

P-LIKE PROPERTIES OF MEAGER IDEALS AND CARDINAL INVARIANTS

ADAM MARTON

Pavol Jozef Šafárik University in Košice, Košice, SLOVAKIA

ABSTRACT. We study two particular modifications of the P-property of ideals and related cardinal invariants $\text{cof}^{\mathcal{J}}(\mathcal{I})$ and $\text{cov}^+(\mathcal{I})$. We give some results on the existence of $\text{P}(\mathcal{J})$ -ideals or non- $\text{P}(\mathcal{J})$ -ideals regarding specific classes of ideals, particularly meager ideals on ω . We also provide values of the cardinal invariant $\text{cof}^{\mathcal{J}}(\mathcal{I})$ describing the smallest families ensuring $\text{P}(\mathcal{J})$ for particular critical ideals. Moreover, we obtain a simple way of proving strict inequalities $\text{Fin} <_K \langle \mathcal{A} \rangle <_K \text{Fin} \times \text{Fin}$ for any MAD family $\mathcal{A}$ using the weak P-ideal notion.

1. Introduction

One of the extensively studied classes of ideals on ω is the class of (analytic) P-ideals, i.e., ideals satisfying the P-property. In this paper, we study a natural modification of the P-property called $\text{P}(\mathcal{J})$ -property:

DEFINITION 1.1. We say that an ideal $\mathcal{I}$ is a $\text{P}(\mathcal{J})$ -ideal if for every family $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ there is $I \in \mathcal{I}$ such that $I_n \subseteq^{\mathcal{J}} I$ for all $n \in \omega$.

This property was introduced in [21] by M. Mačaj and M. Szeziak within their study of ideal convergence in topological spaces. It later appeared in the paper [9] by R. Filipów and M. Staniszewski, who studied ideal convergence of sequences of real-valued functions. They showed that the setting in which (among others) ideal uniform convergence is stronger than ideal quasi-normal¹ convergence is characterized by the relationship “ $\mathcal{I}$ is a $\text{P}(\mathcal{J})$ -ideal”, hence the

© 2023 Mathematical Institute, Slovak Academy of Sciences.

2020 Mathematics Subject Classification: 03E04, 03E17, 03E20, 03E75.

Keywords: ideal, P-ideal, cardinal invariants.

This work was supported by the Slovak Research and Development Agency under the Contract no. APVV-20-0045. The support of the internal faculty grant No. vvgs-2023-2534 is kindly announced.

 Licensed under the Creative Commons BY-NC-ND 4.0 International Public License.

¹Some authors prefer the term ideal equal convergence to ideal quasi-normal convergence.

importance for the real functions field. This is an interesting result regarding the fact that uniform convergence is stronger than quasi-normal convergence (introduced in [4, 5]) in the standard non-ideal setting.

THEOREM 1.2 (R. Filipów, R.—M. Staniszewski [9], 2014). *Let X be a non-empty set and $\mathcal{I}, \mathcal{J}$ be ideals on ω . The following are equivalent:*

- (1) $\mathcal{I}$ is a $P(\mathcal{J})$ -ideal.
- (2) *For any sequence $\langle f_n : n \in \omega \rangle$ of real-valued functions on X , if $\langle f_n : n \in \omega \rangle$ is $\mathcal{I}$ -uniformly convergent to f , then $\langle f_n : n \in \omega \rangle$ is $(\mathcal{I}, \mathcal{J})$ -quasi-normally convergent to f .*

In [10, 25], the same authors proved some simple basic properties of this P -like notion: e.g., “monotonicity” of $P(\mathcal{J})$ with respect to inclusion, i.e., if $\mathcal{I}$ is a $P(\mathcal{J})$ -ideal, then $\mathcal{I}$ is a $P(\mathcal{J}')$ for every $\mathcal{J}' \supseteq \mathcal{J}$, in particular, every P -ideal is a $P(\mathcal{J})$ -ideal for any ideal $\mathcal{J}$. They have also shown that $\mathcal{I}$ is a $P(\mathcal{I})$ -ideal for any ideal $\mathcal{I}$ or the fact that $\mathcal{I}$ is a $P(\mathcal{J})$ and $\mathcal{J}$ is a $P(\mathcal{I})$ for any pair of maximal ideals $\mathcal{I}, \mathcal{J}$.

In this paper, we aim to concentrate on this characterization of the strength of uniform functional convergence from a purely combinatorial point of view, i.e., we intend to study precisely the relationship “ $\mathcal{I}$ is a $P(\mathcal{J})$ -ideal”. First, we give answers to some natural non-trivial² questions about the existence of $P(\mathcal{J})$ or non- $P(\mathcal{J})$ -ideals. In particular, we prove that there is a meager non- $P(\mathcal{J})$ -ideal with cofinality being equal to $\mathfrak{c}$ for any meager ideal $\mathcal{J}$, and also that maximal ideals are extreme cases of $P(\mathcal{J})$ -ideals.

With the $P(\mathcal{J})$ -property being defined, we can introduce a cardinal invariant describing the smallest family ensuring the $P(\mathcal{J})$ -property for a particular ideal $\mathcal{I}$, i.e., something like a base of this property. In addition to the basic properties, we examined values of this cardinal invariant for some pairs of critical ideals on ω having nice representation in $\omega \times \omega$, namely Fin , $\text{Fin} \times \emptyset$, $\emptyset \times \text{Fin}$, $\text{Fin} \times \text{Fin}$, Sel , $\mathcal{ED}$, see Table 2. The last three of them are determined somehow by the family of all sequences of natural numbers, that is, ${}^\omega\omega$ is a part of a generating family. All these ideals are meager, see e.g. [6, 23].³

We consider a particular modification of the $P(\mathcal{J})$ -property in the last section. After introducing a cardinal invariant $\text{cov}^+(\mathcal{A})$, where $\mathcal{A}$ is a MAD family on ω , we show that there is an easy way of proving strict inequalities in Katětov order regarding ideals generated by MAD families: $\text{Fin} <_K \langle \mathcal{A} \rangle <_K \text{Fin} \times \text{Fin}$.

²Some of the questions related to the existence are trivial, e.g. any ideal $\mathcal{I}$ is a $P(\mathcal{I})$ -ideal, therefore for any ideal $\mathcal{I}$ there is an ideal $\mathcal{J}$ such that $\mathcal{I}$ is a $P(\mathcal{J})$ -ideal.

³ Fin , $\text{Fin} \times \emptyset$, $\emptyset \times \text{Fin}$, $\text{Fin} \times \text{Fin}$, $\mathcal{ED}$ are meager because they are Borel (see [15, 23]), hence analytic. Thus, they have the Baire property, and consequently, they are meager (see, e.g. [2] Theorem 4.1.1). The ideal Sel is not mentioned in these works, but since $\text{Sel} \subseteq \mathcal{ED}$ and $\mathcal{ED}$ is meager, Sel is meager as well.

2. Preliminaries

We refer the reader to [15, 23] for the basic facts about ideals. Let M be a countable set (usually ω). A family $\mathcal{I} \subseteq \mathcal{P}(M)$ is an **ideal** on M if

- 1) $\emptyset \in \mathcal{I}$, $M \notin \mathcal{I}$,
- 2) if $A \in \mathcal{I}$, $B \subseteq A$, then $B \in \mathcal{I}$,
- 3) if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$,
- 4) $\{x\} \in \mathcal{I}$ for every $x \in M$.

