to be *Cesàro convergent in measure* in Pringshim's sense to $\zeta(\gamma)$ if there exists an event Λ with $M(\Lambda) = 1$ and for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that for $\gamma \in \Lambda$,

$$M\left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \zeta_{i,j}(\gamma) - \zeta(\gamma) \right\| \geq \delta \right\} < \varepsilon, \quad \text{for all } n, k \geq n_0 \text{ and } \gamma \in \Lambda.$$

Definition 2.4. The complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ is said to be *Cesàro convergent in mean* in Pringshim's sense to $\zeta(\gamma)$ if there exists an event Λ with $M(\Lambda) = 1$ and for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$E\left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \zeta_{i,j}(\gamma) - \zeta(\gamma) \right\| \right\} < \varepsilon, \quad \text{for all } n, k \geq n_0 \text{ and } \gamma \in \Lambda.$$

Definition 2.5. If $\phi_{n,k}(\gamma)$ and $\phi(\gamma)$ are distribution functions of $\zeta_{n,k}(\gamma)$ and $\zeta(\gamma)$, then the complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ *Cesàro converges in distribution* to $\zeta(\gamma)$ if for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \phi_{i,j}(\gamma) - \phi(\gamma) \right\| < \varepsilon, \quad \text{for all } n, k \geq n_0 \text{ and } \gamma \in \Lambda.$$

3. Main results

In this section, we present the results of this article. We obtain a relationship between different notions of Cesàro convergence of double sequences of complex uncertain variables.

Theorem 3.1. *If the complex uncertain double sequence* $\{\zeta_{n,k}(\gamma)\}$ *Cesàro converges in mean to* ζ*, then* $\{\zeta_{n,k}(\gamma)\}$ *Cesàro converges in measure to* ζ*.*

P r o o f. It follows from the Markov inequality that for any given number

$$\delta > \eta > 0 \quad \text{and} \quad \varepsilon > 0$$

we have

$$M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}(\gamma)-\zeta(\gamma)\right\| \geq \delta\right\} \leq \frac{1}{\delta}E\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}(\gamma)-\zeta(\gamma)\right\|\right\}.$$

Since $\{\zeta_{n,k}(\gamma)\}$ Cesàro converges in mean to ζ, therefore

$$E\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}(\gamma)-\zeta(\gamma)\right\|\right\} < \eta.$$

Hence,

$$M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}(\gamma)-\zeta(\gamma)\right\| \geq \delta\right\} < \frac{\eta}{\delta} < \varepsilon.$$

This implies

$$M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}(\gamma)-\zeta(\gamma)\right\| \geq \delta\right\} < \varepsilon.$$

Thus, $\{\zeta_{n,k}(\gamma)\}$ Cesàro converges in measure to ζ and the theorem is proved. □

EXAMPLE 3.1. Consider the uncertainty space

$$(\Gamma,\, L,\, M) \quad \text{to be} \quad \{\gamma_1,\gamma_2\} \quad \text{with} \quad M\{\gamma_1\} = M\{\gamma_2\} = \frac{1}{2}.$$

We define complex uncertain variables by

$$\zeta_{n,k}(\gamma) = \begin{cases} n & \text{if } \gamma = \gamma_1, \\ 1+\frac{1}{n+k} & \text{if } \gamma = \gamma_2, \end{cases}$$

for $n, k \in N$. We also define ζ by

$$\zeta(\gamma) = \begin{cases} n & \text{if } \gamma = \gamma_1, \\ 1 & \text{if } \gamma = \gamma_2. \end{cases}$$

Then for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$E\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}(\gamma)-\zeta(\gamma)\right\|\right\} < \varepsilon, \qquad \text{for all } n,k \geq n_0 \text{ and } \gamma \in \Lambda.$$

It follows from Theorem 3.1 that $\{\zeta_{n,k}(\gamma)\}$ Cesàro converges in measure. We can easily get

$$M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}(\gamma)-\zeta(\gamma)\right\| \geq \delta\right\} < \varepsilon, \quad \text{for all } n,k \geq n_0 \text{ and } \gamma \in \Lambda,$$

which implies that $\{\zeta_{n,k}\}$ Cesàro converges in measure to ζ.

