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ON DENSITY AND INDEPENDENCE OF SETS

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ABSTRACT. The paper deals with subsets of $\mathbb{N}$ independent with respect to given density. We apply their properties to the reminder systems and to the statistically independent uniformly distributed sequences.

Introduction

Suppose that $\mathbb{N}$ is the set of natural numbers. If $A \subset \mathbb{N}$, then the value

$$\overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [1,N]|}{N} \tag{1}$$

is called *upper asymptotic density* of A. The system of sets S, satisfying the equality

$$\overline{d}(S) + \overline{d}(\mathbb{N} \setminus S) = 1$$

will be denoted as $\mathcal{D}$. Denoting $d = \overline{d}|_{\mathcal{D}}$ the equality

$$d(S) = \lim_{N\to\infty} \frac{|A \cap [1,N]|}{N}$$

holds and the value $d(S)$ is called the *asymptotic density* of S.

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In 1951 Ivan Niven published in the paper [9] the following significant result:

Theorem 1. *Denote for $S \subset \mathbb{N}$ and prime number p*

$$S_p = \{s \in S; p|s \wedge p^2 \nmid s\}.$$

Let $p_1, p_2, p_3, \dots$ be a sequence of primes such that

$$\sum_{n=1}^{\infty} \frac{1}{p_n} = \infty,$$

then $d(S) = 0$ if and only if $d(S_{p_n}) = 0$ for each $n = 1, 2, 3, \dots$

The proof of this result in the mentioned paper is based on the estimation of the value $|S \cap [1, N]|, N \in \mathbb{N}$. The aim of this paper is to study the independence of sets and sequences with respect to the certain type of finitely additive measure which will be called density. We derive the "density" analogy of Borrel Cantelli lemmas. This leads, inter alia, to the Theorem of Niven formulated above.

The asymptotic density allows to describe the distribution of sequences in certain cases. A sequence $\{v(n)\}$ of elements of $[0, 1]$ is called *uniformly distributed modulo* 1 if for each subinterval $I \subset [0, 1]$ the set $v^{-1}(I)$ belongs to $\mathcal{D}$ and $d\big(v^{-1}(I)\big) = |I|$, (see [14], [8], [2], [12], [13]). Weyl's criterion proven in the books and articles cited above provides that for irrational number α the sequence of fractional parts $\{\{n\alpha\}\}$ is uniformly distributed modulo 1. This allows to calculate the asymptotic density of some simple sets associated with diophantine approximations.

If x is a non integer number then for each integer p we have

$$|x - p| < 1 \iff p = [x] \vee p = [x] + 1. \tag{2}$$

This implies that for $c \in (0, \frac{1}{2})$ we have that an integer p such that $|x - p| < c$ exists if and only if $\{x\} < c \vee \{x\} > 1 - c$. Let us consider an irrational number α. The previous considerations give that for given positive integer n such integer p exists that

$$\left|\alpha - \frac{p}{n}\right| < \frac{c}{n} \tag{3}$$

if and only if

$$\{n\alpha\} \in [0, c) \cup (1 - c, 1].$$

Let us denote by $S_c(\alpha)$ the set of all natural numbers n such that the inequality (3) holds for suitable integer p. Taking into account that the sequence $\{\{n\alpha\}\}$ is uniformly distributed we get:

1. $S_c(\alpha) \in \mathcal{D}$ and $d\big(S_c(\alpha)\big) = 2c$.

This leads to the following complement of Dirichlet theorem on approximation[1]:

Theorem 2. *Let $f(n)$ be a function such that $\lim_{n\to\infty} \frac{f(n)}{n} = \infty$. If α is an irrational number, then the set of all $n \in \mathbb{N}$ that for suitable integers p_n, the inequalities*

$$\left|\alpha - \frac{p_n}{n}\right| < \frac{1}{f(n)}$$

hold, belongs to $\mathcal{D}$ and its asymptotic density is 0.