Throughout the text, M will always mean a countable set. A family $\mathcal{B} \subseteq \mathcal{I}$ is a **base** of $\mathcal{I}$ if for any $I \in \mathcal{I}$ there is $B \in \mathcal{B}$ such that $I \subseteq B$. The cardinality of a smallest base of $\mathcal{I}$ will be denoted by $\text{cof}(\mathcal{I})$, i.e.,

$$\text{cof}(\mathcal{I}) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } \mathcal{I}\}.$$

For $A, B \subseteq \omega$ we write $A \subseteq^* B$ if and only if $A \setminus B$ is finite and $A \subseteq^{\mathcal{I}} B$ if $B \setminus A \in \mathcal{I}$. Let $\mathcal{E}$ be a family of subsets of M . If there is not a finite subfamily $\mathcal{C}$ of $\mathcal{E}$ with finite $M \setminus \bigcup \mathcal{C}$, then an **ideal generated by $\mathcal{E}$** is the smallest ideal containing $\mathcal{E}$ and Fin and we denote this ideal by $\langle \mathcal{E} \rangle$, i.e.,

$$\langle \mathcal{E} \rangle = \left\{ E \in \mathcal{P}(M) : E \subseteq^* \bigcup_{i < n} E_n \text{ for some } E_0, \dots, E_{n-1} \in \mathcal{E} \right\}.$$

If $\mathcal{I}$ and $\mathcal{J}$ are ideals on the same set, then by $\mathcal{I} \vee \mathcal{J}$ we denote the ideal generated by $\mathcal{I} \cup \mathcal{J}$ (i.e., the supremum of $\{\mathcal{I}, \mathcal{J}\}$ in the family of all ideals ordered by inclusion) if such an ideal exists. If $\mathcal{I} \vee \mathcal{J}$ exists, it is easy to see that $\mathcal{I} \vee \mathcal{J} = \{I \cup J : I \in \mathcal{I}, J \in \mathcal{J}\}$. The family of all **$\mathcal{I}$ -positive sets** (subsets of M which are not in $\mathcal{I}$) is denoted by $\mathcal{I}^+$, i.e., $\mathcal{I}^+ = \mathcal{P}(M) \setminus \mathcal{I}$. For any $E \in \mathcal{I}^+$ we define an ideal $\mathcal{I} \upharpoonright E = \{I \cap E : I \in \mathcal{I}\}$. An ideal $\mathcal{I}$ is **maximal** if either $A \in \mathcal{I}$ or $M \setminus A \in \mathcal{I}$ for every $A \in \mathcal{P}(M)$ (or equivalently, if $\mathcal{I}$ is maximal with respect to the ordering by inclusion). A dual notion to a maximal ideal is a maximal filter, commonly called an ultrafilter. We say that an ideal $\mathcal{I}$ on M is **tall** if for any $X \in [M]^\omega$ there is $I \in \mathcal{I}$ such that $X \cap I$ is infinite.

A family $\mathcal{A} \subseteq \mathcal{J}^+$ is $\mathcal{J}$ -almost disjoint ($\mathcal{J}$ -AD) if $A \cap B \in \mathcal{J}$ for each $A, B \in \mathcal{A}$, $A \neq B$. A $\mathcal{J}$ -AD family is a $\mathcal{J}$ -MAD family if for each $X \in \mathcal{J}^+$ there is an $A \in \mathcal{A}$ such that $X \cap A \in \mathcal{J}^+$ (i.e., $\mathcal{A}$ is $\subseteq$ -maximal among the $\mathcal{J}$ -AD families). It is common to use terms AD family and MAD family instead of Fin-AD family and Fin-MAD family. While (M)AD family is a standard set-theoretic term, $\mathcal{J}$ -(M)AD family is a newer notion which has been studied mostly in recent years, see, e.g. [1, 7].

For the set of all natural numbers we use the standard notation ω . $[X]^{<\kappa}$ denotes the family of all subsets of X of cardinality smaller than κ . Families $[X]^\kappa$, $[X]^{\leq\kappa}$ are defined in a similar manner. Therefore, condition (4) in the definition of ideal can be replaced with $[M]^{<\omega} \subseteq \mathcal{I}$. We use $(\forall^\infty n < \omega)$ to abbreviate ‘for all but finitely many n ’s’. For sets A, B denote by ${}^B A$ the set of all functions from B to A . We usually identify functions with their graphs and therefore we also identify ${}^B A$ with the family of graphs of functions from B to A . The Cantor space ${}^\omega 2$ is viewed as the product of infinitely many copies of $\{0, 1\}$ (with discrete topology) endowed with the standard Tychonoff product topology. We identify $\mathcal{P}(\omega)$ with ${}^\omega 2$ (and their topologies) via characteristic functions. We call an ideal $\mathcal{I}$ on ω meager, analytic, etc. if $\mathcal{I}$ is meager, analytic, etc. as a subset of the Cantor space $\mathcal{P}(\omega)$.

We define a standard $\leq^*$ -ordering on the set ${}^\omega \omega$ by letting

$$f \leq^* g \quad \text{if and only if} \quad (\forall^\infty n < \omega) f(n) \leq g(n).$$

The cardinality of a smallest $\leq^*$ -dominating⁴ family is denoted by $\mathfrak{d}$, i.e.,

$$\mathfrak{d} = \min \{|\mathcal{F}| : \mathcal{F} \subseteq {}^\omega \omega \wedge (\forall g \in {}^\omega \omega)(\exists f \in \mathcal{F}) g \leq^* f\}^5$$

and the cardinality of a smallest $\leq^*$ -unbounded family is denoted by $\mathfrak{b}$, i.e.,

$$\mathfrak{b} = \min \{|\mathcal{F}| : \mathcal{F} \subseteq {}^\omega \omega \wedge (\forall g \in {}^\omega \omega)(\exists f \in \mathcal{F}) f \not\leq^* g\}.$$

We will consider following critical⁶ ideals on ω (although we will use handy representations in $\omega \times \omega$).

$$\begin{aligned} \text{Fin} &= [\omega \times \omega]^{<\omega}. \\ \text{Fin} \times \emptyset &= \{I \subseteq \omega \times \omega : (\forall^\infty n < \omega) \{m : \langle n, m \rangle \in I\} = \emptyset\}. \\ \emptyset \times \text{Fin} &= \{I \subseteq \omega \times \omega : (\forall n < \omega) \{m : \langle n, m \rangle \in I\} \text{ is finite}\}. \\ \text{Fin} \times \text{Fin} &= \{I \subseteq \omega \times \omega : (\forall^\infty n < \omega) \{m : \langle n, m \rangle \in I\} \text{ is finite}\}. \\ \text{Sel} &= \{I \subseteq \omega \times \omega : (\exists k < \omega)(\forall n < \omega) |\{m : \langle n, m \rangle \in I\}| < k\}. \\ \mathcal{ED} &= \{I \subseteq \omega \times \omega : (\exists k < \omega)(\forall^\infty n < \omega) |\{m : \langle n, m \rangle \in I\}| < k\}. \end{aligned}$$

Note that $\text{Fin} \times \emptyset$ is the ideal generated by all vertical sections $\{n\} \times \omega$, $\emptyset \times \text{Fin}$ is generated by all f^- , where $f^- = \{\langle n, m \rangle : m \leq f(n)\}$ for any $f \in {}^\omega \omega$ ⁷, and $\text{Fin} \times \text{Fin}$ is the ideal generated by all vertical slices and all f^- . Note also that $\text{Sel} = \langle {}^\omega \omega \rangle$ and $\mathcal{ED}$ is the ideal generated by all the functions from ${}^\omega \omega$ and all the columns $\{n\} \times \omega$.

⁴i.e., cofinal in ${}^\omega \omega$ preordered by $\leq^*$.

⁵For more details about cardinal characteristics of the continuum see, e.g. [3].

⁶The reason why we call these ideals ‘critical’ can be found, e.g., in [15].

⁷In fact, it suffices to consider any $\leq^*$ -dominating family.

3. Existence results

In this section, we wish to give answers to some naturally arising questions regarding existence of $P(\mathcal{J})$ or non- $P(\mathcal{J})$ -ideals. For example, we wish to know if there is a non- $P(\mathcal{J})$ -ideal for an ideal $\mathcal{J}$ or if there is an ideal $\mathcal{J}$ such that every ideal $\mathcal{I}$ is a $P(\mathcal{J})$ -ideal. Some of the results will be formulated in a general setting with ideals on an arbitrary countable set M , since we will apply these results on ideals with an underlying set that differs from ω , namely $\omega \times \omega$, in other sections. One can easily see that most of the results formulated for ideals on ω can be stated in a general form where ω is replaced with any countable set M .

Ideals $\mathcal{I}, \mathcal{J}$ on ω are said to be **orthogonal**, written $\mathcal{I} \perp \mathcal{J}$, if there is a set $A \subseteq \omega$ such that $A \in \mathcal{I}$ and $\omega \setminus A \in \mathcal{J}$.

