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EXISTENCE RESULT FOR A STOCHASTIC FUNCTIONAL DIFFERENTIAL SYSTEM DRIVEN BY G-BROWNIAN MOTION WITH INFINITE DELAY

El-Hacène Chalabi[1] — Salim Mesbahi[2] — Amar Ouaoua[3]

[1]Salah Boubnider University, Constantine, ALGERIA

[2]Ferhat Abbas University, Setif, ALGERIA

[3]20 August 1955 University, Skikda, ALGERIA

ABSTRACT. In this article, we are interested in the study of a class of stochastic functional differential systems driven by G-Brownian motion with infinite delay. We prove the existence and uniqueness of the solutions when two basic conditions are met: the linear growth condition and the Lipschitz condition.

1. Introduction

Due to their wide applications in various fields, such as science and engineering, stochastic differential systems are receiving increasing attention from scientists, as they are more precise in describing certain physical phenomena, and this is what prompts us to study this kind of problems; see, for example, Kolarova [4, 5]. Among the most important of these systems, we find the class of stochastic functional differential systems driven by G-Brownian motion with infinite delay, used in the mathematical modelling of certain phenomena in branches of science and engineering such as fluid mechanics, thermodynamics and wave propagation. This class has been studied by many mathematicians, we cite, for example, the works of Faizullah et *al.* [2, 3], Bao and Jiang [1], Ren et *al.* [10–12], Wang et *al.* [13], Wei and Wang [14], Wu et *al.* [15] and references therein. They obtained several important results concerning the existence of the solution, its uniqueness and its estimation under appropriate conditions.

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They also discussed stability, boundedness, convergence of the solution and other issues.

The problem we are addressing in this work fits in this context. We are interested in the study of the existence and uniqueness of the solution for a class of stochastic functional differential systems with infinite delay driven by G-Brownian motion (G-ISFDSs).

What's new here is that our problem has been presented in the form of a system, which generalizes the works studied in the form of equations only, and also some new conditions, as we will see later in the assumption **(A1)**. This shift from an equation to a system required several improvements in the techniques used to overcome some of the difficulties. We would like to stress once again the importance of the question under consideration in that it's a mixture of several concepts, which makes it a very important model.

The rest of this paper is organized as follows. In the next section, we provide some notations and definitions necessary to understand the content of this work. In the third section, we pose the problem we want to study with its hypotheses and our main result. The last section is devoted to prove the main result.

2. Preliminary definitions and concepts

In this section, we recall some basic notions and definitions that we use in this work. More details and information related to this section may be found in Peng [6–9].

Let Ω be a given non-empty set, and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω such that any arbitrary constant $c \in \mathcal{H}$ and if $X \in \mathcal{H}$, then $|X| \in \mathcal{H}$. We consider that $\mathcal{H}$ is the space of random variables.

Definition 2.1. A functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ is called sublinear expectation, if for all X, Y in $\mathcal{H}$, c in $\mathbb{R}$ and $\lambda \geq 0$, the following properties are satisfied:

(i) (Monotonicity): if $X \leq Y$, then $\mathbb{E}[X] \leq \mathbb{E}[Y]$,

(ii) (Constant preserving): $\mathbb{E}[c] = c$,

(iii) (Sub-additivity): $\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$,

(iv) (Positive homogeneity): $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called sublinear expectation space.

We assume that if $X_1,\ X_2, \ldots,\ X_n \in \mathcal{H}$, then $\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{\ell,lip}(\mathbb{R}^n)$, where $C_{\ell,\mathrm{Lip}}(\mathbb{R}^n)$ is the space of linear functions φ defined as follows: for all $x,\ y \in \mathbb{R}^n$

$$C_{\ell,\mathrm{Lip}}(\mathbb{R}^n) = \{\varphi : \mathbb{R}^n \to \mathbb{R}; |\varphi(x) - \varphi(y)| \leq C\left(1 + |x|^m + |y|^m\right)|x-y|\},$$

where C is a positive constant and $m \in \mathbb{N}^*$ dependent only on φ.

Let S^d be the space of $d \times d$ symmetric matrices. Γ is a given non-empty, bounded and closed subset of S^d. For $A = (A_{i,j})_{i,j=1}^{d} \in S^d$ given, we define $G : S^d \to \mathbb{R}$ by

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} Tr\left(\gamma \gamma^{Tr} A\right).$$

Definition 2.2. In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a d-dimensional vector of random variables $X \in \mathcal{H}^d$ is G-normal distributed, *if* for each $\varphi \in C_{\ell,\mathrm{Lip}}(\mathbb{R}^d)$, the function

$$u(t,x) = \mathbb{E}\Big(\varphi\big(x + \sqrt{t}X\big)\Big)$$

is the unique viscosity solution of the following G-heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = G(D^2 u), & \\ & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) = \varphi(x), & \end{cases}$$

where $D^2 u = \left(\partial^2_{x_i x_j} u\right)_{i,j}^{d}$ is the Hessian matrix of u.

We denote by $\Omega = C_0(\mathbb{R})$ the space of all $\mathbb{R}$-valued continuous functions ω defined on $\mathbb{R}_+$ such that $\omega(0) = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) = \sum_{k=1}^{\infty} 2^{-k} \max_{t \in [0,k]} \left[\left|\left(\omega^1(t) - \omega^2(t)\right)\right| \wedge 1\right].$$

For each fixed $T > 0$, let $\Omega_T = \{\omega(._{\wedge T}), \omega \in \Omega\}$ with

$$\mathrm{Lip}(\Omega_T) = \left\{\varphi\big(B(t_1), \ldots, B(t_m)\big),\ m \geq 1,\ t_1, \ldots, t_m \in [0,T], \varphi \in C_{\ell,\mathrm{Lip}}(\mathbb{R}^m)\right\}$$

and let $\mathrm{Lip}(\Omega) = \overset{\infty}{\underset{n=1}{\cup}} \mathrm{Lip}(\Omega_n)$. We denote by $L^p_G(\Omega_T)$, $p \geq 1$, the completion of $\mathrm{Lip}(\Omega_T)$ under the norm $||X||_p = (\mathbb{E}[|X|^p])^{\frac{1}{p}}$. Similarly, we denote $L^p_G(\Omega)$ the completion space of $\mathrm{Lip}(\Omega)$. It was shown in Peng [6,7] that there exists a family of probability measures $\mathcal{P}$ on Ω such that

$$\mathbb{E}[X] = \sup_{P \in \mathcal{P}} E^P[X] \quad \text{for } X \in L^1_G(\Omega),$$

where E^P stands for the linear expectation under the probability P. We say that a property holds quasi surely (q.s.) if it holds for each $P \in \mathcal{P}$.