We close Introduction recalling well-known result from elementary analysis:

2. Let us consider a sequence $\{u_n\}$, $u_n \in [0,1)$. Then

$$\sum_{n=1}^{\infty} u_n = \infty \Rightarrow \prod_{n=1}^{\infty} (1 - u_n) = 0.$$

This follows immediately from the inequality

$$1 - x \le e^{-x} \quad \text{for } \ x \in [0,1).$$

1. Density

If $\mathcal{Y}$ is a system of subsets of $\mathbb{N}$, then $\mathcal{Y}$ is called *q-algebra* if and only if

i) $\mathbb{N} \in \mathcal{Y}$,

ii) $A \in \mathcal{Y} \Longrightarrow \mathbb{N} \setminus A \in \mathcal{Y}$,

iii) $A, B \in \mathcal{Y}, A \cap B = \emptyset \Rightarrow A \cup B \in \mathcal{Y}$.

Let us remark that the system $\mathcal{D}$ defined in the Introduction fulfils these conditions.

A finitely additive probability measure defined on q-algebra $\mathcal{Y}$ is called *density* if and only if:

iv) $A \in \mathcal{Y}$ if and only if

$$\forall \varepsilon > 0 \ \ \exists\, A_1, A_2 \in \mathcal{Y}; \quad A_1 \subset A \subset A_2 \wedge \nu(A_2) - \nu(A_1) < \varepsilon.$$

Clearly, d-asymptotic density is a density on $\mathcal{D}$.

[1]Well-known Dirichlet theorem says that for each irrational number α a infinity set of positive integer n exists that

$$\left|\alpha - \frac{p_n}{n}\right| < \frac{1}{n^2} \quad \text{for suitable} \quad p_n \in \mathbb{Z}.$$

Suppose that a q algebra $\mathcal{Y}$ is given and ν is a density on $\mathcal{Y}$. We say that sets $A_1, \dots, A_m \in \mathcal{Y}$ are *ν-independent* if and only if for each $i_1 < \dots < i_k$, such that $1 \le i_1, i_k \le m$, the intersection $\cap_{j=1}^k A_{i_j}$ belongs to $\mathcal{Y}$ and

$$\nu\left(\bigcap_{j=1}^k A_{i_j}\right) = \prod_{j=1}^k \nu(A_{i_j}). \tag{4}$$

A sequence of sets $A_1, A_2, A_3, \dots$ will be called *ν-independent* if and only if the sets $A_1, \dots, A_k$ are *ν-independent* for each $k = 1, 2, 3, \dots$

We start by the following form of Borrel Cantelli lemma for density:

Theorem 3. *If $A_1, A_2, \dots, A_k \in \mathcal{Y}$ are ν-independent sets, then $\cup_{j=1}^k A_j$ belongs to $\mathcal{Y}$ and*

$$\nu\left(\bigcup_{j=1}^k A_j\right) = 1 - \prod_{j=1}^k \bigl(1 - \nu(A_j)\bigr). \tag{5}$$

If the sequence $A_1, A_2, A_3, \dots$ is ν-independent and

$$\sum_{j=1}^\infty \nu(A_j) = \infty, \tag{6}$$

then the set $A = \cup_{j=1}^\infty A_j$ belongs to $\mathcal{Y}$ and $\nu(A) = 1$.

For the proof we derive the following statements.

3. If $A_1, A_2 \in \mathcal{Y}$ are independent sets, then $A_1 \cup A_2 \in \mathcal{Y}$ and

$$\nu(A_1 \cup A_2) = 1 - \bigl(1 - \nu(A_1)\bigr)\bigl(1 - \nu(A_2)\bigr).$$

P r o o f. We have

$$A_1 \cup A_2 = A_1 \cup \bigl(A_2 \setminus (A_1 \cap A_2)\bigr). \tag{7}$$

The set $A_1 \cap A_2$ belongs to $\mathcal{Y}$ thus the set $A_2 \setminus (A_1 \cap A_2)$ belongs to $\mathcal{Y}$. This set is disjoint with A_1, therefore equality (7) we imply $A_1 \cup A_2 \in \mathcal{Y}$. From (7) we get

$$\begin{aligned}\nu(A_1 \cup A_2) &= \nu(A_1) + \nu(A_2) - \nu(A_1)\nu(A_2) \\ &= 1 - \bigl(1 - \nu(A_1)\bigr)\bigl(1 - \nu(A_2)\bigr). \qquad \square\end{aligned}$$

4. If the sets $A_1, A_2, A_3 \in \mathcal{Y}$ are *ν-independent*, then the sets $A_1 \cup A_2, A_3$ are *ν-independent* also.

