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# NEW FIXED POINT THEOREM ON EXTENDED $b$-METRIC SPACE USING A CLASS OF CONTRACTION MAPPINGS

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ABSTRACT. The main purpose of this paper is to investigate a fixed point for a class of contractive maps over a complete extended $b$-metric space. In this article, we used the Bianchini-type contraction and BK-type contractive mapping to find new results in the direction of a complete extended $b$-metric space. To illustrate our results, we give several examples.

## 1. Introduction

Fixed point theory is currently one of the most crucial tools in many scientific fields, including applied science, computer science, engineering, and the growth of nonlinear analysis. In this field, one of the most effective tools is the Banach contraction theorem "Suppose $T$ be a mapping on a complete metric space $(X, d)$ into itself satisfying

$$d(Tx, Ty) \le Kd(x, y) \tag{1}$$

where $K \in [0, 1)\ \forall\ x, y \in X$, then $T$ has a unique fixed point on $X$", which was introduced by Banach in 1922 [4]. Following that, this result was expanded by numerous authors using various contractions and mappings in various metric spaces.


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In addition, in 1968, R. Kannan [14] established that: "Let $(X, d)$ be a complete metric space and $S : X \to X$ be a mapping such that there exists $\alpha \in \left[0, \frac{1}{2}\right)$. The inequality holds for all $x, y \in X$

$$d(Sx, Sy) = \alpha\big(d(x, Sx) + d(y, Sy)\big). \tag{2}$$

Then, there exists a unique fixed point of $S$ on $X$".

Few years later, in 1972, Bianchini [7] established that if $S : X \to X$ is a mapping such that $h \in [0, 1)$, then for all $x, y \in X$, the inequality holds. The inequality

$$d(Sx, Sy) = h \max\big(d(x, Sx), d(y, Sy)\big) \tag{3}$$

is satisfied, then there exists a unique fixed point of $S$ over $X$.

In 1989, the counterpart of the Banach contraction principle was proven in $b$--metric space, which Bakhtin [5] presented as a generalization of metric space. Later, the outcomes of $b$-metric spaces were expanded upon by Czerwik [11] in 1993. Numerous researchers illustrated the famous Banach fixed point theorem in the $b$-metric space using this concept.

In 2009, the fixed point theory for multivalued generalized contraction on a set with two $b$-metric was studied by Boriceanu et al. [10].

S. Agrawal, K. Qureshi, and Jyoti Nema [3] investigated the fixed point theorem by using a class of contractive mappings for $b$-metric space in 2016. Recently in 2024, Bhattacharjee et al. [8] studied the Rus contraction in complete $b$-metric space. Many authors are currently investigating a great deal of new information and developing new fixed point theorems over complete $b$-metric space.

Initially, the notion of extended $b$-metric space was presented by Kamran et al. [15] in 2017, representing a noteworthy expansion of $b$-metric space. By weakening the triangle inequality even more, they achieved this generalized spaces.

Following them, a couple of authors continue investigating the extended $b$--metric space, for instance: common fixed point findings on an extended $b$-metric space were examined in 2018 by Alqahtani et al. [1]; in the same year, Alqahtani et al. [2] disclosed non-unique fixed point results in an extended $b$-metric space; recently, in 2021, Huang et al. [13] investigated rational type contractions in extended $b$-metric spaces, etc.

In this paper, we give several new fixed point theorems that also hold in extended $b$-metric space drawing inspiration from those references. Also, we provide some examples to illustrate our research findings.

The article is structured as follows: in the first section, we introduce our work and go over the historical context. We discussed some previous results and a preliminary definition for our work in the second section. After establishing our findings in the third section, we saw the conclusion of the work and the future direction.

## 2. Mathematical Preliminary

In this section, we have gathered some basic definitions, theorems, and lemmas that are relevant to this research.

**Definition 2.1** ([5]). Let $\bar{Y}$ be a non-empty set and $p \geq 1$ and $\Theta^* : \bar{Y} \times \bar{Y} \to [0, \infty)$. If the mapping $\Theta^*$ satisfies the following conditions:

(b1) $\Theta^*(x^*, y^*) \geq 0$;

(b2) $\Theta^*(x^*, y^*) = 0$ if and only if $x^* = y^*$;

(b3) $\Theta^*(x^*, y^*) = \Theta^*(y^*, x^*)$;

(b4) $\Theta^*(x^*, z^*) \leq p[\Theta^*(x^*, y^*) + \Theta^*(y^*, z^*)]$ for all $x^*, y^*, z^* \in X$,

then $(\bar{Y}, \Theta^*)$ is said to be a $b$-metric space and $\Theta^*$ is called a $b$-metric on $\bar{Y}$.

