

ASYMPTOTIC PROPERTIES OF SOLUTIONS TO FOURTH-ORDER DIFFERENCE EQUATIONS ON TIME SCALES

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ABSTRACT. We provide sufficient criteria for the existence of solutions for fourth-order nonlinear dynamic equations on time scales

$$\left(a(t)x^{\Delta^2}(t)\right)^{\Delta^2} = b(t)f(x(t)) + c(t),$$

such that for a given function $y: \mathbb{T} \rightarrow \mathbb{R}$ there exists a solution $x: \mathbb{T} \rightarrow \mathbb{R}$ to considered equation with asymptotic behaviour $x(t) = y(t) + o\left(\frac{1}{t^\beta}\right)$. The presented result is applied to the study of solutions to the classical Euler–Bernoulli beam equation, which means that it covers the case $\mathbb{T} = \mathbb{R}$.

1. Introduction

We consider the fourth-order nonlinear equation in the form

$$\left(a(t)x^{\Delta^2}(t)\right)^{\Delta^2} = b(t)f(x(t)) + c(t), \quad (1)$$

for $t \in \mathbb{T}$, where $\mathbb{T}$ is a time scale.

Throughout this paper $x: \mathbb{T} \rightarrow \mathbb{R}$ is unknown function and

$$b, c: \mathbb{T} \rightarrow \mathbb{R}, \quad a: \mathbb{T} \rightarrow \mathbb{R}_+, \quad \frac{1}{a} \in C_{rd}(\mathbb{T}, \mathbb{R}_+), \quad f \in C(\mathbb{R}, \mathbb{R})$$

are given. In this paper, we examine the existence of solutions of equation (1) with asymptotic behaviour and approximation of solutions.

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Time scales research area has a tremendous amount of theoretical value, such as differential equations and difference equations can be consolidated within the theoretical framework of dynamics equations on time scales. Due to this, it can avoid repeating studies in difference and differential equations while exploring more similarities and differences between them. Likewise, the investigation of dynamic equations had a wide range of applications. In particular, fourth-order equations often arise in mathematical models in various fields of science and engineering. For instance, the fourth-order equation describes a mathematical model of beam deflection. A special case of Eq. (1), with $b(t) \equiv 0$, takes the form

$$(a(t)x^{\Delta^2}(t))^{\Delta^2} = c(t),$$

and it is a generalization of the classical Euler–Bernoulli beam equation

$$(a(t)x''(t))'' = c(t) \quad \text{on any time scale } \mathbb{T}.$$

To notice the background of dynamic equations on time scales, see [5–7, 15].

As it is commonly known, most dynamical equations with nonconstant coefficients, especially some nonlinear equations, have no analytical solution. Many authors concentrate on the oscillatory and asymptotic behaviour of the second-order dynamic equation on time scales (see, for example, [2–4, 8–11, 14]). So far, they are less interested in studying higher order dynamical equations [18, 21–24, 27]. Oscillation results for fourth-order nonlinear dynamic equations were considered in [1, 12, 13, 20, 25, 26, 28, 29]. Our research concerns the asymptotic behaviour of solutions to fourth-order nonlinear dynamic equations. For the case of the time scale $\mathbb{T} = \mathbb{N}$, our result is formulated as Theorem 3.1 in [19]. In [17], Migda presented a new approach to the theory of asymptotic properties of solutions of difference equations. His method uses the regional topology on the space of all real sequences and the ‘regional’ version of the Schauder Fixed Point Theorem. We adopt this approach for dynamical equations on time scales.

2. Basic facts in time scales calculus

A time scale is a model of time [6, 7], where the step size is a function of time. From a mathematical point of view, it is an arbitrary nonempty closed subset $\mathbb{T}$ of the set $\mathbb{R}$ of real numbers.

The mapping $\sigma : \mathbb{T} \rightarrow \mathbb{T}$, defined by $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$ with $\inf \emptyset = \sup \mathbb{T}$, is called the forward jump operator. Similarly, we define the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$ with $\sup \emptyset = \inf \mathbb{T}$. The following classification of points is used within the theory: a point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense and left-scattered if $\sigma(t) = t$ (for $t < \sup \mathbb{T}$), $\sigma(t) > t$, $\rho(t) = t$ (for $t > \inf \mathbb{T}$) and $\rho(t) < t$, respectively.