PROPOSITION 3.1. *If $\mathcal{J}_m$ is a maximal ideal on ω , then $\mathcal{I}$ is a $P(\mathcal{J}_m)$ -ideal for every ideal $\mathcal{I}$ on ω .*

Proof. It suffices to consider two cases. If $\mathcal{J}_m$ and $\mathcal{I}$ are orthogonal, we are done. If $\mathcal{J}_m$ and $\mathcal{I}$ are not orthogonal, then we have $\omega \setminus I \notin \mathcal{J}_m$ for any $I \in \mathcal{I}$. Hence, from the definition of maximal ideal, $I \in \mathcal{J}_m$ for any $I \in \mathcal{I}$, i.e., $\mathcal{I} \subseteq \mathcal{J}_m$. $\square$

Next, we wish to show that there is a non- $P(\mathcal{J})$ -ideal for a broad class of ideals which includes not only non-tall ideals but also, e.g., asymptotic density zero ideal $\mathcal{Z} = \{A \subseteq \omega : \lim_{n \rightarrow \infty} \frac{|A \cap n|}{n} = 0\}$ (simply take a partition $\mathcal{A} = \{A_n : n \in \omega\}$ of ω such that $d(A_n) = 1/2^n$, see Proposition 3.4) and hence also van der Waerden ideal

$$\mathcal{W} = \{A \subseteq \omega : A \text{ does not contain arbitrary long arithmetic progressions}\}$$

since $\mathcal{W} \subseteq \mathcal{Z}$ (see [26]), $\text{Fin} \times \text{Fin}$, ideals generated by MAD families, generally, meager ideals (see [24] Lemma 1.3), etc. Note that if $\mathcal{X}$ is an AD family, then $\langle \mathcal{X} \rangle$ is tall if and only if $\mathcal{X}$ is a MAD family (see, e.g. [15]).

First, we introduce an important tool for studying ideals generated by two ideals sharing only “very small” sets.

PROPOSITION 3.2 ([22]). *Let $\mathcal{J}$ be an ideal on M and let $\mathcal{I}_1, \mathcal{I}_2$ be non-orthogonal ideals on M such that $\mathcal{I}_1 \cap \mathcal{I}_2 \subseteq \mathcal{J}$. The following statements are equivalent.*

- (1) $\mathcal{I}_1$ and $\mathcal{I}_2$ are $P(\mathcal{J})$ -ideals.
- (2) $\mathcal{I}_1 \vee \mathcal{I}_2$ is a $P(\mathcal{J})$ -ideal.

There is also an easy way to handle the ideals generated by AD families described by the following assertion. By $\mathcal{I} \subseteq^{\uparrow} \mathcal{J}$ we mean condition

$$(\exists E \in \mathcal{I}^*) \mathcal{I} \restriction E \subseteq \mathcal{J},$$

where $\mathcal{I}^* = \{M \setminus I : I \in \mathcal{I}\}$ is the dual filter. Note that if $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{I} \subseteq^{\uparrow} \mathcal{J}$.

PROPOSITION 3.3 ([22]). *Let $\mathcal{J}$ be an ideal on M and $\mathcal{C}$ be an infinite AD family. The following statements are equivalent.*

- (1) $\langle \mathcal{C} \rangle$ is a $P(\mathcal{J})$ -ideal.
- (2) $\mathcal{C} \setminus \mathcal{J}$ is finite.
- (3) $\langle \mathcal{C} \rangle \subseteq^{\uparrow} \mathcal{J}$.

THEOREM 3.4. *If $\mathcal{J}$ is an ideal on ω such that there is a family*

$$\mathcal{X} = \{X_n : n \in \omega\} \subseteq \mathcal{J}^+$$

of pairwise disjoint sets, then there is a tall ideal $\mathcal{I}$ such that $\mathcal{I}$ is not a $P(\mathcal{J})$ -ideal.

Proof. Put $\mathcal{I} = \langle \mathcal{X} \rangle \vee \{Y \subseteq \omega : (\forall n \in \omega) |Y \cap X_n| < \omega\}$. It is easy to see that the family $\{Y \subseteq \omega : (\forall n \in \omega) |Y \cap X_n| < \omega\}$ is an ideal, and moreover, it is a P -ideal—the proof uses a similar technique as the standard proof of $\mathfrak{b} > \omega$ (see [13] Theorem 8.3). Then, from Proposition 3.2 it follows that $\mathcal{I}$ is a $P(\mathcal{J})$ -ideal if and only if $\langle \mathcal{X} \rangle$ is a $P(\mathcal{J})$ -ideal. By Proposition 3.3, $\mathcal{I}$ is a non- $P(\mathcal{J})$ -ideal. The AD family $\mathcal{X}$ can be extended to a MAD $\mathcal{X}'$ by a routine application of Zorn-Kuratowski lemma. Since $\langle \mathcal{X}' \rangle \subseteq \mathcal{I}$, the ideal $\mathcal{I}$ is tall. $\square$

One can easily see that we can replace the assumption that there is a family $\mathcal{X} = \{X_n : n \in \omega\} \subseteq \mathcal{J}^+$ of pairwise disjoint sets with the existence of $\mathcal{J}$ -AD family $\mathcal{A}$ with $|\mathcal{A}| \geq \omega$. Actually, the following weaker assertion with this assumption holds true.

PROPOSITION 3.5. *Let $\mathcal{J}$ be an ideal on ω and $\mathcal{A}$ be an infinite $\mathcal{J}$ -AD family. Then $\langle \mathcal{A} \rangle$ is not a $P(\mathcal{J})$ -ideal.*

Proof. Let $\{A_n : n \in \omega\}$ be a sequence of distinct members of $\mathcal{A}$. Choose $A \in \langle \mathcal{A} \rangle$ arbitrarily. Then, there is a finite $\mathcal{A}' \subseteq \mathcal{A}$ with $A \subseteq^* \bigcup \mathcal{A}'$. Pick $k \in \omega$ such that $A_k \notin \mathcal{A}'$. Note that $A_k \cap A \in \mathcal{J}$ for all $A \in \mathcal{A}'$, and consequently, we have $A_k \cap \bigcup \mathcal{A}' \in \mathcal{J}$. Next, we have $A_k \setminus \bigcup \mathcal{A}' \notin \mathcal{J}$ because otherwise $A_k = (A_k \setminus \bigcup \mathcal{A}') \cup (A_k \cap \bigcup \mathcal{A}')$ would be in $\mathcal{J}$. Since $A \subseteq^* \bigcup \mathcal{A}'$, we have that $A_k \setminus A \notin \mathcal{J}$ as well. Thus, $\{A_n : n \in \omega\}$ is witnessing that $\langle \mathcal{A} \rangle$ is not a $P(\mathcal{J})$ -ideal. $\square$

Consequently, we have that no ideal generated by an infinite AD family is a P -ideal.

We can improve the result in Proposition 3.4 for meager ideals in such a way that the non- $P(\mathcal{I})$ -ideal will also be “large” (of cofinality $\mathfrak{c}$). To prove this, we need the following simple lemma.

LEMMA 3.6. *Every infinite MAD family $\mathcal{A}$ on ω is equipotent to the smallest base of the ideal generated by $\mathcal{A}$, that is,*

$$\text{cof}(\langle\langle\mathcal{A}\rangle\rangle) = |\mathcal{A}|.$$

Proof. Assume not, i.e., assume that there is a MAD family $\mathcal{A}$ such that $\text{cof}(\langle\langle\mathcal{A}\rangle\rangle) = \kappa < |\mathcal{A}|$, and let $\mathcal{B}$ be a base of cardinality κ . By the definition of generated ideals, for every $B \in \mathcal{B}$ there is a finite $\mathcal{A}_B \subseteq \mathcal{A}$ such that $B \subseteq^* \bigcup \mathcal{A}_B$ for any $B \in \mathcal{B}$. Then, $\mathcal{B}' = \{\bigcup \mathcal{A}_B \cup F : B \in \mathcal{B}, F \in [\omega]^{<\omega}\}$ is a base of $\langle\langle\mathcal{A}\rangle\rangle$ as well. A map $f : \mathcal{B} \times [\omega]^{<\omega} \rightarrow \mathcal{B}'$ defined by

$$f(B, F) = \bigcup \mathcal{A}_B \cup F$$

is surjective, hence $|\mathcal{B}'| \leq |\mathcal{B} \times [\omega]^{<\omega}| = |\mathcal{B}| \cdot \aleph_0 = |\mathcal{B}|$. For any $B \in \mathcal{B}'$ let $\mathcal{C}_B = \{A \in \mathcal{A} : A \subseteq B\}$. Clearly, $|\mathcal{C}_B| < \omega$ for all $B \in \mathcal{B}'$. Then,

$$\left| \bigcup_{B \in \mathcal{B}'} \mathcal{C}_B \right| \leq |\mathcal{B}'| = \kappa < |\mathcal{A}|,$$

since $\bigcup_{B \in \mathcal{B}'} \mathcal{C}_B$ is a union of $|\mathcal{B}'|$ many finite sets. Moreover,

$$\bigcup_{B \in \mathcal{B}'} \mathcal{C}_B = \{A \in \mathcal{A} : (\exists B \in \mathcal{B}') A \subseteq B\}$$

and since its cardinality is strictly less than that of $\mathcal{A}$, there exists an $A \in \mathcal{A}$ which is not covered by any $B \in \mathcal{B}'$, which is a contradiction to the fact that $\mathcal{B}'$ is a base of $\langle\langle\mathcal{A}\rangle\rangle$. $\square$

Note that it was not necessary to construct a new base in this proof and we could manage with the original base just as well. The only reason for this step is that it is obvious which elements of the MAD family are covered by particular sets from the new base.