Example 2.1 ([16]). Let $K = L^p[0, 1]$ be the collections of all real functions $z(t)$ such that $\int_0^1 |z(t)|^p dt < \infty$, where $t \in [0, 1]$ and $0 < p < 1$. For the function $d : K \times K \to \mathbb{R}_0^+$ defined by $b^{'}(z, y) := \int_0^1 \left(|z(t) - y(t)|^p dt\right)^{1/p}$, for each $z, y \in L^p[0, 1]$, the ordered pair $(K, b^{'})$ forms a $b$-metric space with $s = 2^{1/p}$.

**Definition 2.2** ([15]). Let $\bar{Y}$ be a non-empty set and $\psi : \bar{Y} \times \bar{Y} \to [1, \infty)$ and $\Theta^*_\psi : \bar{Y} \times \bar{Y} \to [0, \infty)$. If the mapping $\Theta^*_\psi$ satisfies the following conditions:

(e1) $\Theta^*_\psi(x^*, y^*) \geq 0$;

(e2) $\Theta^*_\psi(x^*, y^*) = 0$ if and only if $x^* = y^*$;

(e3) $\Theta^*_\psi(x^*, y^*) = \Theta^*_\psi(y^*, x^*)$;

(e4) $\Theta^*_\psi(x^*, z^*) \leq \psi(x^*, z^*)[\Theta^*_\psi(x^*, y^*) + \Theta^*_\psi(y^*, z^*)]$ for all $x^*, y^*, z^* \in \bar{Y}$,

then $(\bar{Y}, \Theta^*_\psi)$ is said to be an extended $b$-metric space and $\Theta^*_\psi$ is called an extended $b$-metric on $\bar{Y}$.

Example 2.2. Let

$$\bar{Z} = \{-1, 0, 1\} \quad \text{and} \quad \psi : \bar{Z} \times \bar{Z} \to [1, \infty[ \quad \text{such that} \quad \psi(x, y) = 1 + |x| + |y|.$$

Define $\Theta^*_\psi(x, y) : \bar{Z} \times \bar{Z} \to [0, \infty[$ as:

$\Theta^*_\psi(x, y) = \frac{1}{|x|+|y|}$ if $x \neq y$,

$\Theta^*_\psi(x, y) = 0$ if $x = y$,

$\Theta^*_\psi(x, 0) = \Theta^*_\psi(0, x) = \frac{1}{|x|}$ for $x, y \in \bar{Z}$.

Now, we want to show that $(\bar{Z}, \Theta^*_\psi)$ is an extended $b$-metric space. Clearly, we can easily verify that conditions (e1), (e2) and (e3) of definition (2.2) hold for this example. So we check condition (e4) as follows:

**Case 1.** If $x^* \neq y^* \neq z^*$. For $x^*, y^*, z^* \in \bar{Z}$, let $x^* = -1$, $y^* = 1$ and $z^* = 0$, we have $\Theta^*_\psi(-1,1) = \frac{1}{|-1|+|1|} = \frac{1}{2}$ and

$$\psi(-1,1)\Theta^*_\psi(-1,0) + \Theta^*_\psi(0,1) = (1 + |-1| + |1|)\left(\frac{1}{|-1|} + \frac{1}{|1|}\right) = (3)(1+1) = 6.$$

So clearly,

$$\Theta^*_\psi(-1,1) < \psi(-1,1)[\Theta^*_\psi(-1,0) + \Theta^*_\psi(0,1)].$$

A similar calculation holds for $\Theta^*_\psi(-1,0)$ and $\Theta^*_\psi(0,1)$. Hence for $x^*, y^*, z^* \in \bar{Z}$,

$$\Theta^*_\psi(x^*,y^*) < \psi(x^*,y^*)[\Theta^*_\psi(x^*,z) + \Theta^*_\psi(z^*,y^*)] \ \forall \ x^*, y^*, z^* \in \bar{Z}. \tag{4}$$

**Case 2.** If $x^* = y^* \neq z^*$. For $x^*, y^*, z^* \in \bar{Z}$, let $x^* = 0$, $y^* = 0$ and $z^* = 1$, we have $\Theta^*_\psi(0,0) = 0$ and

$$\psi(0,0)\Theta^*_\psi(0,1) + \Theta^*_\psi(1,0) = (1 + |0| + |0|)\left(\frac{1}{|1|} + \frac{1}{|1|}\right) = (1)(1+1) = 2.$$

So clearly,

$$\Theta^*_\psi(0,0) < \psi(0,0)[\Theta^*_\psi(0,1) + \Theta^*_\psi(1,0)].$$

A similar calculation holds for $\Theta^*_\psi(-1,-1)$ and $\Theta^*_\psi(1,1)$. Hence for $x^*, y^*, z^* \in \bar{Z}$

$$\Theta^*_\psi(x^*,y^*) < \psi(x^*,y^*)[\Theta^*_\psi(x^*,z^*) + \Theta^*_\psi(z^*,y^*)] \ \forall \ x^*, y^*, z^* \in \bar{Z}. \tag{5}$$