Theorem 3.2. *Let the double sequence of complex uncertain variables $\{\zeta_{n,k}\}$ have a real part $\{\xi_{n,k}\}$ and an imaginary part $\{\eta_{n,k}\}$, respectively, for $n,k \in N$. If uncertain double sequences $\{\xi_{n,k}\}$ and $\{\eta_{n,k}\}$ Cesàro converge in measure to ξ and η, respectively, then the complex uncertain double sequence $\{\zeta_{n,k}\}$ Cesàro converges in distribution to $\zeta = \xi + i\eta$.*

Proof. Let $c = a + ib$ be a given continuity point of the complex uncertainty distribution ϕ. On the one hand, for any $\alpha > a$, $\beta > b$ we have

$$\left\{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\xi_{i,j} \leq a, \frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\eta_{i,j} \leq b\right\}$$

$$= \left\{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\xi_{i,j} \leq a, \frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\eta_{i,j} \leq b, \xi \leq \alpha, \eta \leq \beta\right\}$$

$$\cup \left\{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\xi_{i,j} \leq a, \frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\eta_{i,j} \leq b, \xi > \alpha, \eta > \beta\right\}$$

$$\cup \left\{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\xi_{i,j} \leq a, \frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\eta_{i,j} \leq b, \xi \leq \alpha, \eta > \beta\right\}$$

$$\cup \left\{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\xi_{i,j} \leq a, \frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\eta_{i,j} \leq b, \xi > \alpha, \eta \leq \beta\right\}$$

$$\subset \{\xi \leq \alpha, \eta \leq \beta\} \cup \left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \xi_{i,j} - \xi \right\| \geq \alpha - a \right\}$$

$$\cup \left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \eta_{i,j} - \eta \right\| ge\beta - b \right\}.$$

It follows from the subadditivity axiom that

$$\phi_{n,k}(c) = \phi_{i,j}(a+ib) \leq \phi(\alpha + i\beta) + M \left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \xi_{i,j} - \xi \right\| \geq \alpha - a \right\}$$

$$+ M \left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \eta_{i,j} - \eta \right\| \geq \beta - b \right\}.$$

Since, $\{\xi_{n,k}\}$ and $\{\eta_{n,k}\}$ are Cesàro convergent in measure to ξ and η, respectively, we have

$$M \left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \xi_{i,j} - \xi \right\| \geq \alpha - a \right\} < \frac{\varepsilon}{2}$$

and

$$M \left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \eta_{i,j} - \eta \right\| \geq \beta - b \right\} < \frac{\varepsilon}{2}$$

$$\begin{aligned} \phi_{i,j}(c) &= \phi_{i,j}(a+ib) \\ &< \phi(\alpha + i\beta) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \phi(\alpha + i\beta) + \varepsilon. \end{aligned}$$

This implies,

$$\phi_{i,j}(c) < \phi(\alpha + i\beta) + \varepsilon.$$

Thus, we obtain

$$\sup \phi_{i,j}(c) \leq \phi(\alpha + i\beta) \quad \text{for any} \quad \alpha > a, \quad \beta > b.$$