P r o o f. It holds

$$(A_1 \cup A_2) \cap A_3 = (A_1 \cap A_3) \cup (A_2 \cap A_3).$$

From **3.** we obtain that $(A_1 \cup A_2) \cap A_3$ belongs to $\mathcal{Y}$, and

$$\begin{aligned}\nu\big((A_1 \cup A_2) \cap A_3\big) &= \nu(A_1 \cap A_3) + \nu(A_2 \cap A_3) - \nu(A_1 \cap A_2 \cap A_3)\\ &= \nu(A_3)\big(\nu(A_1) + \nu(A_2) - \nu(A_1)\nu(A_2)\big)\\ &= \nu(A_3)\nu(A_1 \cup A_2).\end{aligned}$$ □

5. If the sets $A_1, \dots, A_m \in \mathcal{Y}$ are ν-*independent*, then the sets $\mathbb{N} \setminus A_1, A_2, \dots, A_m$ are independent also.

P r o o f. Let $1 < i_2 < \cdots < i_k \leq m$. Then

$$(\mathbb{N} \setminus A_1) \cap \bigcap_{j=2}^{k} A_{i_j} = \bigcap_{j=2}^{k} A_{i_j} \Bigg\backslash \left(A_1 \cap \bigcap_{j=2}^{k} A_{i_j} \right).$$

Thus

$$(\mathbb{N} \setminus A_1) \cap \bigcap_{j=2}^{k} A_{i_j} \in \mathcal{Y}$$

and

$$\begin{aligned}\nu\left((\mathbb{N} \setminus A_1) \cap \bigcap_{j=2}^{k} A_{i_j} \right) &= \nu\left(\bigcap_{j=2}^{k} A_{i_j} \right) - \nu\left(A_1 \cap \bigcap_{j=2}^{k} A_{i_j} \right)\\ &= \prod_{j=2}^{k} \nu(A_{i_j}) - \nu(A_1) \prod_{j=2}^{k} \nu(A_{i_j})\\ &= \big(1 - \nu(A_1)\big) \prod_{j=2}^{k} \nu(A_{i_j})\\ &= \nu(\mathbb{N} \setminus A_1) \prod_{j=2}^{k} \nu(A_{i_j}).\end{aligned}$$ □

P r o o f o f T h e o r e m 3. For $k = 2$, the theorem follows from **3**. Suppose that the theorem holds for $k - 1$. The sets $A_1 \cup \cdots \cup A_{k-1}$ and A_k are independent. By application of **3.** we get that the assertion holds for k.

Put $B_N = \cup_{j=1}^{N} A_j$. Clearly,

$$B_N \in \mathcal{Y} \quad \text{and} \quad B_N \subset A \subset \mathbb{N}.$$

From (5), (6) and point **2.** we get $\lim_{N\to\infty} \nu(B_N) = 1$. Thus the assertion follows from the property iv) of density. □

From this we can generalize the result of Ivan Niven, [9], to the following form:

Theorem 4. *Let* (6) *holds. Then for each* $S \subset \mathbb{N}$ *we have* $S \in \mathcal{Y}$ *and* $\nu(S) = 0$ *if and only if*

$$S \cap A_j \in \mathcal{Y} \quad \textit{and} \quad \nu(S \cap A_j) = 0 \quad \textit{for } j = 1, 2, 3, \dots$$

Arithmetic progressions or in the other words the reminder classes will play important role. In accordance with the usual notation in algebra we shall denote

$$r + (m) = \{n \in \mathbb{N}; n \equiv r \pmod{m}\},$$

where $m \in \mathbb{N}$ and $r \in \mathbb{Z}$ $(0 + (m) := (m))$.

The density ν will be called *arithmetic* if for each $m \in \mathbb{N}, r \in \mathbb{Z}$ the set $r + (m)$ belongs to

$$\mathcal{Y} \quad \text{and} \quad \nu\big(r + (m)\big) = \frac{1}{m}.$$

The Chinese reminder theorem implies that for mutually relatively natural numbers $m_1, \dots, m_k$ and $r_1, \dots, r_k \in \mathbb{Z}$ such $r \in \mathbb{Z}$ exists that

$$\bigcap_{i=1}^{k} r_i + (m_i) = r + (m_1 m_2 \cdots m_k).$$

This yields

6. If ν is arithmetic density, then the sets $r_1 + (m_1), \dots, r_k + (m_k)$, mentioned above, are ν-indpendent.

By an application of Theorem 3 for the case of asymptotic density and reminder classes we get