**Case 3.** If $x^* = y^* = z^*$. For $x^*, y^*, z \in \bar{Z}$, let $x^* = 1$, $y^* = 1$ and $z^* = 1$, we have $\Theta^*_\psi(1,1) = 0$ and

$$\psi(1,1)\Theta^*_\psi(1,1) + \Theta^*_\psi(1,1) = (1 + |1| + |1|)\,(0+0) = (3)(0) = 0.$$

So clearly,

$$\Theta^*_\psi(1,1) = \psi(1,1)[\Theta^*_\psi(1,1) + \Theta^*_\psi(1,1)].$$

A similar calculation holds for $\Theta^*_\psi(-1,-1)$ and $\Theta^*_\psi(0,0)$

$$\Theta^*_\psi(x^*,y^*) = \psi(x^*,y^*)[\Theta^*_\psi(x^*,z^*) + \Theta^*_\psi(z^*,y^*)] \ \forall \ x^*, y^*, z^* \in \bar{Z}. \tag{6}$$

So, from equations (4), (5) and (6), we have,

$$\Theta^*_\psi(x^*,y^*) \leq \psi(x^*,y^*)[\Theta^*_\psi(x^*,z^*) + \Theta^*_\psi(z^*,y^*)] \ \forall \ x^*, y^*, z^* \in \bar{Z}.$$

Hence, $(\bar{Z}, \Theta^*_\psi)$ is an extended $b$-metric space.

The Cauchy and convergent sequences in extended $b$-metric spaces are defined in this way.

**Definition 2.3.** Let $(\bar{Y}, \Theta^*_\psi)$ be an extended $b$-metric space and $\{\bar{\eta}_n\}$ be a sequence in $\bar{Y}$: Then, the sequence $\{\bar{\eta}_n\}$ is called a convergent sequence and converges to some $x^*$ in $\bar{Y}$; if for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $\Theta^*_\psi(\bar{\eta}_n, x^*) < \epsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to\infty} \bar{\eta}_n = x^*$.

**Definition 2.4.** Let $(\bar{Y}, \Theta^*_\psi)$ be an extended $b$-metric space and $\{\bar{\eta}_n\}$ be a sequence in $\bar{Y}$. The sequence $\{\bar{\eta}_n\}$ is called a Cauchy sequence if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_m) < \epsilon$ for all $m, n \geq n_0$.

**Definition 2.5.** Let $(\bar{Y}, \Theta^*_\psi)$ be an extended $b$-metric space and $\{\bar{\eta}_n\}$ be a sequence in $\bar{Y}$. The $b$-metric space $(\bar{Y}, \Theta^*_\psi)$ is called a complete $b$-metric space if every Cauchy sequence $\{\bar{\eta}_n\}$ is convergent in $\bar{Y}$.

**Definition 2.6.** Let $(\bar{Y}, \Theta^*_\psi)$ be an extended $b$-metric space and $\delta : \bar{Y} \to \bar{Y}$ be a function defined on $\bar{Y}$. Then, $A$ is said to be the limit of $f$, as $x^*$ approaches to $p$ if for every real $\epsilon > 0$, there exists a real $\gamma > 0$ such that for all real $x^*$, $0 < d(x^*, p) < \gamma$ implies $d(\delta(x^*), A) < \epsilon$. Symbolically, we can write as $\lim_{x^* \to p} \delta x^* = A$.

**Definition 2.7.** Let $(\bar{Y}, \Theta^*_\psi)$ be an extended $b$-metric space and $\{\bar{\eta}_n\}$ be a sequence in $\bar{Y}$ and converges to $x^* \in X$ as $n \to \infty$. Let $\delta$ be a self mapping defined on $\bar{Y}$. Then, $\delta$ is said to be the continuous at $x^*$ if for every real $\epsilon > 0$, there exists a real $\gamma > 0$ such that $d(\bar{\eta}_n, x^*) < \gamma$ implies $d(\delta\bar{\eta}_n, \delta x^*) < \epsilon$. Symbolically, we can write it as $\lim_{\bar{\eta}_n \to x^*} \delta {x^*}_n = \delta x^*$.

**Lemma 2.8** ( [10]). *Let $(\bar{Y}, \Theta^*_\psi)$ be an extended $b$-metric space. If there exists $k \in [0, 1)$ such that the sequence $\{\bar{\eta}_n\}$ for an arbitrary $x_0 \in \bar{Y}$ satisfies $\lim_{n,m\to\infty} \psi(\bar{\eta}_n, \bar{\eta}_m) < \frac{1}{k}$, and also*

$$0 < \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) \leq k\Theta^*_\psi\big(\bar{\eta}_{n-1}, \bar{\eta}_n\big) \tag{7}$$

*for any $n \in \mathbb{N}$, then the sequence $\{\bar{\eta}_n\}$ is a Cauchy sequence in $\bar{Y}$.*

## 3. Main results

In this section, we have establish the main results of this work.