Ideals generated by MAD families are meager (see e.g. [6]). We will need the following corollary (see [23]) of a well-known theorem due to M. Talagrand and S.-A. Jalali-Naini.

PROPOSITION 3.7 ([18, 28]). *For any meager ideal $\mathcal{I}$ on ω and any $\mathcal{I}$ -positive set X there is an almost disjoint family $\mathcal{A}$ of $\mathcal{I}$ -positive subsets of X of cardinality $\mathfrak{c}$.*

Now, we can formulate the intended improvement. The interesting thing about the following theorem is the fact that the non- $P(\mathcal{I})$ -ideal, where $\mathcal{I}$ is a meager

ideal, behaves differently from different points of view — it is topologically negligible, large in (set-theoretic) terms of base and also “dense” in some set-theoretic sense since it is tall.

THEOREM 3.8. *If $\mathcal{J}$ is a meager ideal on ω , then there is a tall meager non- $\text{P}(\mathcal{J})$ -ideal $\mathcal{I}$ with $\text{cof}(\mathcal{I}) = \mathfrak{c}$.*

Proof. By Proposition 3.7, there is an AD family $\mathcal{A} \subseteq \mathcal{J}^+$ with $|\mathcal{A}| = \mathfrak{c}$. This family can be easily, similarly as in the proof of the Theorem 3.4, extended to a MAD family $\mathcal{A}'$. Since $\mathcal{A}' \setminus \mathcal{J}$ is infinite, it follows from Proposition 3.3 that $\langle \mathcal{A}' \rangle$ is not a $\text{P}(\mathcal{J})$ -ideal. Moreover, $\langle \mathcal{A}' \rangle$ is meager and tall, because $\mathcal{A}'$ is a MAD family. Since $|\mathcal{A}'| = \mathfrak{c}$, by applying Lemma 3.6, we get also $\text{cof}(\langle \mathcal{A}' \rangle) = \mathfrak{c}$. $\square$

4. Smallest subfamilies ensuring $\text{P}(\mathcal{J})$ -property

Here, we shall engage cardinal invariants in our study of the $\text{P}(\mathcal{J})$ -property. Cardinal invariants proved to be a very useful and also interesting tool for studying characteristics of various families, mostly ideals – probably the most famous example is the Cichoń’s diagram. Considering ideals on ω , we shall mention at least [14, 15, 23]. We wish to contribute to this area by defining the following cardinal invariant for a pair of ideals.

For any pair of ideals $\mathcal{I}, \mathcal{J}$ on ω such that $\mathcal{I}$ is a $\text{P}(\mathcal{J})$ -ideal we define the following cardinal invariant:

$$\text{cof}_{\text{ct}}^{\mathcal{J}}(\mathcal{I}) = \min \{ |\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall \mathcal{C} \in [\mathcal{I}]^\omega) (\exists A \in \mathcal{A}) (\forall C \in \mathcal{C}) C \subseteq^{\mathcal{J}} A \}.$$

If $\mathcal{J} = \text{Fin}$, we write $\text{cof}_{\text{ct}}^*(\mathcal{I})$ instead of $\text{cof}_{\text{ct}}^{\text{Fin}}(\mathcal{I})$. One can see that this invariant describes the smallest family which ensures the relationship “ $\mathcal{I}$ is a $\text{P}(\mathcal{J})$ -ideal” to be valid, i.e., it describes something like a base for the $\text{P}(\mathcal{J})$ -property of an ideal $\mathcal{I}$.

We shall justify the notation $\text{cof}_{\text{ct}}^{\mathcal{J}}(\mathcal{I})$ for the introduced cardinal invariant. In [14], F. Hernández-Hernández and M. Hrušák introduced the following notation for tall ideals

$$\text{cof}^*(\mathcal{I}) = \min \{ |\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I}) (\exists A \in \mathcal{A}) I \subseteq^* A \}.$$

If we replace Fin with an arbitrary ideal $\mathcal{J}$, we can consider the following cardinal invariant

$$\text{cof}^{\mathcal{J}}(\mathcal{I}) = \min \{ |\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I}) (\exists A \in \mathcal{A}) I \subseteq^{\mathcal{J}} A \}.$$

One can easily check that $\text{cof}^{\mathcal{J}}(\mathcal{I}) = \text{cof}_{\text{ct}}^{\mathcal{J}}(\mathcal{I})$ whenever $\mathcal{I}$ is a $\text{P}(\mathcal{J})$ -ideal. Indeed, let $\mathcal{I}$ be a $\text{P}(\mathcal{J})$ -ideal. Then clearly,

$$\begin{aligned} \{ \mathcal{A} : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall \mathcal{C} \in [\mathcal{I}]^\omega) (\exists A \in \mathcal{A}) (\forall C \in \mathcal{C}) C \subseteq^{\mathcal{J}} A \} \\ = \{ \mathcal{A} : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I}) (\exists A \in \mathcal{A}) I \subseteq^{\mathcal{J}} A \}. \end{aligned}$$

The “ $\subseteq$ ” part is trivial. For the converse, let $\mathcal{A}$ be a member of the latter set, and let $C \in [\mathcal{I}]^\omega$. Since $\mathcal{I}$ is a $P(\mathcal{J})$ -ideal, there is $I \in \mathcal{I}$ such that $C \subseteq^{\mathcal{J}} I$ for all $C \in \mathcal{C}$. By the assumption, there is $A \in \mathcal{A}$ such that $I \subseteq^{\mathcal{J}} A$. Therefore, $C \subseteq^{\mathcal{J}} A$ for any $C \in \mathcal{C}$ by the transitivity of $\subseteq^{\mathcal{J}}$.

Therefore, to study $\text{cof}_{\text{ct}}^{\mathcal{J}}(\mathcal{I})$ means to study $\text{cof}^{\mathcal{J}}(\mathcal{I})$ whenever $\mathcal{I}$ is a $P(\mathcal{J})$ -ideal. So, even if our goal is to investigate $\text{cof}_{\text{ct}}^{\mathcal{J}}(\mathcal{I})$, that is, the smallest families ensuring the $P(\mathcal{J})$ -property, it is sufficient to study $\text{cof}^{\mathcal{J}}(\mathcal{I})$ instead. Note that in the case $\mathcal{J} = \text{Fin}$, we obtain exactly $\text{cof}^*(\mathcal{I})$ introduced in [14]. In the same paper, one can also find the following useful lemma.

LEMMA 4.1 ([14]). *$\text{cof}^*(\mathcal{I}) = \text{cof}(\mathcal{I})$ for any uncountably generated ideal $\mathcal{I}$.*

In the following assertion, we list some of the basic properties of the invariant.

LEMMA 4.2. *Let $\mathcal{I}, \mathcal{J}, \mathcal{J}'$ be ideals on ω . Then, the following holds true.*

- (i) *If $\mathcal{I} \subseteq \mathcal{J}$, then $\text{cof}^{\mathcal{J}}(\mathcal{I}) = 1$, in particular, $\text{cof}^{\mathcal{I}}(\mathcal{I}) = 1$.*
- (ii) *$\text{cof}^{\mathcal{J}}(\mathcal{I}) \leq \text{cof}(\mathcal{I})$.*
- (iii) *If $\mathcal{J} \subseteq \mathcal{J}'$, then $\text{cof}^{\mathcal{J}'}(\mathcal{I}) \leq \text{cof}^{\mathcal{J}}(\mathcal{I})$.*
- (iv) *Either $\text{cof}^{\mathcal{J}}(\mathcal{I}) = 1$ or $\text{cof}^{\mathcal{J}}(\mathcal{I}) \geq \omega$.*
- (v) *If $\mathcal{I}$ is a $P(\mathcal{J})$ -ideal, then either $\text{cof}^{\mathcal{J}}(\mathcal{I}) = 1$ or $\text{cof}^{\mathcal{J}}(\mathcal{I}) \geq \omega_1$.*

Proof.