For a finite partition of $[0,T]$, $\pi_T = \{t_0, t_1, \ldots, t_N\}$, we set

$$\mu(\pi_T) = \max\{|t_{ki+1} - t_k|,\ k = 0, 1, \ldots, N-1\}.$$

Consider the collection $M_G^{p,0}(0,T)$ of simple processes defined by

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1}[}(t),$$

where

$$\begin{aligned} \xi_k &\in L_G^p(\mathcal{F}_{t_k}),\ 0 \le k \le N-1,\ \ p \ge 1, \\ \mathcal{F}_t &= \sigma\{B(s),\ \ 0 \le s \le t\} \end{aligned}$$

is the filtration generated by the canonical process $\big(B(t)\big)_{t\ge 0}$, and

$$L_G^p(\mathcal{F}_t) = \left\{\xi \in L_G^1(\mathcal{F}_t);\ \mathbb{E}(\xi) < \infty\right\}.$$

The completion of $M_G^{p,0}(0,T)$ under the norm

$$\|\eta\| = \left(\frac{1}{T}\int\limits_0^T \mathbb{E}\left(|\eta_t|^p\right)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

is denoted by $M_G^p(0,T)$. Note that $M^qG(0,T) \subset M_G^p(0,T)$, for $1 \le p \le q$.

3. Statement of the problem and main result

3.1. Assumptions

To study our problem, assume that $\big(B(t)\big)_{t\ge 0}$ is an m-dimensional G-Brownian motion defined on complete probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions. Let

$$f_1, g_1 :\ [0,T] \times \Big(BC\big((-\infty,T],\mathbb{R}^d\big)\Big)^2 \to \mathbb{R}^d,$$

$$f_2,\ f_3,\ g_2,\ g_3 :\ [0,T] \times \Big(BC\big((-\infty,T],\mathbb{R}^d\big)\Big)^2 \to \mathbb{R}^{d\times m}$$

are measurable Borel's functions. Consider the following d-dimensional G-ISFDSs

$$\begin{cases} dX(t) &= f_1\big(t, X(t), Y(t)\big)\,\mathrm{d}t + f_2\big(t, X(t), Y(t)\big)\,\mathrm{d}\left\langle B, B\right\rangle(t) \\ & \quad + f_3\big(t, X(t), Y(t)\big)\,\mathrm{d}B(t), \\ dY(t) &= g_1\big(t, X(t), Y(t)\big)\,\mathrm{d}t + g_2\big(t, X(t), Y(t)\big)\,\mathrm{d}\left\langle B, B\right\rangle(t) \\ & \quad + g_3\big(t, X(t), Y(t)\big)\,\mathrm{d}B(t), \end{cases} \tag{1}$$

where $0 \le t \le T$. The initial condition $\big(\alpha(0), \beta(0)\big) \in \mathbb{R}^{2d}$ is given, $\big(\langle B, B\rangle(t)\big)$ for $t \ge 0$; is the quadratic variation process of G-Brownian motion $\big(B(t)\big)_{t\ge 0}$, and for $1 \leq i \leq 3$, $f_i(\cdot, x, y)$, $g_i(\cdot, x, y) \in M_G^2\big((-\infty, T]; \mathbb{R}^{2d}\big)$ for all x, $y \in \mathbb{R}^d$. Next, we give the initial data of (1) as follows

$$\begin{aligned}\big(X(t_0), Y(t_0)\big) = (\alpha, \beta) = \big\{\big(\alpha(\zeta), \beta(\zeta)\big) : -\infty < \zeta \leq 0\big\} \\ \in M_G^2\left((-\infty, 0]\,; \mathbb{R}^d\right) \times M_G^2\left((-\infty, 0]\,; \mathbb{R}^d\right) \end{aligned} \tag{2}$$

such that $X\big(t_0\big), Y\big(t_0\big)$ are $\mathcal{F}_0$ measurable and $BC\left((-\infty, 0]\,; \mathbb{R}^d\right)$-values random variables.

Thus, (1) with initial value (2) can be written in the following form:

$$\begin{cases} X(t) & = & \alpha(0) + \int\limits_0^t f_1\big(s, X(s), Y(s)\big)\,\mathrm{d}s + \int\limits_0^t f_2\big(s, X(s), Y(s)\big)\,\mathrm{d}\langle B, B\rangle(s) \\ & & + \int f_3\big(s, X(s), Y(s)\big)\,\mathrm{d}B(s), \\ Y(t) & = & \beta(0) + \int\limits_0^t g_1\big(s, X(s), Y(s)\big)\,\mathrm{d}s + \int\limits_0^t g_2\big(s, X(s), Y(s)\big)\,\mathrm{d}\langle B, B\rangle(s) \\ & & + \int\limits_0^t g_3\big(s, X(s), Y(s)\big)\,\mathrm{d}B(s). \end{cases} \tag{3}$$

We first introduce the notion of solution to the problem (3) used here.

Definition 3.1. $\big(X(t), Y(t)\big)$ on $-\infty < t \leq T$ is called the solution of (3) with initial data (2), if:

(i) $\big(X(t), Y(t)\big)$ is continuous and for all $0 \leq t \leq T$; $X(t)$ and $Y(t)$ are $\mathcal{F}_t$- -adapted.

(ii)
$$\begin{aligned} f_1,\ g_1 &\in L^1\left([0, T]; \mathbb{R}^d\right), \\ f_2,\ f_3,\ g_2,\ g_3 &\in L^2\left([0, T]; \mathbb{R}^{d \times m}\right). \end{aligned}$$

(iii) $\big(X(0), Y(0)\big) = \big(\alpha, \beta\big)$, and for each $t \in [0, T]$, we have $\big(X(t), Y(t)\big)$ is solution of (1) in the quasi surely sense.