We say that t is isolated if $\rho(t) < t < \sigma(t)$, and that t is dense if $\rho(t) = t = \sigma(t)$. The function $\mu: \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ and called the graininess function. The delta (or Hilger) derivative of $f: \mathbb{T} \rightarrow \mathbb{R}$ at a point $t \in \mathbb{T}^\kappa$, where

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty, \end{cases}$$

is defined in the following way.

DEFINITION 2.1 ([6]). The delta derivative $f^\Delta(t)$ is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left| (f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U.$$

DEFINITION 2.2 ([6]). A function $f: \mathbb{T} \rightarrow \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$, and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $f: \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$.

Let $C(A, B)$, $C_{rd}(A, B)$ denote the set of all continuous functions mapping A to B , the set of all rd-continuous functions mapping A to B , respectively.

DEFINITION 2.3 ([6]). Assume $f: \mathbb{T} \rightarrow \mathbb{R}$ is a regulated function. We define the indefinite integral of a regulated function f by $\int f(t) \Delta t = F(t) + C$, where C is an arbitrary constant and F is a pre-antiderivative of f . We define the Cauchy integral by $\int_a^b f(t) \Delta t = F(b) - F(a)$ for all $a, b \in \mathbb{T}$.

Since we investigate the asymptotic behaviour of (1), we assume, throughout this paper, that $\mathbb{T}$ is an arbitrary time scale such that $\inf \mathbb{T} = 0$ and $\sup \mathbb{T} = \infty$. It implies that $\mathbb{T}^\kappa = \mathbb{T}$.

3. Preliminaries

Below we present five lemmas necessary in the proof of the main theorem.

LEMMA 3.1. *If $y: \mathbb{T} \rightarrow \mathbb{R}$ and $\left(a(t)y^{\Delta^2}(t)\right)^{\Delta^2} = 0$, then there exist real constants c_1, c_2, c_3, c_4 such that*

$$y(t) = c_1 \int_0^t \left(\int_0^r \frac{w}{a(w)} \Delta w \right) \Delta r + c_2 \int_0^t \left(\int_0^r \frac{1}{a(w)} \Delta w \right) \Delta r + c_3 t + c_4 \quad (2)$$

for any $t \in \mathbb{T}$. If $c_1, c_2, c_3, c_4 \in \mathbb{R}$ and $y : \mathbb{T} \rightarrow \mathbb{R}$ is defined by (2), then $(a(t)y^{\Delta^2}(t))^{\Delta^2} = 0$.

LEMMA 3.2. Assume $x \in C_{rd}(\mathbb{T}, \mathbb{R})$, $r, s, t, w, z \in \mathbb{T}$ and the integral

$$\int_0^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty x(z) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s$$

is convergent. Then, for any $t \in \mathbb{T}$, we have

$$\left\{ a(t) \left[\int_t^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty x(z) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \right]^{\Delta^2} \right\}^{\Delta^2} = x(t). \quad (3)$$

LEMMA 3.3. Assume $U_1, W_1 \in C_{rd}(\mathbb{T}, (0, \infty))$, $\int_0^\infty W_1(s) \left(\int_s^\infty U_1(r) \Delta r \right) \Delta s < \infty$ and $r, s, t \in \mathbb{T}$. Then

$$\int_0^\infty U_1(r) \left(\int_0^r W_1(s) \Delta s \right) \Delta r < \infty,$$

and

$$\int_t^\infty W_1(s) \left(\int_s^\infty U_1(r) \Delta r \right) \Delta s \leq \int_t^\infty U_1(r) \left(\int_0^r W_1(s) \Delta s \right) \Delta r. \quad (4)$$

LEMMA 3.4. Assume $u \in C_{rd}(\mathbb{T}, \mathbb{R})$, $r, s, t, w, z \in \mathbb{T}$,

$$Q(t) = \int_0^t \left(\int_0^r \frac{w}{a(w)} \Delta w \right) \Delta r, \quad (5)$$

and

$$\int_0^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty u(z) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s < \infty. \quad (6)$$