Taking $\alpha + i\beta \to a + ib$, we get

$$\lim_{i,j \to \infty} \sup \phi_{i,j}(c) \leq \phi(c). \tag{1}$$

On the other hand, for any $x < a$, $y < b$, we have

$$\begin{aligned}
\{\xi \leq x, \eta \leq y\} = & \left\{ \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \xi_{i,j} \leq a, \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \eta_{i,j} \leq b, \xi \leq x, \eta \leq y \right\} \\
& \cup \left\{ \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \xi_{i,j} \leq a, \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \eta_{i,j} > b, \xi \leq x, \eta \leq y \right\} \\
& \cup \left\{ \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \xi_{i,j} > a, \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \eta_{i,j} \leq b, \xi \leq x, \eta \leq y \right\} \\
& \cup \left\{ \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \xi_{i,j} > a, \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \eta_{i,j} > b, \xi \leq x, \eta \leq y \right\} \\
& \subset \left\{ \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \xi_{i,j} \leq a, \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \eta_{i,j} \leq b \right\} \\
& \cup \left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \xi_{i,j} - \xi \right\| \geq a - x \right\} \cup \left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \eta_{i,j} - \eta \right\| \geq b - y \right\},
\end{aligned}$$

which implies

$$\begin{aligned}
\phi(x+iy) \leq \phi_{i,j}(a+ib) + M & \left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \xi_{i,j} - \xi \right\| \geq a - x \right\} \\
+ M & \left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \eta_{i,j} - \xi \right\| \geq b - y \right\}.
\end{aligned}$$

Since,

$$M \left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \xi_{i,j} - \xi \right\| \geq a - x \right\} < \frac{\varepsilon}{2}$$

and

$$M \left\{ \left\| \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} \eta_{i,j} - \eta \right\| \geq b - y \right\} < \frac{\varepsilon}{2},$$

we set

$$\phi(x+iy) < \phi_{i,j}(a+ib) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \quad \text{for any} \quad x < a, y < b.$$

$$\phi(x+iy) < \phi_{i,j}(a+ib) + \varepsilon.$$

This implies $\phi(x+iy) \leq \inf \phi_{i,j}(c)$, and from $x+iy \to a+ib$, we get

$$\phi(c) \leq \lim_{i,j\to\infty} \inf \phi_{i,j}(c). \tag{2}$$

It follows from (1) and (2) that

$$\phi_{i,j}(c) \to \phi(c) \quad \text{as} \quad i,j \to \infty.$$

That is, the complex uncertain double sequence $\{\zeta_{n,k}\}$ is Cesàro convergent in distribution to $\zeta = \xi + i\eta$. This establishes the theorem. □

Example 3.2. Cesàro convergence in distribution does not imply Cesàro convergence in measure in general. Take an uncertainty space

$$(\Gamma, L, M) \quad \text{to be} \quad \{\gamma_1, \gamma_2\} \quad \text{with} \quad M\{\gamma_1\} = M\{\gamma_2\} = 1/2.$$

Define a complex uncertain variable by

$$\zeta(\gamma) = \begin{cases} i & \text{if} \quad \gamma = \gamma_1, \\ -i & \text{if} \quad \gamma = \gamma_2. \end{cases}$$

Define $\zeta_{n,k} = -\,\zeta$ for $n \in N$. Then, $\zeta_{n,k}$ and ζ have the same distribution

$$\phi_{n,k}(c) = \phi_{n,k}(a+ib) = \begin{cases} 0 & \text{if} \quad a < 0, & b < +\infty, \\ 0 & \text{if} \quad a \geq 0, & b < \ -1, \\ 1/2 & \text{if} \quad a \geq 0, & -1 \leq b \leq \ \ 1, \\ 1 & \text{if} \quad a \geq 0, & b \geq \ \ 0. \end{cases}$$

Then, $\{\zeta_{n,k}\}$ Cesàro converges in distribution to ζ. However, for given $\delta > 0$ and $\varepsilon > 0$, we have

$$M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}(\gamma) - \zeta(\gamma)\right\| \geq \delta\right\} = M\left\{\gamma\colon \left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j} - \zeta\right\| \geq \delta\right\}$$
$$= 1 > \varepsilon.$$

That is, the double sequence $\{\zeta_{n,k}\}$ does not *Cesàro* converge in measure to ζ.