Now, we assume that for all function $h = f_i$ or g_i, $1 \leq i \leq 3$, there exist two positive and continuous functions on $[0, T]$, φ and ψ, such that,

(A1): For all x, $y \in BC\left((-\infty, 0]\,; \mathbb{R}^d\right)$, and $t \in [0, T]$

$$|h(t, x, y)|^2 \leq \varphi(t)\left(1 + |x|^2 + |y|^2\right).$$

(A2): For all x_1, y_1, x_2, $y_2 \in BC\big((-\infty, 0]; \mathbb{R}^d\big)$, and $t \in [0, T]$

$$|h(t, x_2,\ y_2) - h(t, x_1, y_1)|^2 \leq \psi(t)\left(|x_2 - x_1|^2 + |y_2 - y_1|^2\right).$$

In the following we equip the space of processes in

$$M_G^2\left((-\infty,T];\mathbb{R}^d\right)\times M_G^2\left((-\infty,T];\mathbb{R}^d\right)$$

with the norm

$$||(X,Y)||=\mathbb{E}^{\frac{1}{2}}\left[\sup_{-\infty<t\leq T}\left(|X(t)|^2+|Y(t)|^2\right)\right].$$

Note that this space is indeed of Banach.

3.2. The main result

Now, we can state the main result of this paper:

Theorem 3.2. *Under the assumptions* **(A1)** *and* **(A2)***, with initial data* (2) *the G-ISFDSs* (3) *has a unique solution*

$$\big(X(t),Y(t)\big)\in M_G^2\big((-\infty,T];\mathbb{R}^d\big)\times M_G^2\big((-\infty,T];\mathbb{R}^d\big).$$

4. Proof of the main result

To prove the Theorem 3.2, we will need the following Lemma:

Lemma 4.1. *Let* **(A1)** *and* **(A2)** *hold. If* $\big(X(t),Y(t)\big)\in M_G^2\big((-\infty,T];\mathbb{R}^d\big)\times M_G^2\big((-\infty,T];\mathbb{R}^d\big)$ *is the solution of G-ISFDSs* (3) *with initial data* (2)*, and* $0\leq s<t\leq T$*, then*

$$\mathbb{E}\left(\sup_{-\infty<t\leq T}\left[|X(t)|^2+|Y(t)|^2\right]\right)\leq$$

$$\mathbb{E}\left[|\alpha|^2\right]+\mathbb{E}\left[|\beta|^2\right]+K_1\exp\left(TK_2\sup_{0\leq t\leq T}[\psi(t)]\right),$$

where

$$\begin{aligned}K_1&=4\left(\mathbb{E}\left[|\alpha(0)|^2\right]+\mathbb{E}\left[|\beta(0)|^2\right]\right)+16T\left[\left(C_1+C_2\right)T+C_3\right]\sup_{0\leq t\leq T}\left[\varphi(t)\right],\\ K_2&=16\left[(C_1+C_2)T+C_3\right],\end{aligned}$$

and $C_1,\ C_2,C_3$ *are positive arbitrary constants.*

P r o o f. Let $\big(X(t), Y(t)\big)$ be a solution for (3). So, for all $t \in [0,T]$, we have

$$\begin{aligned}|X(t)|^2 \quad \leq \quad & 4\,|\alpha(0)|^2 + 4\left|\int_0^t f_1\big(s, X(s), Y(s)\big)\,\mathrm{d}s\right|^2 \\ & + 4\left|\int_0^t f_2\big(s, X(s), Y(s)\big)\,\mathrm{d}\langle B, B\rangle(s)\right|^2 + 4\left|\int_0^t f_3\big(s, X(s), Y(s)\big)\,\mathrm{d}B(s)\right|^2\end{aligned}$$

and

$$\begin{aligned}|Y(t)|^2 \quad \leq \quad & 4\,|\beta(0)|^2 + 4\left|\int_0^t g_1\big(s, X(s), Y(s)\big)\,\mathrm{d}s\right|^2 \\ & + 4\left|\int_0^t g_2\big(s, X(s), Y(s)\big)\,\mathrm{d}\langle B, B\rangle(s)\right|^2 + 4\left|\int_0^t g_3\big(s, X(s), Y(s)\big)\,\mathrm{d}B(s)\right|^2.\end{aligned}$$

Taking the G-expectation, and by using BDG's and Hölder's inequalities, we obtain

$$\begin{aligned}\sup_{0\leq s\leq t} \mathbb{E}\left[|X(s)|^2\right] \leq\ & 4\mathbb{E}\left[|\alpha(0)|^2\right] + 4C_1 t\mathbb{E}\int_0^t \left|f_1\big(s, X(s), Y(s)\big)\right|^2\,\mathrm{d}s \\ & + 4C_2 t\mathbb{E}\int_0^t \left|f_2\big(s, X(s), Y(s)\big)\right|^2\,\mathrm{d}s + 4C_3\mathbb{E}\int_0^t \left|f_3\big(s, X(s), Y(s)\big)\right|^2\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}\sup_{0\leq s\leq t} \mathbb{E}\left[|X(s)|^2\right] \leq\ & 4\mathbb{E}\left[|\alpha(0)|^2\right] \\ & + 8C_1 t\mathbb{E}\int_0^t \Big(\left|f_1\big(s, X(s), Y(s)\big) - f_1\,(s,0,0)\right|^2 + |f_1\,(s,0,0)|^2\Big)\,\mathrm{d}s \\ & + 8C_2 t\mathbb{E}\int_0^t \Big(\left|f_2\big(s, X(s), Y(s)\big) - f_2\,(s,0,0)\right|^2 + |f_2\,(s,0,0)|^2\Big)\,\mathrm{d}s \\ & + 8C_3\mathbb{E}\int_0^t \Big(\left|f_3\big(s, X(s), Y(s)\big) - f_3\,(s,0,0)\right|^2 + |f_3\,(s,0,0)|^2\Big)\,\mathrm{d}s.\end{aligned}$$

By using the assumptions **(A1)** and **(A2)**, we have

$$\mathbb{E}\left(\sup_{0\leq s\leq t}|X(s)|^2\right)\leq 4\mathbb{E}\left[|\alpha(0)|^2\right]$$
$$+\ 8[(C_1+C_2)t+C_3]\mathbb{E}\int_0^t\left(\varphi(s)+\psi(s)\,|X(s)|^2\right)\,\mathrm{d}s,$$

so

$$\mathbb{E}\left(\sup_{0\leq s\leq t}|X(s)|^2\right)\leq 4\mathbb{E}\left[|\alpha(0)|^2\right]+8\left[(C_1+C_2)\,t+C_3\right]\int_0^t\varphi(s)\,\mathrm{d}s$$
$$+\ 8[(C_1+C_2)t+C_3]\int_0^t\psi(s)\mathbb{E}\left(\sup_{0\leq u\leq s}|X\,(u)|^2\right)\,\mathrm{d}s.$$