Then

$$\int_0^\infty Q(z) |u(z)| \Delta z < \infty,$$

and

$$\int_t^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty |u(z)| \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \leq \int_t^\infty Q(z) |u(z)| \Delta z.$$

The following lemma is a version of the Schauder Fixed Point Theorem

LEMMA 3.5. *Assume $y, \rho \in C(\mathbb{T}, \mathbb{R})$, $\rho(t) \geq 0$ for $t \in \mathbb{T}$, and $\lim_{t \rightarrow \infty} \rho(t) = 0$.*

In the set

$$X = \{C(\mathbb{T}, \mathbb{R}) \ni x : |x(t) - y(t)| \leq |\rho(t)|, t \in \mathbb{T}\}$$

we define the metric by the formula $d(x, z) = \sup_{t \in \mathbb{T}} |x(t) - z(t)|$. Then every continuous map $H : X \rightarrow X$ has a fixed point.

The proof of Lemma 3.5 is analogous to the proof of Lemma 4.7 [16].

4. Approximate solutions

In this section, we present sufficient conditions for the existence of a solution x to equation (1) such that for any constant $\tilde{C} > 0$ there exists $t_0 \in \mathbb{T}$ that

$$|x(t) - y(t)| \leq \frac{\tilde{C}}{t^\beta}, \quad t \geq t_0,$$

where $\beta \geq 0$ and y is a solution of the equation $(a(t)y^{\Delta^2}(t))^{\Delta^2} = 0$. Note that, by Lemma 3.1, a function y is a solution of the equation $(a(t)y^{\Delta^2}(t))^{\Delta^2} = 0$ if and only if there exist real constants c_1, c_2, c_3, c_4 such that

$$y(t) = c_1 \int_0^t \left(\int_0^w \frac{r}{a(r)} \Delta r \right) \Delta w + c_2 \int_0^t \left(\int_0^w \frac{1}{a(r)} \Delta r \right) \Delta w + c_3 t + c_4.$$

Below we present the main result of the article. We will need Lemma 3.5 to prove this theorem.

THEOREM 4.1. *Assume $\beta \geq 0$, y is a solution of the equation*

$$(a(t)y^{\Delta^2}(t))^{\Delta^2} = 0,$$

and a, b, c are the coefficients in equation (1). Moreover, let

$$\int_0^\infty s^\beta \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty (|b(z)| + |c(z)|) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s < \infty, \quad (7)$$

and f be continuous and bounded on a set A where

$$A = \bigcup_{t=q}^\infty [y(t) - \alpha, y(t) + \alpha], \quad \text{for some } q \in \mathbb{T}, \alpha > 0.$$

Then there exists a solution x of (1) such that for any constant $\tilde{C} > 0$ there exists $t_0 \in \mathbb{T}$ that $|x(t) - y(t)| \leq \frac{\tilde{C}}{t^\beta}$ if $t \geq t_0$.

Proof. For $t \in \mathbb{T}$ and $x : \mathbb{T} \rightarrow \mathbb{R}$ let

$$x^*(t) = b(t)f(x(t)) + c(t).$$

There exists a positive constant L , such that $|f(\tau)| \leq L$ for any $\tau \in A$. Define $u : \mathbb{T} \rightarrow \mathbb{R}$ by

$$u(t) = \int_t^\infty s^\beta \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty (L|b(z)| + |c(z)|) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s.$$

Since u is finite (by assumption) and $\beta \geq 0$, we have

$$\begin{aligned} & \int_0^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty (L|b(z)| + |c(z)|) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \\ & \leq \int_0^\infty s^\beta \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty (L|b(z)| + |c(z)|) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s < \infty. \end{aligned}$$

Define $\rho : \mathbb{T} \rightarrow \mathbb{R}$ by

$$\rho(t) = \int_t^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty (L|b(z)| + |c(z)|) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s.$$

Then

$$\begin{aligned} t^\beta \rho(t) &= t^\beta \int_t^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty (L|b(z)| + |c(z)|) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \\ &= \int_t^\infty t^\beta \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty (L|b(z)| + |c(z)|) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s. \end{aligned}$$