Theorem 3.3. *Let $\{\zeta_{n,k}\}$ be a double sequence of complex uncertain variables with a real part $\{\xi_{n,k}\}$ and an imaginary part $\{\eta_{n,k}\}$, respectively, for $n, k \in N$. If the uncertain double sequences $\{\xi_{n,k}\}$ and $\{\eta_{n,k}\}$ Cesàro converge in measure to ξ and η, respectively, then the complex uncertain double sequence $\{\zeta_{n,k}\}$ Cesàro converges in measure to $\zeta = \xi + i\eta$.*

P r o o f. It follows from the definition of Cesàro convergence in measure of uncertain double sequence that for any small numbers $\delta > 0$ and $\varepsilon > 0$,

$$M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\xi_{i,j} - \xi\right\| \geq \frac{\delta}{\sqrt{2}}\right\} < \frac{\varepsilon}{2}$$

and

$$M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\eta_{i,j} - \eta\right\| \geq \frac{\delta}{\sqrt{2}}\right\} < \frac{\varepsilon}{2}.$$

We have

$$\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j} - \zeta\right\| = \sqrt{\left(\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\xi_{i,j} - \xi\right)^2 + \left(\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\eta_{i,j} - \eta\right)^2}.$$

We also have

$$\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j} - \zeta\right\| \geq \delta\right\} \subset \left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\xi_{i,j} - \xi\right\| \geq \frac{\delta}{\sqrt{2}}\right\}$$
$$\cup \left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\eta_{i,j} - \eta\right\| \geq \frac{\delta}{\sqrt{2}}\right\}.$$

Using subadditivity axiom of uncertain measure, we obtain

$$M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}(\gamma) - \zeta(\gamma)\right\| \geq \delta\right\} \leq M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\xi_{i,j}(\gamma) - \xi(\gamma)\right\| \geq \frac{\delta}{\sqrt{2}}\right\}$$
$$+ M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\eta_{i,j}(\gamma) - \eta(\gamma)\right\| \geq \frac{\delta}{\sqrt{2}}\right\}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore,

$$M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}(\gamma) - \zeta(\gamma)\right\| \geq \delta\right\} < \varepsilon.$$

That is, $\{\zeta_{n,k}\}$ Cesàro converges in measure to ζ. □

Theorem 3.4. *Let $\{\zeta_{n,k}\}$ be a double sequence of complex uncertain variables. Then, $\{\zeta_{n,k}\}$ Cesàro converges a.s. to ζ if and only if for any δ, $\varepsilon > 0$ we have*

$$M\left(\bigcap_{p=q=1}^{\infty}\bigcup_{n=p}^{\infty}\bigcup_{k=q}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\| \geq \delta \right\}\right) < \varepsilon.$$

P r o o f. By the definition of Cesàro convergence a.s., we have that there exists an event Λ with $M\{\Lambda\} = 1$ such that

$$\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}(\gamma)-\zeta(\gamma)\right\| < \varepsilon.$$

Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\gamma \in \Lambda$, that is equivalent to

$$M\left(\bigcup_{p=q=1}^{\infty}\bigcap_{n=p}^{\infty}\bigcap_{k=q}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\| < \delta\right\}\right) < 1-\varepsilon.$$

It follows from the duality axiom of uncertain measure that

$$M\left(\bigcap_{p=q=1}^{\infty}\bigcup_{n=p}^{\infty}\bigcup_{k=q}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\| \geq \delta\right\}\right) < \varepsilon.$$

□

This establishes the theorem.

Theorem 3.5. *Let $\{\zeta_{n,k}(\gamma)\}$ be a double sequence of complex uncertain variables. Then, $\{\zeta_{n,k}(\gamma)\}$ Cesàro converges uniformly almost surely to ζ if and only if*

$$M\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\| \geq \delta\right\}\right) < \varepsilon.$$

P r o o f. If $\{\zeta_{n,k}(\gamma)\}$ Cesàro converges uniformly almost surely to ζ, then for any $\delta > 0$ there exists B such that $M\{B\} < \eta$, and $\{\zeta_{n,k}(\gamma)\}$ Cesàro converges uniformly to ζ on $\Gamma - B$. Thus, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}(\gamma)-\zeta(\gamma)\right\| < \delta,$$

where $n, k \geq n_0$ and $\gamma \in B$.