- (i) The family $\mathcal{A} = \{\emptyset\}$ is a witness for $\text{cof}^{\mathcal{J}}(\mathcal{I}) = 1$.
- (ii) If $\mathcal{B}$ is a base for $\mathcal{I}$, then obviously

$$\mathcal{B} \in \{\mathcal{A} \subseteq \mathcal{I} : (\forall I \in \mathcal{I})(\exists A \in \mathcal{A}) I \subseteq^{\mathcal{J}} A\}.$$

- (iii) Let $\mathcal{A}$ witness the definition of $\text{cof}^{\mathcal{J}}(\mathcal{I})$. Since $\mathcal{J} \subseteq \mathcal{J}'$, we also have

$$(\forall I \in \mathcal{I})(\exists A \in \mathcal{A}) I \subseteq^{\mathcal{J}'} A.$$

- (iv) Let $\mathcal{A}$ be a witness for $\text{cof}^{\mathcal{J}}(\mathcal{I})$ and $1 < |\mathcal{A}| < \omega$. Then $\bigcup \mathcal{A} \in \mathcal{I}$, $\{\bigcup \mathcal{A}\}$ satisfies the definition of $\text{cof}^{\mathcal{J}}(\mathcal{I})$ and is of cardinality smaller than $\mathcal{A}$, which is a contradiction.

- (v) A finite cardinality of a witness has been solved in (iv). If $\mathcal{A}$ is a witness for $\text{cof}^{\mathcal{J}}(\mathcal{I})$ and $|\mathcal{A}| = \omega$, then there is $I_{\mathcal{A}} \in \mathcal{I}$ such that $A \subseteq^{\mathcal{J}} I_{\mathcal{A}}$ for all $A \in \mathcal{A}$ because $\mathcal{I}$ is a $P(\mathcal{J})$ -ideal by the assumption. For any $I \in \mathcal{I}$ there is $A \in \mathcal{A}$ such that $I \subseteq^{\mathcal{J}} A$, since $\mathcal{A}$ is a witness for $\text{cof}^{\mathcal{J}}(\mathcal{I})$. The conclusion $I \subseteq^{\mathcal{J}} I_{\mathcal{A}}$ follows immediately from the transitivity of the relation $\subseteq^{\mathcal{J}}$. Then, $\{I_{\mathcal{A}}\}$ is smaller than $\mathcal{A}$ and satisfies the definition of $\text{cof}^{\mathcal{J}}(\mathcal{I})$, which is a contradiction. $\square$

Note that in the last part of this assertion, we prove that the cardinal $\text{cof}^{\mathcal{J}}(\mathcal{I})$ cannot be countably infinite whenever $\mathcal{I}$ is a $\text{P}(\mathcal{J})$ -ideal. In the following example, we shall illustrate the smallest possible infinite case of the family ensuring $\text{cof}^{\mathcal{J}}(\mathcal{I})$, i.e., of cardinality ω_1 .

EXAMPLE. It is known that there exists an increasing chain in $\mathcal{P}(\omega)/\text{Fin}$ of length ω_1 with a chain of representatives, say $b = \langle B_\alpha : \alpha < \omega_1 \rangle$, such that $B_\alpha \subseteq^* B_\beta$ and $|B_\beta \setminus B_\alpha| = \omega$ for any $\alpha < \beta < \omega_1$. Then, the ideal $\mathcal{I} = \langle \{B_\alpha : \alpha < \omega_1\} \rangle$ is clearly a P -ideal.⁸ Indeed, let $\{I_n : n \in \omega\} \subseteq \mathcal{I}$. b is a chain⁹, we have that there is an increasing sequence $\alpha = \langle \alpha_n : n \in \omega \rangle$ such that $I_n \subseteq^* B_{\alpha_n}$ for every n . The ordinal ω_1 is regular, so α is not cofinal in ω_1 and hence $\sup\{\alpha_n : n \in \omega\} = \gamma < \omega_1$. Then, $I_n \subseteq^* B_\gamma$ for any n . We will show that $\text{cof}^*(\mathcal{I}) = \omega_1$ without using the cofinality of $\mathcal{I}$.

Let $C \subseteq \mathcal{I}$ with $|\mathcal{C}| < \omega_1$. Since $\mathcal{C}$ is countable, there is an increasing sequence $\langle \alpha_n : n \in \omega \rangle$ such that every $C \in \mathcal{C}$ is a subset of some B_{α_n} . Again, we have $\sup\{\alpha_n : n \in \omega\} = \beta < \omega_1$. Then, $\mathcal{C} \subseteq \langle \{B_\alpha : \alpha < \beta\} \rangle$ and $B_{\beta+1} \not\subseteq^* C$ for any $C \in \mathcal{C}$.

At this point we wish to investigate relationships between aforementioned critical ideals on $\omega \times \omega$. In the following theorem we evaluate the invariant $\text{cof}^{\mathcal{J}}(\mathcal{I})$ for all pairs of studied ideals if it makes sense, i.e., if $\mathcal{I}$ is a $\text{P}(\mathcal{J})$ -ideal.

First, we list the $\text{P}(\mathcal{J})$ -relationships between the ideals, see [22].

TABLE 1. $\text{P}(\mathcal{J})$ interactions between critical ideals on $\omega \times \omega$.

	P	$\text{P}(\text{Fin} \times \emptyset)$	$\text{P}(\emptyset \times \text{Fin})$	$\text{P}(\text{Fin} \times \text{Fin})$	$\text{P}(\text{Sel})$	$\text{P}(\mathcal{ED})$
Fin	✓	✓	✓	✓	✓	✓
$\text{Fin} \times \emptyset$	✗	✓	✗	✓	✗	✓
$\emptyset \times \text{Fin}$	✓	✓	✓	✓	✓	✓
$\text{Fin} \times \text{Fin}$	✗	✓	✗	✓	✗	✓
Sel	✗	✗	✓	✓	✓	✓
$\mathcal{ED}$	✗	✗	✗	✓	✗	✓

The next theorem shows that the cardinal invariant $\text{cof}^{\mathcal{J}}(\mathcal{I})$ has fairly simple behavior regarding these pairs of ideals since it either collapses all the way down to 1 or it is equal precisely to the dominating number.

⁸It is unclear to the author what the origin of this construction is, but it might be considered folklore.

⁹So, from the fact that we do not have to deal with finite unions. For any I_n there is $k < \omega$ and $\alpha_1, \dots, \alpha_k < \omega_1$ such that $I_n \subseteq^* \bigcup_{i=1}^k B_{\alpha_i}$ by the definition of a generated ideal. Since $B_{\alpha_1}, \dots, B_{\alpha_k} \subseteq^* B_{\max\{\alpha_1, \dots, \alpha_k\}}$, we have that $I_n \subseteq^* B_{\max\{\alpha_1, \dots, \alpha_k\}}$.

THEOREM 4.3.

- (i) $\mathbf{cof}^{\mathcal{J}}(\text{Fin}) = 1$ for any ideal $\mathcal{J}$.
- (ii) $\mathbf{cof}^{\mathcal{J}}(\text{Fin} \times \emptyset) = 1$ for any ideal $\mathcal{J}$ such that $\text{Fin} \times \emptyset$ is a $\text{P}(\mathcal{J})$ -ideal.
- (iii) $\mathbf{cof}^*(\emptyset \times \text{Fin}) = \mathbf{cof}^{\text{Fin} \times \emptyset}(\emptyset \times \text{Fin}) = \mathbf{cof}^{\text{Sel}}(\emptyset \times \text{Fin}) = \mathbf{cof}^{\mathcal{ED}}(\emptyset \times \text{Fin}) = \mathbf{cof}(\emptyset \times \text{Fin}) = \mathfrak{d}$ and $\mathbf{cof}^{\text{Fin} \times \text{Fin}}(\emptyset \times \text{Fin}) = 1$.
- (iv) $\mathbf{cof}^{\text{Fin} \times \emptyset}(\text{Fin} \times \text{Fin}) = \mathbf{cof}^{\mathcal{ED}}(\text{Fin} \times \text{Fin}) = \mathbf{cof}(\text{Fin} \times \text{Fin}) = \mathfrak{d}$.
- (v) $\mathbf{cof}^{\emptyset \times \text{Fin}}(\text{Sel}) = \mathbf{cof}^{\text{Fin} \times \text{Fin}}(\text{Sel}) = \mathbf{cof}^{\mathcal{ED}}(\text{Sel}) = 1$.
- (vi) $\mathbf{cof}^{\text{Fin} \times \text{Fin}}(\mathcal{ED}) = 1$.

Proof.

(i) follows directly from (i) of Lemma 4.2 and the fact that $\text{Fin} \subseteq \mathcal{J}$ for any ideal $\mathcal{J}$.