Since $\beta \geq 0$ we obtain $s \geq t \Rightarrow s^\beta \geq t^\beta$ and

$$t^\beta \rho(t) \leq \int_t^\infty s^\beta \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty (L|b(z)| + |c(z)|) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s = u(t).$$

Hence for any constant $\tilde{C} > 0$ there exists $t_1 \in \mathbb{T}$ that for $t \geq t_1$

$$|\rho(t)| \leq \frac{\tilde{C}}{t^\beta}, \quad t \in \mathbb{T}. \quad (8)$$

Let

$$Y = \{C(\mathbb{T}, \mathbb{R}) \ni x : |x(t) - y(t)| \leq \alpha, \quad t \in \mathbb{T}\},$$

where y is a solution of the equation

$$\left(a(t)y^{\Delta^2}(t) \right)^{\Delta^2} = 0.$$

There exists $p \in \mathbb{T}$ such that

$$\rho(t) \leq \alpha \quad \text{for } t \geq p.$$

Hence, by assumptions, $f(x)$ is bounded for any $x \in Y$. Therefore, if $x \in Y$, then

$$\begin{aligned} & \int_t^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty |x^*(z)| \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \\ & \leq \int_t^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty (L|b(z)| + |c(z)|) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \\ & = \rho(t) < \alpha < \infty. \end{aligned}$$

Let

$$X = \{C(\mathbb{T}, \mathbb{R}) \ni x : |x(t) - y(t)| \leq \rho(t) \text{ for } t \geq p \text{ and } x(t) = y(t) \text{ for } t < p\},$$

where y is a solution of the equation

$$(a(t)y^{\Delta^2}(t))^{\Delta^2} = 0.$$

Moreover, we set

$$H : Y \rightarrow C(\mathbb{T}, \mathbb{R}),$$

$$H(x)(t) = \begin{cases} y(t) & \text{for } t < p \\ y(t) + \int_t^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty x^*(z) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s & \text{for } t \geq p. \end{cases}$$

It is obvious that $X \subset Y$. We will prove that the mapping H has a fixed point in X . If $x \in X$, then for $t \geq p$ we have

$$\begin{aligned} |H(x)(t) - y(t)| &= \left| \int_t^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty x^*(z) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \right| \\ &\leq \int_t^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty |x^*(z)| \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \leq \rho(t). \end{aligned}$$

Therefore $HX \subset X$.

The task is now to show that H is continuous. Let us recall the function Q given by (5), i.e.,

$$Q(t) = \int_0^t \left(\int_0^r \frac{w}{a(w)} \Delta w \right) \Delta r.$$

By Lemma 3.4, we have

$$\int_0^\infty Q(s)|b(s)|\Delta s < \infty.$$

Let $x \in X$, and $\varepsilon > 0$. Choose $M \geq p$ and a positive constant γ such that

$$L \int_M^\infty Q(s)|b(s)|\Delta s < \varepsilon \quad \text{and} \quad \gamma \int_0^M Q(s)|b(s)|\Delta s < \varepsilon.$$

Set

$$A_0 = \bigcup_{t=1}^M [y(t) - \alpha, y(t) + \alpha].$$

Choose a positive $\delta \in \mathbb{R}$ such that if $\tau_1, \tau_2 \in A_0$ and $|\tau_1 - \tau_2| < \delta$, then

$$|f(\tau_1) - f(\tau_2)| < \gamma.$$

Let us take $\tilde{x} \in X$ such that $\sup_{t \in \mathbb{T}} |x(t) - \tilde{x}(t)| < \delta$. Then

$$\begin{aligned} & \sup_{t \in \mathbb{T}} |H(x)(t) - H(\tilde{x})(t)| \\ &= \sup_{t \geq p} \left| \int_n^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty (x^*(z) - \tilde{x}^*(z)) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \right| \\ &\leq \int_p^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty |x^*(z) - \tilde{x}^*(z)| \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \\ &\leq \int_p^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty |b(z)| |f(x(z)) - f(\tilde{x}(z))| \Delta z \right) \Delta r \right) \Delta w \right) \Delta s. \end{aligned}$$