That is,

$$\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\|\geq\delta\right\}\subset B.$$

It follows from the subadditivity axiom of uncertain measure that

$$M\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\|\geq\delta\right\}\right)\leq M\{B\}<\eta<\varepsilon.$$

This implies

$$M\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\|\geq\delta\right\}\right)<\varepsilon.$$

On the contrary, if

$$M\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\|\geq\delta\right\}\right)<\varepsilon$$

for any $\delta>0$, then for any given $\eta>0$ and $p\geq 1$ there exists m_k such that

$$M\left(\bigcup_{n=m_p}^{\infty}\bigcup_{k=m_p}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\|\geq\delta\right\}\right)<\frac{n}{2^p}.$$

Let

$$B=\bigcup_{p=1}^{\infty}\bigcup_{n=m_p}^{\infty}\bigcup_{k=m_p}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\|\geq\delta\right\}.$$

Then,

$$M\{B\}\leq\sum_{p=1}^{\infty}M\left(\bigcup_{n=m_p}^{\infty}\bigcup_{k=m_p}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\|\geq\delta\right\}\right)\leq\sum_{p=1}^{\infty}\frac{\eta}{2^p}=\eta.$$

Furthermore, we have

$$\sup_{\gamma\in\Gamma-B}\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{n,k}(\gamma)-\zeta(\gamma)\right\|<\delta.$$

This establishes the result. □

Theorem 3.6. *If the complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ Cesàro converges uniformly almost surely to ζ, then $\{\zeta_{n,k}(\gamma)\}$ Cesàro converges almost surely to ζ.*

P r o o f. It follows from Theorem 3.5 that if $\{\zeta_{n,k}(\gamma)\}$ Cesàro converges uniformly almost surely to ζ, then

$$M\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\| \geq \delta \right\}\right) < \varepsilon.$$

Since,

$$\begin{aligned} & M\left(\bigcap_{p=q=1}^{\infty}\bigcup_{n=p}^{\infty}\bigcup_{k=q}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\| \geq \delta \right\}\right) \\ \leq \ & M\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\| \geq \delta \right\}\right) < \varepsilon. \end{aligned}$$

This implies

$$M\left(\bigcap_{p=q=1}^{\infty}\bigcup_{n=p}^{\infty}\bigcup_{k=q}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\| \geq \delta\right\}\right) < \varepsilon.$$

Hence, $\{\zeta_{n,k}\}$ Cesàro converges a.s. to ζ. This establishes the result. □

Theorem 3.7. *If the complex uncertain double sequence $\{\zeta_{n,k}\}$ Cesàro converges uniformly almost surely to ζ, then $\{\zeta_{n,k}\}$ Cesàro converges in measure to ζ.*

P r o o f. If $\{\zeta_{n,k}\}$ Cesàro converges uniformly almost surely to ζ, then from Theorem 3.5 we have

$$M\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\| \geq \delta\right\}\right) < \varepsilon$$

and

$$M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\| \geq \delta\right\} \leq M\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\| \geq \delta\right\}\right) < \varepsilon.$$

This implies

$$M\left\{\left\|\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{i,j}-\zeta\right\|\geq\delta\right\}<\varepsilon.$$

Therefore, $\{\zeta_{n,k}\}$ Cesàro converges in measure to ζ. □

4. Conclusion

In this article, we have introduced the notion of Cesàro convergence of double sequences of complex uncertain variables. We have established the relationship between different types of concepts of convergence, such as mean, measure, almost sure, converges in distribution, general convergence, etc. We have provided and discussed counter examples in this article.

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Debasish Datta
Binod Chandra Tripathy
Department of Mathematics
Tripura University
IN-799022 Agartala
West Tripura
INDIA
E-mail: debasishdatta_math@rediffmail.com
debasishdattamath@gmail.com
tripathybc@yahoo.com
tripathybc@gmail.com