(ii) $\text{Fin} \times \emptyset$ is the ideal generated by the partition $\mathcal{C} = \{\{n\} \times \omega : n \in \omega\}$, thus it is generated by an AD family. The ideal $\text{Fin} \times \emptyset$ is a $\text{P}(\mathcal{J})$ -ideal by the assumption, hence it follows from Proposition 3.3 that $\mathcal{C} \setminus \mathcal{J}$ is finite. One can easily check that $\{\bigcup(\mathcal{C} \setminus \mathcal{J})\}$ is a witness for $\mathbf{cof}^{\mathcal{J}}(\text{Fin} \times \emptyset) = 1$.

(iii) First, note that $\mathbf{cof}(\emptyset \times \text{Fin}) = \mathbf{cof}(\text{Fin} \times \text{Fin}) = \mathfrak{d}$ (see, e.g. [23]). The equality $\mathbf{cof}^*(\emptyset \times \text{Fin}) = \mathfrak{d}$ follows from Lemma 4.1.

Let $\mathcal{A} \subseteq \emptyset \times \text{Fin}$ be of cardinality $\kappa < \mathfrak{d}$ and let $\{A_\alpha : \alpha < \kappa\}$ be an enumeration of the family $\mathcal{A}$. Let $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$ be such that $A_\alpha \subseteq f_\alpha^-$ for every $\alpha < \kappa$. The cardinality of $\mathcal{F}$ is strictly less than $\mathfrak{d}$, so there is $g \in {}^\omega\omega$ such that $g \not\leq^* f_\alpha$ (i.e., $|\{n : g(n) > f_\alpha(n)\}| = \omega$) for all $\alpha < \kappa$. Then, also $|g \setminus f_\alpha| = \omega$, and thus, $|g \setminus f_\alpha^-| = \omega$ for all $\alpha < \kappa$. Consequently, $g \setminus A_\alpha$ is infinite for any $\alpha < \kappa$, which implies $g \setminus A_\alpha \notin \text{Fin} \times \emptyset$ since it contains an infinite partial function as a subset. Therefore, $\mathcal{A}$ does not satisfy the definition of $\mathbf{cof}^{\text{Fin} \times \emptyset}(\emptyset \times \text{Fin})$.

To see $\mathbf{cof}^{\text{Sel}}(\emptyset \times \text{Fin}) = \mathfrak{d}$, start the proof analogously as in the previous case. After finding g which is not dominated by any f_α , define $g' \in {}^\omega\omega$ as follows. For any n let

$$g'(n) = g(n) + n.$$

Then, $g^- \setminus f_\alpha^- \notin \text{Sel}$ for every $\alpha < \kappa$. This is actually easy to see, because for any α we have that $N_\alpha = \{n : g'(n) > f_\alpha(n)\}$ is infinite. Then, $|g'(n) \setminus f_\alpha(n)| \geq 1+n$ for any $n \in N_\alpha$. Thus,

$$(\forall k < \omega)(\exists n < \omega) \quad |\{m : \langle n, m \rangle \in g^- \setminus f_\alpha^-\}| \geq k.$$

The proof of $\mathbf{cof}^{\text{Sel}}(\emptyset \times \text{Fin})$ works for $\mathbf{cof}^{\mathcal{ED}}(\emptyset \times \text{Fin})$ as well.

The last result of (iii), $\mathbf{cof}^{\text{Fin} \times \text{Fin}}(\emptyset \times \text{Fin}) = 1$ follows from the fact $\emptyset \times \text{Fin} \subseteq \text{Fin} \times \text{Fin}$, and Lemma 4.2.

(iv) This holds because of the same reason as $\mathbf{cof}^{\mathbf{Fin} \times \emptyset}(\emptyset \times \mathbf{Fin}) = \mathfrak{d}$ and the fact that for any $I \in \mathbf{Fin} \times \mathbf{Fin}$ there are $I_0 \in \mathbf{Fin} \times \emptyset$ and $I_1 \in \emptyset \times \mathbf{Fin}$ such that $I = I_0 \cup I_1$. Let $\{I_\alpha^0 \cup I_\alpha^1 : \alpha < \kappa\} \subseteq \mathbf{Fin} \times \mathbf{Fin}$, where $\kappa < \mathfrak{d}$, with $I_\alpha^0 \in \mathbf{Fin} \times \emptyset$, $I_\alpha^1 \in \emptyset \times \mathbf{Fin}$ for any $\alpha < \kappa$, and let $\{f_\alpha : \alpha < \kappa\}$ be such that $I_\alpha^1 \subseteq f_\alpha^-$ for every $\alpha < \kappa$. Again, there is g such that $g \not\leq^* f_\alpha$ and consequently $g \setminus f_\alpha^- \notin \mathbf{Fin} \times \emptyset$ for any $\alpha < \kappa$. Moreover, $g \setminus (f_\alpha^- \cup I) \notin \mathbf{Fin} \times \emptyset$ for any $I \in \mathbf{Fin} \times \emptyset$ since I covers only finitely many points of an infinite function g , hence the residue $g \setminus (f_\alpha^- \cup I)$ contains an infinite partial function, which is not in $\mathbf{Fin} \times \emptyset$. Therefore, $g \setminus (I_\alpha^0 \cup I_\alpha^1) \notin \mathbf{Fin} \times \emptyset$ for every $\alpha < \kappa$.

To see $\mathbf{cof}^{\mathcal{ED}}(\mathbf{Fin} \times \mathbf{Fin}) = \mathfrak{d}$, just start the proof as in the previous case. After finding g , construct g' and define N_α for any $\alpha < \kappa$ as in (iii). Then, for all $n \in N_\alpha$ we have $|g'(n) \setminus f_\alpha(n)| \geq 1 + n$. Hence, for all but finitely many $n \in N_\alpha$ we have $|(\{n\} \times \omega) \cap (g'^- \setminus (f_\alpha^- \cup I_\alpha^0))| \geq 1 + n$.

(v), (vi) All the equalities follow from the fact that $\mathcal{Sel} \subseteq \emptyset \times \mathbf{Fin}$, $\mathbf{Fin} \times \mathbf{Fin}$, $\mathcal{ED}$ and $\mathcal{ED} \subseteq \mathbf{Fin} \times \mathbf{Fin}$. $\square$

The following table shows $\mathbf{cof}_{\text{ct}}^{\mathcal{J}}(\mathcal{I})$ -cardinal numbers (i.e., $\mathbf{cof}^{\mathcal{J}}(\mathcal{I})$) for all pairs of aforementioned ideals if it makes sense (if the $\mathbf{P}(\mathcal{J})$ -condition is satisfied), i.e., the results following from Theorem 4.3. If the ideal $\mathcal{I}$ is not a $\mathbf{P}(\mathcal{J})$ -ideal, then we left the corresponding place in the table empty. The diagonal is filled with ones by Lemma 4.2.

TABLE 2. Cardinal $\mathbf{cof}_{\text{ct}}^{\mathcal{J}}(\mathcal{I}) = \mathbf{cof}^{\mathcal{J}}(\mathcal{I})$ for all pairs of ideals considered in this work.

$\downarrow \mathcal{I} / \mathcal{J} \rightarrow$	$\mathbf{Fin}$	$\mathbf{Fin} \times \emptyset$	$\emptyset \times \mathbf{Fin}$	$\mathbf{Fin} \times \mathbf{Fin}$	$\mathcal{Sel}$	$\mathcal{ED}$
$\mathbf{Fin}$	1	1	1	1	1	1
$\mathbf{Fin} \times \emptyset$		1		1		1
$\emptyset \times \mathbf{Fin}$	$\mathfrak{d} = \mathbf{cof}(\emptyset \times \mathbf{Fin})$	$\mathfrak{d} = \mathbf{cof}(\emptyset \times \mathbf{Fin})$	1	1	$\mathfrak{d} = \mathbf{cof}(\emptyset \times \mathbf{Fin})$	$\mathfrak{d} = \mathbf{cof}(\emptyset \times \mathbf{Fin})$
$\mathbf{Fin} \times \mathbf{Fin}$		$\mathfrak{d} = \mathbf{cof}(\mathbf{Fin} \times \mathbf{Fin})$		1		$\mathfrak{d} = \mathbf{cof}(\mathbf{Fin} \times \mathbf{Fin})$
$\mathcal{Sel}$			1	1	1	1
$\mathcal{ED}$				1		1

It is an interesting observation that the $\mathbf{cof}^{\mathcal{J}}(\mathcal{I})$ -cardinal numbers for every particular pair of ideals mentioned in this work was either 1 or the cofinality of an ideal itself. So, the question is whether there may exist ideals $\mathcal{I}, \mathcal{J}$ with $1 < \omega_1 \leq \mathbf{cof}^{\mathcal{J}}(\mathcal{I}) < \mathbf{cof}(\mathcal{I})$.¹⁰ The positive answer to this question follows from the simple observation by M. Repický. First, if $\mathcal{I}, \mathcal{J}$ are ideals on ω , we denote by $\mathcal{I} \oplus \mathcal{J}$ the ideal on $\omega \times \{0, 1\}$ given by

$$\mathcal{I} \oplus \mathcal{J} = \{X \subseteq \omega \times \{0, 1\} : \{n : \langle n, 0 \rangle \in X\} \in \mathcal{I} \wedge \{m : \langle m, 1 \rangle \in X\} \in \mathcal{J}\}.$$

Note that $\mathbf{cof}(\mathcal{I} \oplus \mathcal{J}) = \max\{\mathbf{cof}(\mathcal{I}), \mathbf{cof}(\mathcal{J})\}$.