**Definition 3.1.** A self-map $\delta$ on a metric space $\bar{Y}$ is said to be of Bianchini type contraction if there exists a number $h$, where $0 \leq h < 1$, such that, for each $x^*, y^* \in \bar{Y}$,

$$\Theta^*(\delta x^*, \delta y^*) \leq h \max\{\Theta^*(x^*, \delta x^*), \Theta^*(y^*, \delta y^*), \Theta^*(x^*, y^*)\}. \tag{8}$$

**Definition 3.2.** A self-map $\delta$ on a metric space $\bar{Y}$ is said to be of BK-type (Bianchini-Kannan type) contraction if there exists a non-negative number $a, b$ satisfying $a + 2b < 1$, such that, for each $x^*, y^* \in \bar{Y}$

$$\Theta^*(\delta x^*, \delta y^*) \leq a \max\{\Theta^*(x^*, \delta x^*), \Theta^*(y^*, \delta y^*), \Theta^*(x^*, y^*)\} + b\{\Theta^*(x^*, \delta y^*) + \Theta^*(y^*, \delta x^*)\}. \tag{9}$$

**Theorem 3.3.** *Let $(X, \Theta^*_\psi)$ be a complete extended b-metric space. If $\delta : \bar{Y} \to \bar{Y}$ is a continuous mapping that satisfies*

$$\Theta^*_\psi(\delta\bar{\eta}, \delta\bar{y}) \leq h \max\{\Theta^*_\psi(\bar{\eta}, \delta\bar{\eta}), \Theta^*_\psi(\bar{y}, \delta\bar{y}), \Theta^*_\psi(\bar{\eta}, \bar{y})\}, \tag{10}$$

*where $0 \leq h < 1$ is such that for each $\bar{\eta}_0 \in X, \lim_{n,m\to\infty} \psi(\bar{\eta}_n, \bar{\eta}_m) < \frac{1}{h}$ and $\bar{\eta}_n = \delta^n \bar{\eta}_0$ for all $n \in \mathbb{N}$. Then, $\delta$ has a unique fixed point on $X$.*

P r o o f. We select any $\bar{\eta}_0 \in X$ to be arbitrary. We construct the Picard iterative sequence $\bar{\eta}_n$ by

$$\bar{\eta}_1 = \delta\bar{\eta}_0, \quad \bar{\eta}_2 = \delta\bar{\eta}_1 = \delta^2\bar{\eta}_0, \ \ldots, \ \bar{\eta}_n = \delta\bar{\eta}_{n-1} = \delta^n\bar{\eta}_0 \ \ldots$$

If $\bar{\eta}_n = \bar{\eta}_{n+1}$, then $\bar{\eta}_n$ is a fixed point of $\delta$. So, suppose that $\bar{\eta}_n \neq \bar{\eta}_{n+1}$ for all $n \geq 0$. Now, we will show that the sequence $\{\bar{\eta}_n\}$ is a Cauchy sequence. Then, we have

$$\begin{aligned}
\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) &= \Theta^*_\psi(\delta\bar{\eta}_{n-1}, \delta\bar{\eta}_n) \\
&\leq h \max\{\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n), \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}), \Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n)\} \\
&= h \max\{\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n), \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1})\} \\
&= hM_1,
\end{aligned}$$

where

$$M_1 = \max\{\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n), \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1})\}. \tag{11}$$

**Case 1.** If we consider $M_1 = \Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n)$, then from (11) we have

$$\begin{aligned}
\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) &\leq h\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n) \\
&\ \ \vdots \\
&\leq h^n\Theta^*_\psi(\bar{\eta}_0, \bar{\eta}_1).
\end{aligned}$$

By using triangular inequality and (8), for all $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned}
\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_m) &\leq \psi(\bar{\eta}_n, \bar{\eta}_m)\{\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) + \Theta^*_\psi(\bar{\eta}_{n+1}, \bar{\eta}_m)\} \\
&\leq \psi(\bar{\eta}_n, \bar{\eta}_m)\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) \\
&\quad + \psi(\bar{\eta}_n, \bar{\eta}_m)\psi(\bar{\eta}_{n+1}, \bar{\eta}_m)\{\Theta^*_\psi(\bar{\eta}_{n+1}, \bar{\eta}_{n+2}) + \Theta^*_\psi(\bar{\eta}_{n+2}, \bar{\eta}_m)\}.
\end{aligned}$$