By Lemma 3.4,

$$\begin{aligned} & \int_p^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty |b(z)| |f(x(z)) - f(\tilde{x}(z))| \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \\ &\leq \int_p^\infty Q(s)|b(s)| |f(x(s)) - f(\tilde{x}(s))| \Delta s. \end{aligned}$$

Hence

$$\sup_{t \in \mathbb{T}} |H(x)(t) - H(\tilde{x})(t)| \leq \gamma \int_0^M Q(s)|b(s)|\Delta s + 2L \int_M^\infty Q(s)|b(s)|\Delta s < 3\varepsilon.$$

Therefore the map $H : X \rightarrow X$ is continuous. By Theorem 3.5 there exists a point $x \in X$ such that $x = Hx$. Then for $t \geq p$ we have

$$x(t) = y(t) + \int_t^\infty \left(\int_s^\infty \left(\frac{1}{a(w)} \int_w^\infty \left(\int_r^\infty x^*(z) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s.$$

Finally, we show that the fixed point x is a solution of equation (1). Using Lemma 3.2 and Lemma 3.1, for $t \geq p$ we get

$$\begin{aligned} \left(a(t)x^{\Delta^2}(t) \right)^{\Delta^2} &= \left(a(t)y^{\Delta^2}(t) \right)^{\Delta^2} \\ &\quad + \left(a(t) \left(\int_t^\infty \left(\int_s^\infty \left(\frac{1}{a(w)} \int_w^\infty \left(\int_r^\infty x^*(z) \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \right)^{\Delta^2} \right)^{\Delta^2} \\ &= 0 + x^*(t) = b(t)f(x(t)) + c(t). \end{aligned}$$

Therefore x is a solution of (1). Since $x \in X$ and $|\rho(t)| \leq \frac{\tilde{C}}{t^\beta}$ for $t \in \mathbb{T}$ and $t \geq t_1$, we have

$$x(t) = y(t) + o\left(\frac{1}{t^\beta}\right).$$

It means that the thesis holds. $\square$

EXAMPLE. Consider equation (1) on the time scale $\mathbb{T} = \mathbb{R}_+ \cup \{0\}$ in the following form

$$\left(\frac{1}{2^t} x''(t) \right)'' = \frac{1}{4^t} \sin(x(t)) + \frac{1}{4^t}. \quad (9)$$

Let $\beta = 10$. The condition (7) is satisfied since

$$\int_0^\infty s^{10} \left(\int_s^\infty 2^w \left(\int_w^\infty \left(\int_r^\infty \frac{2}{4^z} dz \right) dr \right) dw \right) ds = \frac{1814400}{\ln^{14} 2} < \infty.$$

One of the solutions to the equation

$$\left(\frac{1}{2^t} y''(t) \right)'' = 0,$$

is the function

$$y(t) = \frac{t 2^t}{\ln^2 2} + \left(\frac{1}{\ln 2} - \frac{2}{\ln^3 2} \right) 2^t + \left(\frac{1}{\ln^2 2} - \frac{1}{\ln 2} \right) t + \frac{2}{\ln^3 2} - \frac{1}{\ln^2 2}.$$

Then by Theorem 4.1 there exists a solution x of (1) such that for any constant $\tilde{C} > 0$ there exists $t_0 \in \mathbb{T}$ that $|x(t) - y(t)| \leq \frac{\tilde{C}}{t^{10}}$ if $t \geq t_0$.

EXAMPLE. Let $\mathbb{T} = \{2^n, n \in \mathbb{N}\} \cup \{0\}$ and take into account the following equation (1)

$$x^{\Delta^4}(t) = \frac{t}{2^t(t+1)^7} \arctan(x(t)) + \frac{t}{2^t(t+1)^7}. \quad (10)$$

Assume $\beta = 1$. The condition (7) takes the form

$$\int_0^\infty s \left(\int_s^\infty \left(\int_w^\infty \left(\int_r^\infty \frac{2z}{2^z(z+1)^7} \Delta z \right) \Delta r \right) \Delta w \right) \Delta s < \infty.$$

Below we prove that condition (7) is satisfied.