Proposition 1. *Suppose that an arithmetic density* ν *is given. Let* p_k, $k = 1, 2, 3, \dots$ *be such primes that*

$$\sum_{k=1}^{\infty} \frac{1}{p_k} = \infty.$$

Denote for $A \subset \mathbb{N}, p, r, r'$ *by* $A_{p,r,r'}$ *the set of all* $a \in A$ *such that*

$$a \equiv r \pmod{p}, \quad a \not\equiv r' \pmod{p^2}.$$

If $r_k, r'_k, k = 1, 2, 3, \dots$ *is arbitrary sequence of integers and*

$$A_{p_k, r_k, r'_k} \in \mathcal{Y}, \quad \nu\big(A_{p_k, r_k, r'_k}\big) = 0, \quad k = 1, 2, 3, \dots,$$

then

$$A \in \mathcal{Y} \quad \textit{and} \quad \nu(A) = 0.$$

P r o o f. The sets $r_k + (p_k) \setminus r'_k + (p_k^2)$ can be written in the form

$$r_k + (p_k) \setminus r'_k + \big(p_k^2\big) = \bigcup_{j=1}^{s} \ell_j + (p_k^2)$$

for suitable s, ℓ_j depending on

$$p_k, r_k, r'_k, \quad k = 1, 2, 3, \dots$$

And so we see that these sets are ν-independent. The assertion follows now from Theorem 4 taking account that

$$A_{p_k, r_k, r'_k} = A \cap \Big(r_k + (p_k) \setminus r'_k + \big(p_k^2\big)\Big),$$

the sets $r_k + (p_k) \setminus r'_k + (p_k^2), k = 1, 2, 3, \dots$ are independent. Moreover,

$$\nu\Big(r_k + (p_k) \setminus r'_k + \big(p_k^2\big)\Big) \geq \frac{1}{p_k} - \frac{1}{p_k^2}.$$

Thus

$$\sum_{k=1}^{\infty} \nu\Big(r_k + \big(p_k\big) \setminus r'_k + \big(p_k^2\big)\Big) = \infty.$$

□

2. Outer density, decompositions

Let us denote by $P(\mathbb{N})$ the system of all subsets of $\mathbb{N}$. A function

$\overline{\nu} : P(\mathbb{N}) \to [0,1]$ is called *outer density* if and only if:

v) $\overline{\nu}(\emptyset) = 0,\ \overline{\nu}(\mathbb{N}) = 1$,

vi) $A \subset B \Longrightarrow \overline{\nu}(A) \leq \overline{\nu}(B)$,

vii) $\overline{\nu}(A \cup B) \leq \overline{\nu}(A) + \overline{\nu}(B)$,

viii) $A \cup B = \mathbb{N} \Rightarrow 1 + \overline{\nu}(A \cap B) \leq \overline{\nu}(A) + \overline{\nu}(B)$.

It is easy to check that $\overline{d}$ fulfils the conditions of outer density.

By the standard procedure can be proven:

Proposition 2. *If $\overline{\nu}$ is outer density then the system of sets*

$$\mathcal{D}_\nu = \big\{A \subset \mathbb{N}; \overline{\nu}(A) + \overline{\nu}(\mathbb{N} \setminus A) = 1\big\}$$

is a q-algebra and the restriction $\nu = \overline{\nu}|_{\mathcal{D}_\nu}$ is density.

Proposition 3. *Let $\overline{\nu}$ be outer density and*

$$A_1, A_2, A_3, \dots, A_n \in \mathcal{D}_\nu, n \in \mathbb{N}$$

a sequence of ν-independent sets. If $A \subset \mathbb{N}$ is such set that $\overline{\nu}(A) > 0$ and $\nu(A \cap A_n) = 0, n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} \nu(A_n) < \infty.$$

This assertion follows immediately from Theorem 4, because in the opposite case should be $\nu(A) = 0$.