Likewise, we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq s\leq t}|Y(s)|^2\right) &\leq 4\mathbb{E}\left[|\beta(0)|^2\right]+8[(C_1+C_2)t+C_3]\int_0^t\varphi(s)\,\mathrm{d}s\\ &\quad +8[(C_1+C_2)t+C_3]\int_0^t\psi(s)\mathbb{E}\left(\sup_{0\leq u\leq s}|X\,(u)|^2\right)\,\mathrm{d}s\end{aligned}$$

which give

$$\mathbb{E}\left(\sup_{0\leq s\leq t}\left[|X(s)|^2+|Y(s)|^2\right]\right)$$
$$\leq K_1+K_2\int_0^t\psi\,(u)\,\mathbb{E}\left(\sup_{0\leq u\leq s}\left[|X\,(u)|^2+|Y\,(u)|^2\right]\right)\,\mathrm{d}s,$$

where

$$\begin{aligned}K_1 &= 4\left(\mathbb{E}\,|\alpha(0)|^2+\mathbb{E}\,|\beta(0)|^2\right)+16T[(C_1+C_2)t+C_3]\sup_{0\leq t\leq T}\varphi(t),\\ K_2 &= 16[(C_1+C_2)t+C_3].\end{aligned}$$

We conclude, by Gronwall's Lemma that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left[|X(t)|^2+|Y(t)|^2\right]\right)\leq K_1\exp\left(TK_2\sup_{0\leq t\leq T}[\psi(t)]\right).$$

Noting that the fact

$$\begin{cases} \sup\limits_{-\infty<s\le t} |X(s)|^2 \le |\alpha|^2 + \sup\limits_{0<s\le t} |X(s)|^2, \\ \sup\limits_{-\infty<s\le t} |Y(s)|^2 \le |\beta|^2 + \sup\limits_{0<s\le t} |X(s)|^2, \end{cases}$$

then

$$\begin{aligned} &\mathbb{E}\left(\sup_{-\infty<s\le t}\left[|X(s)|^2 + |Y(s)|^2\right]\right) \\ &\le \mathbb{E}\left[|\alpha|^2\right] + \mathbb{E}\left[|\beta|^2\right] + \mathbb{E}\left(\sup_{0\le s\le t}\left[|X(s)|^2 + |Y(s)|^2\right]\right) \\ &\le \mathbb{E}\left[|\alpha|^2\right] + \mathbb{E}\left[|\beta|^2\right] + K_1 \exp\left(TK_2 \sup_{0\le t\le T}[\psi(t)]\right) \end{aligned}$$

which proves our Lemma. □

Now, we can prove the main result of this work.

Proof of Theorem 3.2.

(i) Uniqueness: Suppose that $\big(X_1(t), Y_1(t)\big)$ and $\big(X_2(t), Y_2(t)\big)$ are two solutions of the system (3), then we have

$$\begin{cases} X_1(t) - X_2(t) = \alpha_1(0) - \alpha_2(0) \\ + \int_0^t f_1\big(s, X_1(s), Y_1(s)\big) - f_1\big(s, X_2(s), Y_2(s)\big)\,\mathrm{d}s \\ + \int_0^t f_2\big(s, X_1(s), Y_1(s)\big) - f_2\big(s, X_2(s), Y_2(s)\big)\,\mathrm{d}\langle B, B\rangle(s) \\ + \int_0^t f_3\big(s, X_1(s), Y_1(s)\big) - f_3\big(s, X_2(s), Y_2(s)\big)\,\mathrm{d}B(s)\,, \\ Y_1(t) - Y_2(t) = \beta_1(0) - \beta_2(0) \\ + \int_0^t g_1\big(s, X_1(s), Y_1(s)\big) - g_1\big(s, X_2(s), Y_2(s)\big)\,\mathrm{d}s \\ + \int_0^t g_2\big(s, X_1(s), Y_1(s)\big) - g_2\big(s, X_2(s), Y_2(s)\big)\,\mathrm{d}\langle B, B\rangle(s) \\ + \int_0^t g_3\big(s, X_1(s), Y_1(s)\big) - g_3\big(s, X_2(s), Y_2(s)\big)\,\mathrm{d}B(s). \end{cases} \tag{4}$$

Taking the G-expectation of the first equation of (4), by using the BDG's and Hölder's inequalities and **(A2)**, we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le s\le t}|X_1(s)-X_2(s)|^2\right) &\le 4\mathbb{E}\left(|\alpha_1(0)-\alpha_2(0)|^2\right)\\ &+4tC_1\int_0^t\psi(s)\mathbb{E}\left(|X_1(s)-X_2(s)|^2+|Y_1(s)-Y_2(s)|^2\right)\,\mathrm{d}s\\ &+4tC_2\int_0^t\psi(s)\mathbb{E}\left(|X_1(s)-X_2(s)|^2+|Y_1(s)-Y_2(s)|^2\right)\,\mathrm{d}s\\ &+4C_3\int_0^t\psi(s)\mathbb{E}\left(|X_1(s)-X_2(s)|^2+|Y_1(s)-Y_2(s)|^2\right)\,\mathrm{d}s\end{aligned}$$

therefore

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le s\le t}|X_1(s)-X_2(s)|^2\right)\le 4\mathbb{E}\left(|\alpha_1(0)-\alpha_2(0)|^2\right)\\ &+4[(C_1+C_2)t+C_3]\int_0^t\psi(s)\mathbb{E}\left[\left(|X_1(s)-X_2(s)|^2+|Y_1(s)-Y_2(s)|^2\right)\right]\,\mathrm{d}s.\end{aligned}$$

Likewise, we have

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le s\le t}|Y_1(s)-Y_2(s)|^2\right)\le 4\mathbb{E}\left(|\beta_1(0)-\beta_2(0)|^2\right)\\ &\quad+4[(C_1+C_2)t+C_3]\int_0^t\psi(s)\mathbb{E}\left[\left(|X_1(s)-X_2(s)|^2+|Y_1(s)-Y_2(s)|^2\right)\right]\,\mathrm{d}s.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le s\le t}|X_1(s)-X_2(s)|^2+|Y_1(s)-Y_2(s)|^2\right)\\ &\le 4\left[\mathbb{E}\left(|\alpha_1(0)-\alpha_2(0)|^2\right)+\mathbb{E}\left(|\beta_1(0)-\beta_2(0)|^2\right)\right]\\ &\quad+8[(C_1+C_2)t+C_3]\int_0^t\psi(s)\mathbb{E}\left(\sup_{0\le u\le s}|X_1(u)-X_2(u)|^2+|Y_1(u)-Y_2(u)|^2\right)\,\mathrm{d}s.\end{aligned}$$