Note that, for the considered time scale, we have

$$\mu(t) = \begin{cases} 2, & \text{if } t = 0 \\ t, & \text{if } t \in \{2^n, n \in \mathbb{N}\}. \end{cases}$$

To show the convergence of the integral appearing in (7) we start with

$$\begin{aligned} \int_r^\infty \frac{2z}{2^z(z+1)^7} \Delta z &= 2 \sum_{z \in [r, \infty)_{\mathbb{T}}} \frac{z}{2^z(z+1)^7} \mu(z) = 2 \sum_{z \in [r, \infty)_{\mathbb{T}}} \frac{z^2}{2^z(z+1)^7} \\ &< 2 \sum_{z \in [r, \infty)_{\mathbb{T}}} \frac{1}{2^z(z+1)^5} < \frac{2}{(r+1)^5} \sum_{z \in [r, \infty)_{\mathbb{T}}} \frac{1}{2^z} \\ &< \frac{2}{(r+1)^5} \sum_{n=r}^\infty \frac{1}{2^n} = \frac{4}{2^r(r+1)^5}, \end{aligned}$$

and the above inequality yields the following one

$$\begin{aligned} \int_w^\infty \left(\int_r^\infty \frac{2z}{2^z(z+1)^7} \Delta z \right) \Delta r &\leq \int_w^\infty \frac{4}{2^r(r+1)^5} \Delta r = \sum_{r \in [w, \infty)_{\mathbb{T}}} \frac{4\mu(r)}{2^r(r+1)^5} \\ &= \sum_{r \in [w, \infty)_{\mathbb{T}}} \frac{4r}{2^r(r+1)^5} < \sum_{r \in [w, \infty)_{\mathbb{T}}} \frac{4}{2^r(r+1)^4} \\ &< \frac{4}{(r+1)^4} \sum_{r \in [w, \infty)_{\mathbb{T}}} \frac{1}{2^r} < \frac{4}{(w+1)^4} \sum_{n=w}^\infty \frac{1}{2^n} \\ &= \frac{8}{2^w(w+1)^4}. \end{aligned}$$

Analogously, we have

$$\int_s^\infty \left(\int_w^\infty \left(\int_r^\infty \frac{2z}{2^z(z+1)^7} \Delta z \right) \Delta r \right) \Delta w \leq \frac{16}{2^s(s+1)^3}.$$

Finally,

$$\begin{aligned} & \int_0^\infty s \left(\int_s^\infty \left(\int_w^\infty \left(\int_r^\infty \frac{2z}{2^z(z+1)^7} \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \leq \int_0^\infty \frac{16s}{2^s(s+1)^3} \Delta s \\ &= \sum_{s \in [0, \infty)_{\mathbb{T}}} \frac{16s\mu(s)}{2^s(s+1)^3} = \sum_{s \in (0, \infty)_{\mathbb{T}}} \frac{16s^2}{2^s(s+1)^3} \\ &< \sum_{s \in (0, \infty)_{\mathbb{T}}} \frac{16}{2^s(s+1)} < \sum_{n=1}^\infty \frac{16}{2^n(n+1)} < \infty, \end{aligned}$$

which shows that condition (7) holds. One of the solutions to the equation

$$y^{\Delta^4}(t) = 0$$

is, by (3.2), the function

$$y(t) = \int_0^t \left(\int_0^r w \Delta w \right) \Delta r + \int_0^t \left(\int_0^r 1 \Delta w \right) \Delta r. \quad (11)$$

Let us calculate the second integral in (11). Since

$$\begin{aligned} \int_0^r 1 \Delta w &= \sum_{w \in [0, r)_{\mathbb{T}}} \mu(w) = 2 + \sum_{w \in (0, r)_{\mathbb{T}}} \mu(w) \\ &= 2 + \sum_{w \in (0, r)_{\mathbb{T}}} w = 2 + 2 + 2^2 + \dots + 2^{n_r-1} = 2^{n_r} = r, \end{aligned}$$

then we obtain

$$\begin{aligned} \int_0^t \left(\int_0^r 1 \Delta w \right) \Delta r &= \int_0^t r \Delta r = \sum_{r \in [0, t)_{\mathbb{T}}} r \mu(r) = 0 \cdot 2 + \sum_{r \in (0, t)_{\mathbb{T}}} r^2 \\ &= 2^2 + (2^2)^2 + \dots + (2^{n_t-1})^2 \\ &= \frac{1}{3}(2^{n_t})^2 - \frac{4}{3} = \frac{1}{3}t^2 - \frac{4}{3}. \end{aligned}$$