Continuing this process, we get

$$\begin{aligned}
&\leq \psi(\bar{\eta}_n, \bar{\eta}_m)\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) + \psi(\bar{\eta}_n, \bar{\eta}_m)\psi(\bar{\eta}_{n+1}, \bar{\eta}_m)\Theta^*_\psi(\bar{\eta}_{n+1}, \bar{\eta}_{n+2}) \\
&\quad + \psi(\bar{\eta}_n, \bar{\eta}_m)\psi(\bar{\eta}_{n+1}, \bar{\eta}_m) \cdots \Theta^*_\psi(x_{m-1}, \bar{\eta}_m) + \Theta^*_\psi(x_{m-1}, \bar{\eta}_m)\} \\
&\leq h^n \psi(\bar{\eta}_n, \bar{\eta}_m)\Theta^*_\psi(\bar{\eta}_0, \bar{\eta}_1) + h^{(n+1)}\psi(\bar{\eta}_n, \bar{\eta}_m)\psi(\bar{\eta}_{n+1}, \bar{\eta}_m) \\
&\quad \{\Theta^*_\psi(\bar{\eta}_0, \bar{\eta}_1) + \cdots + h^{(m-1)}\psi(\bar{\eta}_n, \bar{\eta}_m)\psi(\bar{\eta}_{n+1}, \bar{\eta}_m) \\
&\quad \ldots \psi(x_{m-1}, \bar{\eta}_m)\Theta^*_\psi(\bar{\eta}_0, \bar{\eta}_1)\} \\
&\leq \sum_{t=n}^{m-1} h^t \prod_{t=n}^{m-1} \psi(\bar{\eta}_i, \bar{\eta}_m)\Theta^*_\psi(\bar{\eta}_0, \bar{\eta}_1).
\end{aligned}$$

Since

$$h \lim_{n\to\infty} \psi(\bar{\eta}_n, \bar{\eta}_m) < 1,$$

the series

$$\sum_{n=1}^{\infty} h^n \prod_{i=1}^{n} \psi(\bar{\eta}_i, \bar{\eta}_m)$$

converges by the ratio test for each $m \in \mathbb{N}$. Let

$$M = \sum_{n=1}^{\infty} h^n \prod_{i=1}^{n} \psi(\bar{\eta}_i, \bar{\eta}_m)$$

with the partial sum

$$M_n = \sum_{j=1}^{n} h^j \prod_{i=1}^{j} \psi(\bar{\eta}_i, \bar{\eta}_m).$$

Thus, for $m > n$, the above inequaility implies

$$\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_m) \leq \Theta^*_\psi(\bar{\eta}_0, \bar{\eta}_1)[M_{m-1} - M_n] \to 0 \quad \text{as } \ n \to \infty.$$

So, we conclude that

$$\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_m) = 0 \implies \{\bar{\eta}_n\}$$

is a Cauchy sequence.

**Case 2.** If we consider $M_1 = \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1})$, then from (11) we have

$$\begin{aligned}
&\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) = h\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) \\
&\implies (1-h)\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) \leq 0 \\
&\implies \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) = 0 \quad \text{as} \quad (1-h) > 0, \ \ n \in \mathbb{N}.
\end{aligned}$$

By the triangular inequality and (8), for $m > n$ we have

$$\begin{aligned}\Theta^*_\psi(\bar\eta_n,\bar\eta_m) &\leq \psi(\bar\eta_n,\bar\eta_m)\{\Theta^*_\psi(\bar\eta_n,\bar\eta_{n+1})+\Theta^*_\psi(\bar\eta_{n+1},\bar\eta_m)\}\\ &\leq \psi(\bar\eta_n,\bar\eta_m)\Theta^*_\psi(\bar\eta_n,\bar\eta_{n+1})\\ &\quad +\psi(\bar\eta_n,\bar\eta_m)\psi(\bar\eta_{n+1},\bar\eta_m)\{\Theta^*_\psi(\bar\eta_{n+1},\bar\eta_{n+2})+\Theta^*_\psi(\bar\eta_{n+2},\bar\eta_m)\}.\end{aligned}$$

Continuing this process, we get

$$\begin{aligned} &\leq \psi(\bar\eta_n,\bar\eta_m)\Theta^*_\psi(\bar\eta_n,\bar\eta_{n+1})\\ &\quad +\psi(\bar\eta_n,\bar\eta_m)\psi(\bar\eta_{n+1},\bar\eta_m)\Theta^*_\psi(\bar\eta_{n+1},\bar\eta_{n+2})+\cdots\\ \cdots &\quad +\psi(\bar\eta_n,\bar\eta_m)\psi(\bar\eta_{n+1},\bar\eta_m)\cdots\psi(\bar\eta_{m-1},\bar\eta_m)\Theta^*_\psi(\bar\eta_{m-1},\bar\eta_m)\\ &\to 0 \quad \text{as } \Theta^*_\psi(\bar\eta_n,\bar\eta_{n+1})=0,\ \ \forall\ n\in\mathbb{N}.\end{aligned}$$

So, we conclude that

$$\Theta^*_\psi(\bar\eta_n,\bar\eta_m)=0 \implies \{\bar\eta_n\} \text{ is a Cauchy sequence.}$$