¹⁰Note that this cannot be proved for ideals on ω within ZFC.

Now, let $\mathcal{I}, \mathcal{J}$ be two uncountably generated ideals on ω such that

$$\omega_1 \leq \text{cof}(\mathcal{J}) < \text{cof}(\mathcal{I}).$$

Clearly,

$$\text{cof}^{\mathcal{I} \oplus \text{Fin}}(\mathcal{I} \oplus \mathcal{J}) = \text{cof}^*(\mathcal{J}) = \text{cof}(\mathcal{J}) < \text{cof}(\mathcal{I}) = \text{cof}(\mathcal{I} \oplus \mathcal{J}).$$

5. Weak P-ideal and almost disjoint families

There is a slight modification of the P-property ($\text{P}(\mathcal{J})$ -property), called weak-P-property (weak- $\text{P}(\mathcal{J})$ -property) which is closely related to the Katětov order. We define another cardinal invariant and prove it to be uncountable, hence we get interesting strict inequalities between MAD-generated ideals and Fin-based ideals with respect to the Katětov order.

We say that $\mathcal{I}$ is a **weak $\text{P}(\mathcal{J})$ -ideal**¹¹ if for any family $\{A_n : n \in \omega\} \subseteq \mathcal{I}$ there is $A \in \mathcal{I}^+$ such that $A_n \cap A \in \mathcal{J}$ for each $n \in \omega$.¹² Every $\text{P}(\mathcal{J})$ -ideal is a weak $\text{P}(\mathcal{J})$ -ideal and consequently every P-ideal is a weak P-ideal.

In [8, 12], the authors studied the cardinal invariant $\text{cov}^+(\mathcal{I})$ ¹³ defined for any tall ideal $\mathcal{I}$ as follows:

$$\text{cov}^+(\mathcal{I}) = \min\{|\mathcal{E}| : \mathcal{E} \subseteq \mathcal{I} \wedge (\forall A \in \mathcal{I}^+)(\exists E \in \mathcal{E}) |A \cap E| = \omega\}.$$

We are interested in studying $\text{cov}^+(\mathcal{I})$ when $\mathcal{I}$ is an ideal generated by a MAD family.

If $\mathcal{A}$ is a MAD family, we use the notation $\text{cov}^+(\mathcal{A})$ instead of $\text{cov}^+(\langle \mathcal{A} \rangle)$. Note that $\text{cov}^+(\mathcal{A}) \geq \omega$ for any MAD family $\mathcal{A}$ because for any finite $\mathcal{E}$ we have $\bigcup \mathcal{E} \in \langle \mathcal{A} \rangle$, $A := \omega \setminus \bigcup \mathcal{E} \in \langle \mathcal{A} \rangle^+$ and $A \cap E = \emptyset$ for any $E \in \mathcal{E}$.

Clearly, for any MAD family $\mathcal{A}$ we have that

$$\text{cov}^+(\mathcal{A}) = \min\{|\mathcal{E}| : \mathcal{E} \subseteq \mathcal{A} \wedge (\forall A \in \langle \mathcal{A} \rangle^+)(\exists E \in \mathcal{E}) |A \cap E| = \omega\}.$$

The “ $\leq$ ” part is trivial since any subset of $\mathcal{A}$ is a subset of $\langle \mathcal{A} \rangle$ as well. For the converse, let $\{E_\alpha : \alpha < \kappa\}$ be an enumeration of $\mathcal{E}$, where $\mathcal{E}$ is a family of cardinality κ satisfying the definition of $\text{cov}^+(\mathcal{A})$. By the definition of generated ideals, for any $\alpha < \kappa$ there is $\mathcal{A}_\alpha \subseteq \mathcal{A}$ such that $E_\alpha \subseteq^* \bigcup \mathcal{A}_\alpha$ and $|\mathcal{A}_\alpha| < \omega$. Let $\mathcal{E}' = \bigcup_{\alpha < \kappa} \mathcal{A}_\alpha$. Note that $\mathcal{E}'$ satisfies the definition of $\text{cov}^+(\mathcal{A})$ and since it is an infinite union of κ many finite sets, its cardinality is at most κ . Obviously, we also have $|\mathcal{E}'| \geq |\mathcal{E}|$.

This cardinal invariant is closely related to the weak P-property. In fact, $\text{cov}^+(\mathcal{A}) \leq \omega$ if and only if $\langle \mathcal{A} \rangle$ is a non-weak P-ideal. Actually, a witness for $\text{cov}^+(\mathcal{A})$ would be a witness for $\langle \mathcal{A} \rangle$ being a non-weak P-ideal in that case. However, we will prove that this is not the case for any MAD family $\mathcal{A}$.

¹¹Introduced in [27], inspired probably by [20].

¹²Again, if $\mathcal{J} = \text{Fin}$, we use the term weak P-ideal instead of weak $\text{P}(\text{Fin})$ -ideal.

¹³In [8], denoted by $\text{cov}^*(\mathcal{I}, +)$.

THEOREM 5.1. $\text{cov}^+(\mathcal{A}) \geq \omega_1$ for any infinite MAD family $\mathcal{A}$.

Proof. We show that if $|\mathcal{E}| = \omega$ for $\mathcal{E} \subseteq \mathcal{A}$, then there is $X \in \langle \mathcal{A} \rangle^+$ such that $|X \cap E| < \omega$ for every $E \in \mathcal{E}$. Let $\mathcal{E} = \{E_n : n \in \omega\}$. $|\mathcal{A}| \geq \mathfrak{a} > \omega$ since $\mathcal{A}$ is MAD family. So, there is some subfamily $\mathcal{A}' = \{A_k : k \in \omega\} \subseteq \mathcal{A} \setminus \mathcal{E}$. Let us define

$$X_k = A_k \setminus \bigcup_{j < k} E_j,$$

and

$$X = \bigcup_{k \in \omega} X_k.$$

For any k

$$E_k \cap X = \bigcup_{i \leq k} (E_k \cap X_i) \cup \bigcup_{j > k} (E_k \cap X_j).$$

Now, $E_k \cap X_j = E_k \cap (A_j \setminus (E_0 \cup \dots \cup E_k \cup \dots \cup E_j)) = \emptyset$ for any $j > k$, and $E_k \cap X_i$ is finite for each $i \leq k$ since $X_i \subseteq A_i$ and $|E_k \cap A_i| < \omega$ since $A_i, E_k \in \mathcal{A}$. So, $E_k \cap X$ is finite for any $k \in \omega$. It suffices to show that $X \notin \langle \mathcal{A} \rangle$.

$$X = \bigcup_{i \in \omega} X_i = \bigcup_{i \in \omega} \left(A_i \setminus \bigcup_{k < i} E_k \right) = \bigcup_{i \in \omega} \bigcap_{k < i} A_i \setminus E_k.$$

$A_i \cap E_k$ is finite for each $k < i$ and hence $X_i = \bigcap_{k < i} A_i \setminus E_k$ is infinite. Moreover, $|X_i \cap A_i| = \omega$ for every $i \in \omega$. Then, $|X \cap A_i| = \omega$ holds for $X = \bigcup_{j \in \omega} X_j$ and for all $i \in \omega$. If X was a set in $\langle \mathcal{A} \rangle$, $X \subseteq^* \bigcup \mathcal{B}$, where $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| < \omega$ and also $|X \cap A| < \omega$ for every $A \in \mathcal{A} \setminus \mathcal{B}$. Since $\mathcal{A}' \subseteq \mathcal{A}$ is countably infinite, there is $m \in \omega$ such that $A_m \notin \mathcal{B}$. Then, $X \cap A_m$ is infinite, a contradiction. $\square$

M. Repický has improved this result by proving the following statement in a simple and elegant way (unpublished):

THEOREM 5.2 (M. Repický). $\text{cov}^+(\mathcal{A}) \geq \mathfrak{b}$ for any infinite MAD family $\mathcal{A}$.