By using Gronwall's Lemma, we find

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le t}|X_1(s)-X_2(s)|^2+|Y_1(s)-Y_2(s)|^2\right)\\
&\le 4\left[\mathbb{E}\left(|\alpha_1(0)-\alpha_2(0)|^2\right)+\mathbb{E}\left(|\beta_1(0)-\beta_2(0)|^2\right)\right]\exp\left(\frac{K_2}{2}t\sup_{0\le s\le t}[\psi(s)]\right).
\end{aligned}$$

Now, taking $\big(\alpha_1(0),\beta_1(0)\big)=\big(\alpha_2(0),\beta_2(0)\big)$, we obtain for $t=T$:

$$\mathbb{E}\left(\sup_{0\le t\le T}\left[|X_1(t)-X_2(t)|^2+|Y_1(t)-Y_2(t)|^2\right]\right)=0$$

which gives

$$\big(X_1(t),Y_1(t)\big)=\big(X_2(t),Y_2(t)\big),\quad \text{for each } t\in[0,T].$$

Therefore, for all $t\in(-\infty,T]$, $\big(X_1(t),Y_1(t)\big)=\big(X_2(t),Y_2(t)\big)$ q.s.

(ii) Existence: Let $\big(X_0^0,Y_0^0\big)=(\alpha,\beta)$ and $\big(X_0(t),Y_0(t)\big)=\big(\alpha(0),\beta(0)\big)$ for $t\in[0,T]$. Let $\big(X_0^n,Y_0^n\big)=(\alpha,\beta)$, $n\ge 1$ and for $t\in[0,T]$, we define the following Picard sequence:

$$\begin{cases}
X^n(t) &= \alpha(0)+\int_0^t f_1\left(s,X^{n-1}(s),Y^{n-1}(s)\right)\,\mathrm{d}s\\
&\quad+\int_0^t f_2\left(s,X^{n-1}(s),Y^{n-1}(s)\right)\,\mathrm{d}\langle B,B\rangle(s)\\
&\quad+\int f_3\left(s,X^{n-1}(s),Y^{n-1}(s)\right)\,\mathrm{d}B(s),\\
Y^n(t) &= \beta(0)+\int_0^t g_1\left(s,X^{n-1}(s),Y^{n-1}(s)\right)\,\mathrm{d}s\\
&\quad+\int_0^t g_2\left(s,X^{n-1}(s),Y^{n-1}(s)\right)\,\mathrm{d}\langle B,B\rangle(s)\\
&\quad+\int_0^t g_3\big(s,X^{n-1}(s),Y^{n-1}(s)\big)\,\mathrm{d}B(s).
\end{cases}\tag{5}$$

Obviously,

$$\big(X_0(t),Y_0(t)\big)\in M_G^2\left((-\infty,T],\mathbb{R}^d\right)\times M_G^2\left((-\infty,T],\mathbb{R}^d\right),$$

and by induction

$$\big(X^n(t),Y^n(t)\big)\in M_G^2\left((-\infty,T],\mathbb{R}^d\right)\times M_G^2\left((-\infty,T],\mathbb{R}^d\right).$$

Indeed,
$$\begin{aligned}\left|X^n(t)\right|^2 \le{}& 4\left|\alpha(0)\right|^2\\ &+4\left|\int_0^t f_1\left(s, X^{n-1}(s), Y^{n-1}(s)\right)\,\mathrm{d}s\right|^2\\ &+4\left|\int_0^t f_2\left(s, X^{n-1}(s), Y^{n-1}(s)\right)\,\mathrm{d}\left\langle B\right\rangle(s)\right|^2\\ &+4\left|\int f_3\left(s, X^{n-1}(s), Y^{n-1}(s)\right)\,\mathrm{d}B(s)\right|^2.\end{aligned}$$

Taking the G-expectation, by BDG's, Hölder's and Dobe's martingale inequalities, we have
$$\begin{aligned}\mathbb{E}\left|X^n(t)\right|^2 \le{}& 4\mathbb{E}\left|\alpha(0)\right|^2\\ &+8C_1t\mathbb{E}\int_0^t\Big(\big|f_1\big(s, X^{n-1}(s), Y^{n-1}(s)\big)-f_1\big(s,0,0\big)\big|^2+\left|f_1\left(s,0,0\right)\right|^2\Big)\,\mathrm{d}s\\ &+8C_2t\mathbb{E}\int_0^t\Big(\big|f_2\big(s, X^{n-1}(s), Y^{n-1}(s)\big)-f_2\left(s,0,0\right)\big|^2+\left|f_2\left(s,0,0\right)\right|^2\Big)\,\mathrm{d}s\\ &+8C_3\mathbb{E}\int_0^t\Big(\big|f_3\left(s, X^{n-1}(s), Y^{n-1}(s)\right)-f_3\left(s,0,0\right)\big|^2+\left|f_3\left(s,0,0\right)\right|^2\Big)\,\mathrm{d}s\end{aligned}$$
and by **(A1)**, **(A2)**, it results
$$\begin{aligned}\mathbb{E}\left|X^n(t)\right|^2 \le{}& 4\mathbb{E}\left|\alpha(0)\right|^2+8\left[(C_1+C_2)T+C_3\right]\\ &\times\mathbb{E}\int_0^t\Big(\varphi(s)+\psi(s)\Big(\left|X^{n-1}(s)\right|^2+\left|Y^{n-1}(s)\right|^2\Big)\Big)\,\mathrm{d}s.\end{aligned}$$

In the same way, we find
$$\begin{aligned}\mathbb{E}\left|Y^n(t)\right|^2 \le{}& 4\mathbb{E}\left|\beta(0)\right|^2+8[(C_1+C_2)T+C_3]\\ &\times\mathbb{E}\int_0^t\Big(\varphi(s)+\psi(s)\Big(\left|X^{n-1}(s)\right|^2+\left|Y^{n-1}(s)\right|^2\Big)\Big)\,\mathrm{d}s\end{aligned}$$
so
$$\mathbb{E}\left[\left|X^n(t)\right|^2+\left|Y^n(t)\right|^2\right]\le K_1+K_2\int_0^t\Big(\psi(s)\mathbb{E}\left[\left|X^{n-1}(s)\right|^2+\left|Y^{n-1}(s)\right|^2\right]\Big)\,\mathrm{d}s.$$

Hence, for any $k \geq 1$, it results

$$\max_{1\leq n\leq k}\left(\mathbb{E}\left[|X^n(t)|^2+|Y^n(t)|^2\right]\right)\leq K_1+K_2\int_0^t\left(\psi(s)\max_{1\leq n\leq k}\left(\mathbb{E}\left[\left|X^{n-1}(s)\right|^2+\left|Y^{n-1}(s)\right|^2\right]\right)\right)\mathrm{d}s.$$