Proposition 4. *Let $A_1, A_2, A_3, \dots,$ be sets as in Proposition 3 and*

$$\lim_{N\to\infty} \overline{\nu}\left(\bigcup_{n=N}^{\infty} A_n\right) = 0, \tag{8}$$

then the set $\bigcup_{n=1}^{\infty} A_n$ belongs to $\mathcal{D}_\nu$ and

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 - \prod_{n=1}^{\infty}\big(1 - \nu(A_n)\big).$$

Proof. Analogously, as in proof of Theorem 3 we put

$$B_N = \cup_{n=1}^{N-1} A_n,$$

from (8) we get

$$\lim_{N\to\infty} \overline{\nu}(A \setminus B_N) = 0. \qquad \square$$

Suppose that an outer density $\overline{\nu}$ is given. Let us consider a system of decompositions

$$\mathcal{E}_n = \left\{E_1^{(n)}, \dots, E_{k_n}^{(n)}\right\}, \quad E_i^{(n)} \in \mathcal{D}_\nu, \quad \nu\left(E_i^{(n)}\right) = \frac{1}{k_n} \tag{9}$$

for $n = 1, 2, 3, \dots$, such that the sets $E_{i_1}^{(1)}, \dots, E_{i_n}^{(n)}$ are ν-independent, $n \in \mathbb{N}$. Denote for $S \subset \mathbb{N}$ by the symbol $I(S : \mathcal{E}_n)$ the number of such $i \le k_n$ that $\overline{\nu}\big(S \cap E_i^{(n)}\big) > 0$. In this case we have

$$S \subset \bigcup_{\ell=1}^{I(S:\mathcal{E}_n)} E_{i_\ell}^{(n)} \cup S'_n,$$

where $\nu(S'_n) = 0$, $n = 1, 2, 3, \dots$ This yields

$$S \subset \bigcap_{n=1}^{N} \bigcup_{\ell=1}^{I(S:\mathcal{E}_n)} E_{i_\ell}^{(n)} \cup S'_n, \quad N = 1, 2, 3, \dots$$

From the independence mentioned above we get

$$\overline{\nu}(S) \le \prod_{n=1}^{N} \frac{I(S:\mathcal{E}_n)}{k_n}, \quad N = 1, 2, 3, \dots$$

Considering $N \to \infty$ we get

$$\overline{\nu}(S) \le \prod_{n=1}^{\infty} \frac{I(S:\mathcal{E}_n)}{k_n}. \tag{10}$$

This implies:

Proposition 5. *If* $\overline{\nu}(S) > 0$, *then*

$$\lim_{n\to\infty} \frac{I(S:\mathcal{E}_n)}{k_n} = 1.$$

P r o o f. Clearly,

$$\frac{I(S,\mathcal{E}_n)}{k_n} \le 1.$$

Let $c \in [0,1)$. If

$$\frac{I(S:\mathcal{E}_n)}{k_n} < c$$

for infinitely many n then (10) yields $\overline{\nu}(S) = 0$ — a contradiction. □

Proposition 6. *If* $\overline{\nu}(S) > 0$ *and* $I(S:\mathcal{E}_n) < k_n$ *for* $n \in \mathbb{N}$, *then*

$$\sum_{n=1}^{\infty} \frac{1}{k_n} < \infty.$$

P r o o f. If $I(S:\mathcal{E}_n) < k_n$, then $I(S:\mathcal{E}_n) \le k_n - 1$. From (10) we get

$$\overline{\nu}(S) \le \prod_{n=1}^{\infty} \left(1 - \frac{1}{k_n}\right),$$

and the assertion follows. □

3. Reminder systems

A finite set $R \subset \mathbb{N}$ is called a *complete reminder system* modulo m, for given $m \in \mathbb{N}$ if for each $s \in \{0, \dots, m-1\}$ such uniquely determined $r \in R$ exists that $r \equiv s \pmod{m}$. Clearly, a subset of positive integers is a complete reminder system modulo m if and only if it has m elements, and these elements are incongruent modulo m. Let us come back to the system of decomposition defined in (9). Suppose that a sequence of mutually relative primes natural numbers $m_1, m_2, m_3, \dots$ is given. Put

$$\mathcal{E}_k = \{j + (m_k); j = 0, \dots, m_k - 1\}, \quad k = 1, 2, 3, \dots$$

In this case we shall write $I(A : m_k)$ instead of $I(A : \mathcal{E}_k)$. Suppose that $\overline{\nu}$ is an outer density such that $\nu = \overline{\nu}|_{\mathcal{D}_\nu}$ is an arithmetic density.

From Proposition 5 we get

7. If $A \subset \mathbb{N}$ and $\overline{\nu}(A) > 0$, then

$$\lim_{n\to\infty} \frac{I(A : m_n)}{m_n} = 1,$$

where $m_1, m_2, m_3, \dots$ is a sequence of relatively primes natural numbers.