Let $\mathcal{I}, \mathcal{J}$ be ideals on ω . We say that $\mathcal{I}$ is Katětov below $\mathcal{J}$ (we write $\mathcal{I} \leq_K \mathcal{J}$) if there is a function $f \in {}^\omega \omega$ such that $f^{-1}[I] \in \mathcal{J}$ for all $I \in \mathcal{I}$. An ideal $\mathcal{I}$ is Katětov equivalent to $\mathcal{J}$ (written $\mathcal{I} \simeq_K \mathcal{J}$) if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$. We write $\mathcal{I} <_K \mathcal{J}$ whenever $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{I} \not\leq_K \mathcal{J}$. This definition may be easily extended to ideals on arbitrary sets. We say that $\mathcal{I}$ on M contains and isomorphic copy of $\mathcal{J}$ on N ($\mathcal{J} \sqsubseteq \mathcal{I}$, in short) if there is a bijection $f: N \rightarrow M$ such that $f[J] \in \mathcal{I}$ for each $J \in \mathcal{J}$.

In [17, Proposition 2.1] the authors proved that for every MAD family $\mathcal{A}$, $\text{Fin} \leq_K \langle \mathcal{A} \rangle \leq_K \text{Fin} \times \text{Fin}$. Now, we shall show that the reverse inequality with $\text{Fin} \times \text{Fin}$ holds for no MAD family $\mathcal{A}$.

THEOREM 5.3 ([11, 15, 19, 20, 23]). *Let $\mathcal{I}$ be an ideal on ω . The following statements are equivalent.*

- (1) $\mathcal{I}$ is not a weak P -ideal.
- (2) $\text{Fin} \times \text{Fin} \leq_K \mathcal{I}$.
- (3) $\text{Fin} \times \text{Fin} \sqsubseteq \mathcal{J}$.

It follows directly from Theorem 5.1 and Theorem 5.3 that no ideal generated by MAD family is Katětov above $\text{Fin} \times \text{Fin}$, i.e.,

COROLLARY 5.4. *For any infinite MAD family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ the ideal $\langle \mathcal{A} \rangle$ is a weak P -ideal. In particular $\text{Fin} \times \text{Fin} \not\leq_K \langle \mathcal{A} \rangle$.*

One can easily see that $\mathcal{I} \simeq_K \text{Fin}$ if and only if $\mathcal{I}$ is not tall (see [16]). Hence, by Theorem 5.1 and using aforementioned known facts, we have that for any MAD family $\mathcal{A}$ the following strict inequalities hold

$$\text{Fin} <_K \langle \mathcal{A} \rangle <_K \text{Fin} \times \text{Fin}.$$

Acknowledgements.

I would like to thank the participants of Košice Set Theory & Topology seminar, mainly Jaroslav Šupina and Miroslav Repický, for their valuable comments and suggestions that led to the improvement of this text.

REFERENCES

- [1] HAGA, K. B.—SCHRITTESSER, D.—TÖRNQUIST, A.: *MAD families, determinacy, and forcing*, J. Math. Log. **22** (2021), no. 1, 2150026.
- [2] BARTOSZYŃSKI, T.—JUDAH, H.: *Set Theory: On the Structure of the Real Line*. CRC Press (Taylor & Francis Group) London, 1995
- [3] BLASS, A.: *Combinatorial cardinal characteristics of the continuum*, In: (M. Foreman, A. Kanamori, eds.), *Handbook of Set Theory*. Springer-Verlag, Berlin 2010, pp. 395–489.
- [4] BUKOVSKÁ, Z.: *Thin sets in trigonometrical series and quasinormal convergence*, Math. Slovaca **40** (1990), no. 1, 53–62.
- [5] Császár, Á.—Laczkovich, M. *Discrete and equal convergence*, Stud. Sci. Math. Hungar. **10** (1975), 463–472.
- [6] FARKAS, B.: *Combinatorics of Borel Ideals* Ph.D. Thesis, Budapest University of Technology and Economics Department of Algebra, Institute of Mathematics Supervisor: Prof. Lajos Soukup, Alfréd Rényi Institute of Mathematics Hungarian Academy of Sci. 2011.
- [7] FARKAS, B.—SOUKUP, L.: *More on cardinal invariants of analytically P -ideals*, Commentationes Mathematicae Universitatis Carolinae **50** (2009), no. 2, 281–295.
- [8] FARKAS, B.—ZDOMSKYY, L.: *Ways of destruction*, J. Symb. Log. **87** (2022), no. 3, 938–966.
- [9] FILIPÓW, R.—STANISZEWSKI, M.: *On ideal equal convergence*, Cent. Eur. J. Math. **12** (2014), no. 6, 896–910.

- [10] R. FILIPÓW, R.—STANISZEWSKI, M.: *Pointwise versus equal (quasi-normal) convergence via ideals*, J. Math. Anal. Appl. **422** (2015), no. 2, 995–1006.
- [11] R. FILIPÓW, R.—SZUCA, P.: *Three kinds of convergence and the associated $\mathcal{I}$ -Baire classes*, J. Math. Anal. Appl. **391** (2012), no. 1, 1–9.
- [12] GUZMÁN-GONZÁLEZ, O.—HRUŠÁK, M.—MARTÍNEZ-RANERO, C. A.—RAMOS-GARCÍA, U. A.: *Generic existence of MAD families*, J. Symb. Log. **82** (2017), no. 1, 303–316.
- [13] HALBEISEN, L. J.: *Combinatorial Set Theory With a Gentle Introduction to Forcing*. Springer-Verlag, Berlin, 2012.
- [14] HERNÁNDEZ-HERNÁNDEZ, F.—HRUŠÁK, M.: *Cardinal invariants of analytic P -ideals*, Canad. J. Math. **59** (2007), no. 3, 575–595.
- [15] HRUŠÁK, M.: *Combinatorics of filters and ideals*, Contemp. Math. **533** (2011), 29–69.
- [16] HRUŠÁK, M.: *Katětov order on Borel ideals*, Arch. Math. Logic **56** (2017), no. 3, 831–847.
- [17] HRUŠÁK, M.—GARCÍA-FERREIRA, S.: *Ordering MAD families a la Katětov*, J. Symb. Log. **68** (2003), no. 4, 1337–1353.
- [18] JALALI-NAINI, S.-A.: *The Monotone Subsets of Cantor Space, Filters and Descriptive Set Theory*: Ph.D. Thesis, University of Oxford, 1976.
- [19] LACZKOVICH, M.—RECLAW, I.: *Ideal limits of sequences of continuous functions*, Fundam. Math. **203** (2009), no. 1, 39–46.
- [20] LAFLAMME, C.: *Filter games and combinatorial properties of winning strategies*, Contemp. Math. **192** (1995), 51–67.
- [21] MAČAJ, M.—SLEZIAK, M.: $\mathcal{I}^{\mathcal{K}}$ -convergence, Real Anal. Exchange **36** (2010/11), no. 1, 177–193.
- [22] MARTON, A.—ŠUPINA, J.: *On P -like ideals induced by disjoint families*, J. Math. Anal. Appl. **528** (2023), no. 2, Paper no. 127551.
- [23] MEZA ALCÁNTARA, D.: *Ideals and filters on countable sets*: Ph. D. Thesis, Universidad Nacional Autónoma de México, México, 2009.
- [24] REPICKÝ, M.: *Cardinal invariants and the collapse of the continuum by Sacks forcing*, J. Symb. Log. **73** (2008), no. 2, 711–727.
- [25] STANISZEWSKI, M.: *On ideal equal convergence II*, J. Math. Anal. Appl. **451** (2017), no. 2, 1179–1197.
- [26] SZEMERÉDI, E.: *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), no. 1, 199–245.
- [27] ŠUPINA, J.: *Ideal QN -spaces*, J. Math. Anal. Appl. **435** (2016), no. 1, 477–491.
- [28] TALAGRAND, M.: *Compacts de fonctions mesurables et filtres non mesurables*, Acta Arith. **67** (1980), no. 1, 13–43.

Received November 10, 2022

*Institute of Mathematics
Faculty of Science
Pavol Jozef Šafárik University in Košice
Jesenná 5
040 01 Košice,
SLOVAKIA
E-mail: adam.marton@student.upjs.sk*