Noting that

$$\begin{aligned}&\max_{1\leq n\leq k}\left(\mathbb{E}\left[\left|X^{n-1}(s)\right|^2+\left|Y^{n-1}(s)\right|^2\right]\right)\\&\leq\max\left\{\mathbb{E}\left|\alpha(0)\right|^2+\mathbb{E}\left|\beta(0)\right|^2,\max_{1\leq n\leq k}\left(\mathbb{E}\left[|X^n(t)|^2+|Y^n(t)|^2\right]\right)\right\}\\&\leq\mathbb{E}\left|\alpha(0)\right|^2+\mathbb{E}\left|\beta(0)\right|^2+\max_{1\leq n\leq k}\left(\mathbb{E}\left[|X^n(t)|^2+|Y^n(t)|^2\right]\right)\end{aligned}$$

we have

$$\begin{aligned}&\max_{1\leq n\leq k}\left(\mathbb{E}\left[\left|X^{n-1}(s)\right|^2+\left|Y^{n-1}(s)\right|^2\right]\right)\\&\leq K_1+K_2\int_0^t\left(\psi(s)\left(\mathbb{E}+\mathbb{E}\left|\beta(0)\right|^2+\max_{1\leq n\leq k}\left(\mathbb{E}\left[|X^n(s)|^2+|Y^n(s)|^2\right]\right)\right)\right)\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}&\max_{1\leq n\leq k}\left(\mathbb{E}\left[|X^n(t)|^2+|Y^n(t)|^2\right]\right)\\&\leq K_3+K_2\int_0^t\psi(s)\max_{1\leq n\leq k}\left(\mathbb{E}\left[|X^n(s)|^2+|Y^n(s)|^2\right]\right)\mathrm{d}s,\end{aligned}$$

where

$$K_3=K_1+K_2T\left(\mathbb{E}\left|\alpha(0)\right|^2+\mathbb{E}\left|\beta(0)\right|^2\right)\sup_{0\leq t\leq T}\left[\psi(t)\right].$$

From the Gronwall's inequality, we have

$$\max_{1\leq n\leq k}\left(\mathbb{E}\left[|X^n(t)|^2+|Y^n(t)|^2\right]\right)\leq K_3\exp\left(K_2T\sup_{0\leq t\leq T}\left[\psi(t)\right]\right).$$

Because k is arbitrary, we obtain

$$\mathbb{E}\Big[|X^n(t)|^2+|Y^n(t)|^2\Big]\leq K_3\exp\left(K_2T\sup_{0\leq t\leq T}\left[\psi(t)\right]\right),\quad t\in[0,T],\quad n\geq 1.\qquad(6)$$

This shows that

$$\big(X^n(t),Y^n(t)\big)\in M_G^2\big((-\infty,T];\mathbb{R}^d\big)\times M_G^2\big((-\infty,T];\mathbb{R}^d\big).$$

From the BDG's and Hölder's inequalities, **(A1)** and **(A2)**, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le t}\left|X^1(s)-X^0(s)\right|^2\right)\\
&\le 6[(C_1+C_2)t+C_3]\int_0^t\left(\varphi(s)+\psi(s)\mathbb{E}\left|X^0(s)\right|^2\right)\mathrm{d}s\\
&\le 6[(C_1+C_2)t+C_3]t\sup_{0\le s\le t}\varphi(s)+6[(C_1+C_2)t+C_3]t\mathbb{E}\left|\alpha\right|^2\sup_{0\le s\le t}\psi(s).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le s\le t}\left|Y^1(s)-Y^0(s)\right|^2\right)&\le 6[(C_1+C_2)t+C_3]t\sup_{0\le s\le t}\varphi(s)\\
&+6[(C_1+C_2)t+C_3]t\mathbb{E}\left|\beta\right|^2\sup_{0\le s\le t}\psi(s)
\end{aligned}$$

then

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le t}\left[\left|X^1(s)-X^0(s)\right|^2+\left|Y^1(s)-Y^0(s)\right|^2\right]\right)\\
&\le 12[(C_1+C_2)t+C_3]t\sup_{0\le s\le t}\varphi(s)\\
&\quad+6[(C_1+C_2)t+C_3]t\sup_{0\le s\le t}\psi(s)\left(\mathbb{E}\left|\alpha\right|^2+\mathbb{E}\left|\beta\right|^2\right).
\end{aligned}$$

Taking $t=T$, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le t}\left[\left|X^1(s)-X^0(s)\right|^2+\left|Y^1(s)-Y^0(s)\right|^2\right]\right)\\
&\le\frac{3}{8}TK_2\left(2\sup_{0\le t\le T}\varphi(t)+\sup_{0\le t\le T}\psi(t)\left(\mathbb{E}\left|\alpha\right|^2+\mathbb{E}\left|\beta\right|^2\right)\right)
\end{aligned}$$

then

$$\mathbb{E}\left(\sup_{0\le s\le t}\left[\left|X^1(s)-X^0(s)\right|^2+\left|Y^1(s)-Y^0(s)\right|^2\right]\right)\le\widetilde{K},$$

where

$$\widetilde{K}=\frac{3}{8}TK_2\left(2\sup_{0\le t\le T}\varphi(t)+\sup_{0\le t\le T}\psi(t)\left(\mathbb{E}\left|\alpha\right|^2+\mathbb{E}\left|\beta\right|^2\right)\right).$$

We also obtain

$$\mathbb{E}\left(\sup_{0\le s\le t}\left[\left|X^2(s)-X^1(s)\right|^2+\left|Y^2(s)-Y^1(s)\right|^2\right]\right)$$

$$\le \frac{3}{8}K_2\int\limits_0^t \psi(s)\mathbb{E}\left(\sup_{0\le u\le s}\left[\left|X^1(u)-X^0(u)\right|^2+\left|Y^1(u)-Y^0(u)\right|^2\right]\right)\mathrm{d}s$$

$$\le \frac{3}{8}K_2\widetilde{K}\int\limits_0^t \psi(s)\,\mathrm{d}s \le \frac{3}{8}K_2\widetilde{K}t\sup_{0\le s\le t}\psi(s).$$

$$\mathbb{E}\left(\sup_{0\le s\le t}\left[\left|X^3(s)-X^2(s)\right|^2+\left|Y^3(s)-Y^2(s)\right|^2\right]\right)$$