This yields

Proposition 7. *Let $m_1, m_2, m_3, \dots,$ be such sequence of relatively primes natural numbers greater than 1 that*

$$\sum_{n=1}^{\infty} \frac{1}{m_n} = \infty. \tag{11}$$

If $A \subset \mathbb{N}$ and the inequalities $I(A : m_k) < m_k$ hold for $k = 1, 2, 3, \dots,$ then

$$A \in \mathcal{D}_\nu \quad and \quad \mu(A) = 0.$$

Theorem 5. *Let $A \subset \mathbb{N}$ is such set that $\overline{\nu}(A) > 0$. If $m_1, m_2, m_3, \dots$ be positive integers mutually relative prime such that A does not contain the complete reminder system modulo $m_k, k = 1, 2, 3, \dots,$ then $\sum_{j=1}^{\infty} \frac{1}{m_j} < \infty$.*

Proof. The assumption gives $I(A : m_j) \le m_j - 1$. From (10) we get

$$0 < \overline{\nu}(A) \le \prod_{j=1}^{\infty} \left(1 - \frac{1}{m_j}\right),$$

and the assertion follows. □

4. Distribution of sequences

We say that a bounded sequence of real numbers $\{v(n)\}$ is ν-measurable if for each real number x the set $v^{-1}((-\infty, x))$ belongs to $\mathcal{D}_\nu$. In this case the function $F(x) = \nu\big(v^{-1}((-\infty, x))\big)$ is called *ν-distribution function* of this sequence. Given ν-measurable sequences $\{v_1(n)\}, \dots \{v_k(n)\}$ will be called *ν-independent* if for each $x_1, \dots, x_k$ the sets $v((-\infty, x_1)), \dots, v((-\infty, x_k))$ are ν-independent.

Let $\{v_1(n)\}, \dots, \{v_s(n)\}, \dots,$ be such ν-measurable sequences that for each $k \in \mathbb{N}$ the sequences $\{v_1(n)\}, \dots, \{v_s(n)\}$ are ν-independent. Let us denote by F_i the ν-distribution function of $\{v_i(n)\}, i \in \mathbb{N}$.

By an application of Theorem 3 we get

8. Let $[x_i, y_i), i = 1, 2, 3, \ldots$ be subintervals of $\mathbb{R}$. Then

$$\cup_{i=1}^{k} v_i^{-1}([x_i, y_i)) \in \mathcal{D}_\nu$$

and

$$\nu\left(\bigcup_{i=1}^{k} v_i^{-1}([x_i, y_i)\right) = 1 - \prod_{i=1}^{k}\Big(1 - \big(F_i(y_i) - F_i(x_i)\big)\Big),$$

Therefore

9. If

$$\sum_{i=1}^{\infty} F_i(y_i) - F_i(x_i) = \infty,$$

then the set

$$V = \cup_{i=1}^{\infty} v_i^{-1}([x_i, y_i)) \quad \text{belongs to } \mathcal{D}_\nu \text{ and } \nu(V) = 1.$$

We say that a sequence $\{v(n)\}$ of elements of the interval $[0, 1]$ is *ν-uniformly distributed* modulo 1, if it is ν-measurable and its ν-distribution function is

$$F(x) = 0, \ \ x \le 0, \quad F(x) = x, \ \ x \in [0, 1], \quad F(x) = 1, \ \ x \ge 1.$$

Proposition 8. *Let $\{v_k(n)\}, k = 1, 2, 3, \ldots$ be ν-independent sequences, ν-uniformly distributed modulo 1. Suppose that $A \subset \mathbb{N}$ is such set, that $\overline{\nu}(A) > 0$. Then for each $x \in [0, 1]$ and $\varepsilon > 0$ such k_0 exists that for $k > k_0$ such $n_k \in A$ exists that $|x - v_k(n_k)| < \varepsilon$.*

Proof. Let m be such natural number that $\frac{1}{m} < \varepsilon$. Consider the system of decompositions

$$\mathcal{E}_k = \left\{E_j^{(k)}, \quad j = 0, \ldots, m-1\right\}, \ \ k = 1, 2, 3, \ldots,$$

where

$$E_j^{(k)} = v_k^{-1}\left(\left[\frac{j}{m}, \frac{j+1}{m}\right)\right), \quad j = 0, \ldots, m-2, \quad E_{m-1}^{(k)} = v_k^{-1}\left(\left[\frac{j}{m}, 1\right)\right).$$