Since $X$ is complete. Therefore, there exists

$$x^*\in X \quad \text{such that} \quad \bar\eta_n\to x^*.$$

Currently, we wish to demonstrate that $x^*$ will be a fixed point of $\delta$. Let $\delta\bar\eta_n = x^*$, then

$$\begin{aligned}\Theta^*_\psi(\delta x^*,x^*) &= \Theta^*_\psi(\delta x^*,\delta\bar\eta_n)\\ &\leq h\max\{\Theta^*_\psi(x^*,\delta x^*),\Theta^*_\psi(\bar\eta_n,\delta\bar\eta_n),\Theta^*_\psi(x^*,{x^*}_n)\}\\ \text{when } n\to\infty,\ \ {x^*}_n\to x^* &\implies \delta {x^*}_n\to\delta x^* \qquad\qquad [\text{as } T \text{ is continuous}]\\ &= h\max\{\Theta^*_\psi(x^*,\delta x^*),\Theta^*_\psi(x^*,\delta x^*),\Theta^*_\psi(x^*,x^*)\}\\ &= h\max\{\Theta^*_\psi(x^*,\delta x^*)\} \text{ as } \Theta^*_\psi(x^*,x^*)=0\\ &= h\{\Theta^*_\psi(\delta x^*,x^*)\}\\ &\implies (1-h)\{\Theta^*_\psi(\delta x^*,x^*)\}\leq 0. \end{aligned} \tag{12}$$

Since $h<1$, then it is obvious that $(1-h)>0$ and $\Theta^*_\psi(\delta x^*,x^*)\geq 0$, so we get

$$(1-h)\Theta^*_\psi(\delta x^*,x^*)\geq 0. \tag{13}$$

Therefore, from inequalities (12) and (13) we conclude that

$$(1-h)\Theta^*_\psi(\delta x^*,x^*)=0 \implies \Theta^*_\psi(\delta x^*,x^*)=0 \implies \delta x^*=x^*. \tag{14}$$

Hence, $x^*$ is a fixed point of $\delta$.

**Uniqueness of the fixed point**

We have assumed that $y^* \in X$ is another fixed point of $T$ such that $y^* \neq x^*$, then

$$\delta y^* = y^* \implies \Theta^*_\psi(\delta y^*, y^*) = 0. \tag{15}$$

Now, using (10), we get

$$\begin{aligned}
\Theta^*_\psi(x^*, y^*) &= \Theta^*_\psi(\delta x^*, \delta y^*) \\
&\leq h \max\{\Theta^*_\psi(x^*, \delta x^*), \Theta^*_\psi(y^*, \delta y^*), \Theta^*_\psi(x^*, y^*)\} \\
&\leq h \max\{0, 0, \Theta^*_\psi(x^*, y^*)\} \quad \text{[by using (14) and (15)]} \\
&\leq h\Theta^*_\psi(x^*, y^*),
\end{aligned}$$

which is a contradiction. Thus, we conclude that $x^* = y^* \implies x^*$ is a unique fixed point of $\delta$. Hence, it is proved. □

Example 3.1. Let

$$X = \{p \in \mathbb{Z} : |p| \leq 1\} \quad \text{and} \quad \psi : X \times X \to [1, \infty[$$

such that

$$\psi(p, q) = 1 + |p| + |q| \; \forall \; p, q \in X.$$

Define, $\Theta^*_\psi(p, q) : X \times X \to [0, \infty[$ as

$$\Theta^*_\psi(p, q) = \begin{cases} \frac{1}{|p|+|q|} & if \;\; p \neq q, \\ 0 & if \;\; p = q. \end{cases} \tag{16}$$

Then, $(X, \Theta^*_\psi)$ is a complete extended $b$-metric space. Let $\delta : X \to X$ be a function defined by $\delta(p) = p^2 \;\forall\; p \in X$. Then, for $h = \frac{1}{2} < 1$, condition (8) of Theorem (3.3) is satisfied. Here, $p = 1$ is the unique fixed point on X.

**Theorem 3.4.** *Let $(X, \Theta^*_\psi)$ be a complete extended b-metric space. If $\delta : X \to X$ is a continuous mapping that satisfies*

$$\Theta^*_\psi(\delta\bar{\eta}, \delta\bar{y}) \leq a \max\{\Theta^*_\psi(\bar{\eta}, \delta\bar{\eta}), \Theta^*_\psi(\bar{y}, \delta\bar{y}), \Theta^*_\psi(\bar{\eta}, \bar{y})\} + b\{\Theta^*_\psi(\bar{\eta}, \delta\bar{y}) + \Theta^*_\psi(\bar{y}, \delta\bar{\eta})\}, \tag{17}$$

*where $a, b \in [0, 1)$ such that*

$$0 \leq a + 2b\psi(\bar{\eta}, \bar{y}) < 1 \quad \textit{and} \quad a + 2b < 1$$

*for each*

$$\bar{\eta}_0 \in X, \lim_{n,m\to\infty} \psi(\bar{\eta}_n, \; \bar{\eta}_m) < \frac{1}{k}, \;\; k \in [0, 1)$$