$$\le \frac{3}{8}K_2\int\limits_0^t \psi(s)\mathbb{E}\left(\sup_{0\le u\le s}\left[\left|X^2(u)-X^1(u)\right|^2+\left|Y^2(u)-Y^1(u)\right|^2\right]\right)\mathrm{d}s$$

$$\le \frac{3}{8}K_2\times\frac{3}{8}K_2\widetilde{K}\sup_{0\le t\le T}\psi(t)\int\limits_0^t s\psi(s)\,\mathrm{d}s \le \widetilde{K}\left(\frac{3}{8}K_2\sup_{0\le s\le t}\psi(s)\right)^2\frac{t^2}{2!}$$

and

$$\mathbb{E}\left(\sup_{0\le s\le t}\left[\left|X^4(s)-X^3(s)\right|^2+\left|Y^4(s)-Y^3(s)\right|^2\right]\right)$$

$$\le \frac{3}{8}K_2\int\limits_0^t \psi(s)\mathbb{E}\left(\sup_{0\le u\le s}\left[\left|X^3(u)-X^2(u)\right|^2+\left|Y^3(u)-Y^2(u)\right|^2\right]\right)\mathrm{d}s$$

$$\le \frac{3}{8}K_2\widetilde{K}\left(\frac{3}{8}K_2\sup_{0\le s\le t}\psi(s)\right)^2\int\limits_0^t \frac{s^2}{2!}\psi(s)\,\mathrm{d}s \le \widetilde{K}\left(\frac{3}{8}K_2\sup_{0\le s\le t}\psi(s)\right)^3\frac{t^3}{3!}.$$

We easily deduce, by recurrence, that for any positive integer n and for all $t\in[0,T]$,

$$\mathbb{E}\left(\sup_{0\le s\le t}\left[\left|X^{n+1}(s)-X^n(s)\right|^2+\left|Y^{n+1}(s)-Y^n(s)\right|^2\right]\right)$$

$$\le \frac{\widetilde{K}}{n!}\left(\frac{3}{8}K_2t\sup_{0\le s\le t}\psi(s)\right)^n \tag{7}$$

which means that $\big(X^n(t),Y^n(n)\big)_{n\ge1}$ is a Cauchy sequence.

We will prove that $\big(X^n(t),Y^n(t)\big)_{n\ge1}$ converges to the solution $\big(X(t),Y(t)\big)$ at the sense of L_G^2 on $M_G^2\big((-\infty,T];\mathbb{R}^d\big)\times\big((-\infty,T];\mathbb{R}^d\big)$. Taking $t=T$ in (7), we have

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left[\left|X^{n+1}(t)-X^{n}(t)\right|^{2}+\left|Y^{n+1}(t)-Y^{n}(t)\right|^{2}\right]\right)$$

$$\leq\frac{\widetilde{K}}{n!}\left(\frac{3}{8}K_{2}T\sup_{0\leq t\leq T}\psi(t)\right)^{n},$$

then

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|X^{n+1}(t)-X^{n}(t)\right|^{2}\right)\leq\frac{\widetilde{K}}{n!}\left(\frac{3}{8}K_{2}T\sup_{0\leq t\leq T}\psi(t)\right)^{n},$$

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|Y^{n+1}(t)-Y^{n}(t)\right|^{2}\right)\leq\frac{\widetilde{K}}{n!}\left(\frac{3}{8}K_{2}T\sup_{0\leq t\leq T}\psi(t)\right)^{n}.$$

By the Chebyshev inequality, we have

$$\begin{cases}\widehat{C}\left(\sup\limits_{0\leq t\leq T}\left|X^{n+1}(t)-X^{n}(t)\right|>\dfrac{1}{2^{n}}\right)\leq\dfrac{\widetilde{K}}{n!}\left(\dfrac{3}{2}K_{2}T\sup\limits_{0\leq t\leq T}\psi(t)\right)^{n},\\ \widehat{C}\left(\sup\limits_{0\leq t\leq T}\left|Y^{n+1}(t)-Y^{n}(t)\right|>\dfrac{1}{2^{n}}\right)\leq\dfrac{\widetilde{K}}{n!}\left(\dfrac{3}{2}K_{2}T\sup\limits_{0\leq t\leq T}\psi(t)\right)^{n},\end{cases}$$

where $\widehat{C}(\cdot)$ is the capacity associated to $\mathcal{P}$:

From the fact that

$$\sum_{n=0}^{\infty}\frac{\widetilde{K}}{n!}\left(\frac{3}{2}K_{2}T\sup_{0\leq t\leq T}\psi(t)\right)^{n}<\infty,$$

and by Borel-Cantelli Lemma, for all $w\in\Omega$, there exists a positive integer $n_{0}=n_{0}(w)$, such that

$$\begin{cases}\sup\limits_{0\leq t\leq T}\left|X^{n+1}(t)-X^{n}(t)\right|\leq\dfrac{1}{2^{n}},\ \text{as } n\geq n_{0},\\ \sup\limits_{0\leq t\leq T}\left|Y^{n+1}(t)-X^{n}(t)\right|\leq\dfrac{1}{2^{n}},\ \text{as } n\geq n_{0},\end{cases}$$

or

$$\begin{cases}X^{n}(t)=X^{0}(t)+\sum\limits_{k=1}^{n}\left[X^{k}(t)-X^{k-1}(t)\right],\\ Y^{n}(t)=Y^{0}(t)+\sum\limits_{k=1}^{n}\left[Y^{k}(t)-Y^{k-1}(t)\right],\end{cases}$$

which give

$$\begin{cases}\sum\limits_{k=1}^{\infty}\left|X^{k}(t)-X^{k-1}(t)\right|\leq\sum\limits_{k=0}^{\infty}\dfrac{1}{2^{k}}=2,\\ \sum\limits_{k=1}^{\infty}\left|Y^{k}(t)-Y^{k-1}(t)\right|\leq\sum\limits_{k=0}^{\infty}\dfrac{1}{2^{k}}=2.\end{cases}$$