Clearly, the set $E_j^{(k)}$ s ν-measurable and

$$\nu\left(E_j^{(k)}\right) = \frac{1}{m} \quad \text{for each } k$$

and

$$j = 0, \ldots, m-1.$$

Thus Proposition 6 yields that

$$A \cup E_j^{(k)} \neq \emptyset, \quad j = 0, \ldots m-1$$

with except of finite number of k. $\square$

Let us denote
$$E_N(v) = \frac{1}{N}\sum_{n=1}^{N} v(n),$$
where $\{v(n)\}$ is a sequence and $N \in \mathbb{N}$.

Sequences $\{v_j(n)\}, j = 1, \dots, k$ are called *statistical independent* if and only if for every continuous function $f_1, \dots, f_k$ we have
$$E_N\left(\prod_{j=1}^{k} f_j(v_j)\right) - \prod_{j=1}^{k} E_N\big(f_j(v_j)\big) \to 0$$
for $N \to \infty$.

The bounded sequences of real numbers $\{v_1(n)\}, \dots, \{v_k(n)\}$ which are ν-measurable are called *weakly independent* if for each $x_1, \dots, x_k \in \mathbb{R}$ such that ν-distribution function of $\{v_j(n)\}$ is continuous in $x_j, j = 1, \dots, k$, we have
$$\cap_{j=1}^{k} v_j^{-1}((-\infty, x_j)) \in \mathcal{Y}$$
and
$$\nu\left(\bigcap_{j=1}^{k} v_j^{-1}\big((-\infty, x_j)\big)\right) = \prod_{j=1}^{k} \nu\left(\bigcap_{j=1}^{k} v_j^{-1}\big((-\infty, x_j)\big)\right). \tag{12}$$

In the paper [11] is proven the following statement

10. (Theorem 3 of [11]) If the sequences $\{v_1(n)\}, \dots, \{v_k(n)\}$ of elements of some interval $[0, 1]$ are sequences uniformly distributed modulo 1, then these sequences are statistically independent if and only if for every interval $I_1, \dots, I_k$, subintervals of $[0, 1]$ the sets $v_1^{-1}(I_1) \dots v_k^{-1}(I_k)$ are d-independent.

11. If $\{x_k(n)\}$, $k = 1, 2, 3, \dots$ are statistically independent uniformly distributed sequences mod 1 and I_k are subintervals of $[0, 1]$, then the set
$$R(N) = \{n \in \mathbb{N}; \forall k \le N; x_k(n) \in I_k\} \quad \text{belongs to } \mathcal{D}$$
and
$$d\big(R(N)\big) = |I_1| \cdots |I_N|, \quad \text{for } N = 1, 2, 3, \dots$$

Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be irrational numbers linearly independent over the field of rational numbers, then the sequences $\{\{\alpha_i n\}\}, i = 1, \dots, k$ are statistically independent, (see [13]). Let us denote
$$S_c(\alpha_1, \alpha_2, \dots, \alpha_k) = \bigcap_{j=1}^{k} S_c(\alpha_j).$$
This implies

Proposition 9. *Let $\alpha_1, \dots, \alpha_k$ be irrational numbers linearly independent over the field of rational numbers. Then $S_c(\alpha_1, \dots, \alpha_k)$ belongs to $\mathcal{D}$ and*
$$d\big(S_c(\alpha_1, \dots, \alpha_k)\big) = (2c)^k.$$

Proposition 10. *Let $\alpha_1, \alpha_2, \ldots$ be irrational numbers linearly independent over the field of rational numbers. If S_c is the set of such positive integers n that for suitable integers $p_{n,j}$, the inequalities*

$$\left|\alpha_j - \frac{p_{n,j}}{n}\right| < \frac{c}{n} \quad \textit{hold,}$$

S_c has asymptotic density and $d(S) = 0$.

Proof. We recall that in Introduction is supposed that $c \in [0, \frac{1}{2})$. Thus the assertion follows from the previous proposition taking into account that

$$S_c = \cap_{k=1}^{\infty} S_c(\alpha_1, \ldots, \alpha_k)\,. \qquad \square$$

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