*and*

$$\bar{\eta}_n = \delta^n \bar{\eta}_0 \quad \textit{for all} \quad n \in \mathbb{N}.$$

*Then, $\delta$ has a unique fixed point on $X$.*

Proof. We choose any $\bar{\eta}_0 \in X$ to be arbitrary. Define the Picard iterative sequence $\bar{\eta}_n$ by

$$\bar{\eta}_1 = \delta\bar{\eta}_0, \bar{\eta}_2 = \delta\bar{\eta}_1 = \delta^2\bar{\eta}_0, \ldots, \bar{\eta}_n = \delta^n\bar{\eta}_0.$$

If $\bar{\eta}_n = \bar{\eta}_{n+1}$, then $\bar{\eta}_n$ is the fixed point of $\delta$. So, suppose that $\bar{\eta}_n \neq \bar{\eta}_{n+1}$ for all $n \geq 0$. Now, we will demonstrate that $\{\bar{\eta}_n\}$ is a Cauchy sequence. So, using (17), we have

$$\begin{aligned}
\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) &= \Theta^*_\psi(\delta\bar{\eta}_{n-1}, \delta\bar{\eta}_n) \\
&\leq a\max\{\Theta^*_\psi(\bar{\eta}_{n-1}, \delta\bar{\eta}_{n-1}), \Theta^*_\psi(\bar{\eta}_n, \delta\bar{\eta}_n), \Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n)\} \\
&\quad + b\{\Theta^*_\psi(\bar{\eta}_{n-1}, \delta\bar{\eta}_n) + \Theta^*_\psi(\bar{\eta}_n, \delta\bar{\eta}_{n-1})\} \\
&= a\max\{\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n), \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}), \Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n)\} \\
&\quad + b\{\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1}) + \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_n)\} \\
&= a\max\{\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n), \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}), \Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n)\} \\
&\quad + b\{\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})\} \quad [\,\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_n) = 0 \quad \text{by using (e2)}\,] \\
&\leq a\max\{\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n), \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1})\} \\
&\quad + b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})\{\Theta^*_\psi(\bar{\eta}_{n-1} + \bar{\eta}_n) + \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1})\} \\
&= aM_2 + b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})\{\Theta^*_\psi(\bar{\eta}_{n-1} + \bar{\eta}_n) + \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1})\}, \quad (18)
\end{aligned}$$

where

$$M_2 = \max\{\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n),\ \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}).$$

**Case 1.** If we consider $M_2 = \Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n)$, then from (18) we have

$$\begin{aligned}
\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) &\leq a\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n) + b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})\{\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n) + \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1})\} \\
&\implies \big(1 - b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})\big)\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) \\
&\quad\leq \big(a + b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})\big)\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n) \\
&\implies \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) \leq \frac{a + b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})}{1 - b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})}\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n) \\
&\implies \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) \leq k_1\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n),
\end{aligned}$$

where

$$k_1 = \frac{a + b\psi(\bar{\eta}_{n-1},\ \bar{\eta}_{n+1})}{1 - b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})}.$$

Clearly, $k < 1$ as $a + 2b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n-1}) < 1$.

**Case 2.** If we consider $M_2 = \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1})$, then from (18) we get

$$\begin{aligned}
&\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) \\
&\quad \leq \; a\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) + b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})\{\Theta^*_\psi(\bar{\eta}_{n-1} + \bar{\eta}_n) + \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1})\} \\
&\implies \big(1 - a - b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})\big)\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) \\
&\quad \leq \; b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n) \\
&\implies \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) \leq \left\{\frac{b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})}{1 - a - b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})}\right\}\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n) \\
&\implies \Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) \leq k_2\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n)
\end{aligned}$$

where,

$$k_2 = \frac{b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})}{1 - a - b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n+1})}.$$

Clearly,

$$k < 1 \quad \text{as} \quad a + 2b\psi(\bar{\eta}_{n-1}, \bar{\eta}_{n-1}) < 1.$$

Let $k = \min\{k_1, k_2\}$ such that $k \in [0, 1)$. Then, from the above cases (1) and (2) it follows

$$\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_{n+1}) \leq k\Theta^*_\psi(\bar{\eta}_{n-1}, \bar{\eta}_n). \tag{19}$$

From Lemma (2.4), it is easy to deduce that $\{\bar{\eta}_n\}$ is a Cauchy sequence in X, i.e.,

$$\Theta^*_\psi(\bar{\eta}_n, \bar{\eta}_m) \to 0 \quad \text{as} \;\; n, m \to \infty.$$