The series

$$\sum_{k=1}^{\infty} \left|X^k(t) - X^{k-1}(t)\right| \quad \text{and} \quad \sum_{k=1}^{\infty} \left|Y^k(t) - Y^{k-1}(t)\right|$$

are convergent on $(-\infty, T]$. Moreover, according to Weierstrass's criterion, the series

$$X^0(t) + \sum_{k=1}^{\infty} \left[X^k(t) - X^{k-1}(t)\right] \quad \text{and} \quad Y^0(t) + \sum_{k=1}^{\infty} \left[Y^k(t) - Y^{k-1}(t)\right]$$

are also convergent on $(-\infty, T]$, then the series

$$X^0(t) + Y^0(t) + \sum_{k=1}^{\infty} \left[X^k(t) - X^{k-1}(t)\right] + \sum_{k=1}^{\infty} \left[Y^k(t) - Y^{k-1}(t)\right]$$

is convergent on $(-\infty, T]$. Furthermore, it's uniformly convergent on $(-\infty, T]$. Therefore, approximate sequence $\big(X^n(t), Y^n(t)\big) \to \big(X(t), Y(t)\big)$ on $(-\infty, T]$, or $\big(X^n(t), Y^n(t)\big)_{n\geq 1}$ is a Cauchy sequence in L^2_G. Hence

$$\lim_{n\to\infty} \big(X^n(t), Y^n(t)\big) = \big(X(t), Y(t)\big)$$

then

$$\lim_{n\to\infty} \mathbb{E}\left[\left|X^n(t) - X(t)\right|^2 + \left|Y^n(t) - Y(t)\right|^2\right] = 0.$$

Letting $n \to \infty$ in (6), it comes for all $t \in [0, T]$

$$\mathbb{E}\left[\left|X(t)\right|^2 + \left|Y(t)\right|^2\right] \leq K_3 \exp\left(K_2 T \sup_{0\leq t\leq T} \left[\psi(t)\right]\right).$$

We integrate the above over $(0, t]$, we get

$$\begin{aligned}
\mathbb{E}\int_{-\infty}^{t} \left(\left|X(s)\right|^2 + \left|Y(s)\right|^2\right) \mathrm{d}s &\leq \mathbb{E}\int_{-\infty}^{0} \left(\left|X_0(s)\right|^2 + \left|Y_0(s)\right|^2\right) \mathrm{d}s \\
&\quad + \int_{0}^{t} K_3 \exp\left(K_2 T \sup_{0\leq t\leq T} \left[\psi(t)\right]\right) \mathrm{d}s \\
&\leq \mathbb{E}\int_{-\infty}^{0} \left(\left|\alpha(s)\right|^2 + \left|\beta(s)\right|^2\right) \mathrm{d}s + T K_3 \exp\left(K_2 T \sup_{0\leq t\leq T} \left[\psi t\right]\right) < \infty.
\end{aligned}$$

Thus

$$\big(X(t), Y(t)\big) \in M^2_G\left(\left(-\infty, T\right], \mathbb{R}^d\right) \times M^2_G\left(\left(-\infty, T\right], \mathbb{R}^d\right).$$

We show $\big(X(t), Y(t)\big)$ solution of (3). For $h_i = f_i,\ g_i$:

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t \big[h_1\big(s, X^n(s), Y^n(s)\big) - h_1\big(s, X(s), Y(s)\big)\big]\,\mathrm{d}s\right|^2 \\
&\quad + \mathbb{E}\left|\int_0^t \big[h_2\big(s, X^n(s), Y^n(s)\big) - h_2\big(s, X(s), Y(s)\big)\big]\,\mathrm{d}\langle B, B\rangle(s)\right|^2 \\
&\quad + \mathbb{E}\left|\int_0^t \big[h_3\big(s, X^n(s), Y^n(s)\big) - h_3\big(s, X(s), Y(s)\big)\big]\,\mathrm{d}B(s)\right|^2 \\
&\leq 4[(C_1 + C_2)t + C_3]\int_0^t \psi(s)\mathbb{E}\left[|X^n(s) - X(s)|^2 + |Y^n(s) - Y(s)|^2\right]\mathrm{d}s.
\end{aligned}$$

Noting the sequence $\big(X^n(t), Y^n(t)\big)_{n\geq 1}$ uniformly converges on $(-\infty, T]$, that for any $\varepsilon > 0$, there exists a positive integer n_0 such that as $n \geq n_0$; for any $t \in (-\infty, T]$

$$\mathbb{E}\left[|X^n(s) - X(s)|^2 + |Y^n(s) - Y(s)|^2\right] < \varepsilon.$$

Furthermore, we have

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t \big[h_1\big(s, X^n(s), Y^n(s)\big) - h_1\big(s, X(s), Y(s)\big)\big]\,\mathrm{d}s\right|^2 \\
&\quad + \mathbb{E}\left|\int_0^t \big[h_2\big(s, X^n(s), Y^n(s)\big) - h_2\big(s, X(s), Y(s)\big)\big]\,\mathrm{d}\langle B, B\rangle(s)\right|^2 \\
&\quad + \mathbb{E}\left|\int_0^t \big[h_3\big(s, X^n(s), Y^n(s)\big) - h_3\big(s, X(s), Y(s)\big)\big]\,\mathrm{d}B(s)\right|^2 \\
&< 4[(C_1 + C_2)t + C_3]\,\varepsilon\int_0^t \psi(s)\,\mathrm{d}s \leq \frac{K_2}{4}\varepsilon\, T\sup_{0\leq t\leq T}\psi(t).
\end{aligned}$$

In other words, for $t \in [0,T]$, we have in L_G^2 since $n \to \infty$

$$\int_0^t h_1\big(s, X^n(s), Y^n(s)\big)\,\mathrm{d}s \to \int_0^t h_1\big(s, X(s), Y(s)\big)\,\mathrm{d}s,$$

$$\int_0^t h_2\big(s, X^n(s), Y^n(s)\big)\,\mathrm{d}\langle B,B\rangle(s) \to \int_0^t h_2\big(s, X(s), Y(s)\big)\,\mathrm{d}\langle B,B\rangle(s),$$

$$\int_0^t h_3\big(s, X^n(s), Y^n(s)\big)\,\mathrm{d}B(s) \to \int_0^t h_3\big(s, X(s), Y(s)\big)\,\mathrm{d}B(s).$$

For $t \in [0,T]$, taking limits of (5), as $n \to \infty$, we obtain $\big(X(t), Y(t)\big)$ is solution of (3), what had to be proven. □

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El-Hacène Chalabi
Department of Pharmacy
Faculty of Medicine
Salah Boubnider University
25016 - Constantine, ALGERIA
E-mail: elhacene25@gmail.com

Salim Mesbahi
Department of Mathematics
Faculty of Sciences
Ferhat Abbas University
19000 - Setif, ALGERIA
E-mail: salimbra@gmail.com

Amar Ouaoua
Department of Sciences and Technology
Faculty of Technology
20 August 1955 University
21000 - Skikda, ALGERIA
E-mail: ouaouaama21@gmail.com