Since $X$ is complete. Therefore, there exists $v^* \in X$ such that $\bar{\eta}_n \to v^*$. Now, we want to show that $v^*$ will be a fixed point of $\delta$. Let $\delta\bar{\eta}_n = v^*$, then

$$\begin{aligned}
\Theta^*_\psi(\delta v^*, v^*) &= \Theta^*_\psi(\delta v^*, \delta\bar{\eta}_n) \\
&\leq a\max\{\Theta^*_\psi(v^*, \delta v^*), \Theta^*_\psi(\bar{\eta}_n, \delta\bar{\eta}_n), \Theta^*_\psi(v^*, \bar{\eta}_n)\} \\
&\quad + b\{\Theta^*_\psi(v^*, \delta\bar{\eta}_n) + \Theta^*_\psi(\bar{\eta}_n, \delta v^*)\}
\end{aligned}$$

when $n \to \infty, \bar{\eta}_n \to v^* \implies \delta\bar{\eta}_n \to \delta v^*$ [as $\delta$ is continuous]

$$\begin{aligned}
&\leq a\max\{\Theta^*_\psi(v^*, \delta v^*), \Theta^*_\psi(v^*, \delta v^*), \Theta^*_\psi(v^*, v^*)\} \\
&\quad + b\{\Theta^*_\psi(v^*, \delta v^*) + \Theta^*_\psi(v^*, \delta v^*)\} \\
&\leq a\Theta^*_\psi(v^*, \delta v^*) + 2b\Theta^*_\psi(v^*, \delta v^*) \;\; \text{as } \Theta^*_\psi(v^*, v^*) = 0 \\
&\leq (a + 2b)\Theta^*_\psi(v^*, \delta v^*) \\
&\leq (a + 2b)\Theta^*_\psi(\delta v^*, v^*) \qquad [\text{by using (e3)}]
\end{aligned}$$

which is a contradiction as $(a + 2b) < 1$.

Thus, we conclude that
$$\Theta^*_\psi(\delta v^*, v^*) = 0 \implies \delta v^* = v^*. \tag{20}$$
Therefore, $v^*$ is a fixed point of $\delta$. Hence $\delta$ has a fixed point.

**Uniqueness of the fixed point**

We assumed that $w^* \in X$ is another fixed point of $\delta$ such that $w^* \neq v^*$, i.e.,
$$\delta w^* = w^* \implies \Theta^*_\psi(\delta w^*, w^*) = 0. \tag{21}$$
Now, using (17), we get
$$\begin{aligned}
\Theta^*_\psi(v^*, w^*) &= \Theta^*_\psi\big(\delta v^*, \delta w^*\big)\\
&\leq a \max\{\Theta^*_\psi(v^*, \delta v^*), \Theta^*_\psi(w^*, \delta w^*), \Theta^*_\psi(v^*, w^*)\}\\
&+\ b\{\Theta^*_\psi(v^*,\ \delta w^*) + \Theta^*_\psi(w^*, \delta v^*)\}\\
&= a\Theta^*_\psi(v^*, w^*) + b\{\Theta^*_\psi(v^*, w^*) + \Theta^*_\psi(w^*, v^*)\} \text{ [by using (20) and (21)]}\\
&= a\Theta^*_\psi(v^*, w^*) + b\{\Theta^*_\psi(v^*, w^*) + \Theta^*_\psi(v^*, w^*)\} \text{ [by using (e3)]},\\
&= (a+2b)\Theta^*_\psi(v^*, w^*),
\end{aligned}$$
which is a contradiction as $(a + 2b) < 1$. Finally, we conclude that
$$\Theta^*_\psi(v^*, w^*) \ = \ 0 \implies v^* = w^*.$$
Therefore, $v^*$ is a unique fixed point of $\delta$. Hence, it is proved. □

Example 3.2. Let

$X = \{x \in \mathbb{Z} : |p| \leq 1\}$ and $\psi : X \times X \to [1, \infty[$ such that $\psi(p, q) = 1+|p|+|q|$. Define $\Theta^*_\psi : X \times X \to [0, \infty[$ as
$$\Theta^*_\psi(p, q) = \begin{cases} \frac{1}{|p|+|q|} & if \ \ p \neq q, \\ 0 & if \ \ p = q. \end{cases} \tag{22}$$
Then, $(X, \Theta^*_\psi)$ is a complete extended $b$-metric space. Let $\delta : X \to X$ be a function defined by $\delta(y) = y^2\ \forall\ y \in X$. Then for $a = \frac{1}{2}$ and $b = 0$, condition (9) of Theorem (3.4) is satisfied. Here, $y = 1$ is the unique fixed point of $\delta$ on $X$.

## 4. Conclusion

Using various well-known contraction mappings, including the Bianchini-type contraction mapping and a combination of the Bianchini-type and Kannan-type contraction mappings, referred to as BK-type contraction mappings over the complete extended $b$-metric space, we developed several fixed-point theorems in this paper. We also give some examples to support our main results.

**Future work**

If we use the ideas presented in this paper and attempt to do similar types of research on alternative metric spaces, such as control $b$-metric space, rectangular $b$-metric space, etc., in the future, then we might uncover some insightful findings. The ideal of this work is also applicable for the controlled metric space and the generalization of different space. Many researchers may find suitable application of this work in integral equation and differential equation.

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