Using the above result, we now proceed to calculate the first integral in (11)

$$\begin{aligned}
 \int_0^t \left(\int_0^r w \Delta w \right) \Delta r &= \int_0^t \left(\frac{1}{3} r^2 - \frac{4}{3} \right) \Delta r = -\frac{4}{3} \int_0^t 1 \Delta r + \frac{1}{3} \int_0^t r^2 \Delta r \\
 &= -\frac{4}{3} t + \frac{1}{3} \sum_{r \in [0, t)_{\mathbb{T}}} r^2 \mu(r) = -\frac{4}{3} t + 0 \cdot 2 + \sum_{r \in (0, t)_{\mathbb{T}}} r^3 \\
 &= -\frac{4}{3} t + \frac{1}{3} \left(2^3 + (2^2)^3 + \dots + (2^{n_t-1})^3 \right) \\
 &= -\frac{4}{3} t + \frac{1}{21} (2^{n_t})^3 - \frac{8}{21} = \frac{1}{21} t^3 - \frac{4}{3} t - \frac{8}{21}
 \end{aligned}$$

and eventually, we get

$$y(t) = \frac{1}{21} t^3 + \frac{1}{3} t^2 - \frac{4}{3} t - \frac{12}{7}.$$

Then by Theorem 4.1 there exists a solution x of (1) such that

$$x(t) = \frac{1}{21} t^3 + \frac{1}{3} t^2 - \frac{4}{3} t - \frac{12}{7} + o\left(\frac{1}{t}\right), \quad t \in \mathbb{T}.$$

5. Proofs of lemmas from the paragraph 3

Proofs of Lemma 3.1 and Lemma 3.2 are omitted since are straightforward and need only common calculations.

Proof of Lemma 3.3. Changing the order of integration, we get the first part of the thesis

$$\int_0^\infty W_1(s) \left(\int_s^\infty U_1(r) \Delta r \right) \Delta s = \int_0^\infty U_1(r) \left(\int_0^r W_1(s) \Delta s \right) \Delta r < \infty.$$

Similarly changing the order of integration, we obtain

$$\begin{aligned}
 \int_t^\infty U_1(r) \left(\int_0^r W_1(s) \Delta s \right) \Delta r &= \int_0^t W_1(s) \left(\int_t^\infty U_1(r) \Delta r \right) \Delta s \\
 + \int_n^\infty W_1(s) \left(\int_s^\infty U_1(r) \Delta r \right) \Delta s &\geq \int_t^\infty W_1(s) \left(\int_s^\infty U_1(r) \Delta r \right) \Delta s,
 \end{aligned}$$

since U_1 and W_1 are positive functions. □

Proof of Lemma 3.4. Define functions W_1, U_1, W_2, U_2 by

$$\begin{aligned} W_1(t) &= 1, & U_1(t) &= \frac{1}{a(t)} \int_t^\infty \left(\int_r^\infty |u(z)| \Delta z \right) \Delta r, \\ W_2(t) &= \frac{t}{a(t)}, & U_2(t) &= \int_t^\infty |u(z)| \Delta z. \end{aligned}$$

Then by assumption (6), there is

$$\int_0^\infty W_1(s) \left(\int_s^\infty U_1(w) \Delta w \right) \Delta s < \infty.$$

Using Lemma 3.3, we get

$$\begin{aligned} \int_0^\infty W_2(s) \left(\int_s^\infty U_2(r) \Delta r \right) \Delta s &= \int_0^\infty \frac{s}{a(s)} \left(\int_s^\infty \left(\int_r^\infty |u(z)| \Delta z \right) \Delta r \right) \Delta s \\ &= \int_0^\infty s \left(\frac{1}{a(s)} \int_s^\infty \left(\int_r^\infty |u(z)| \Delta z \right) \Delta r \right) \Delta s \\ &= \int_0^\infty s U_1(s) \Delta s \\ &= \int_0^\infty U_1(s) \left(\int_0^s W_1(w) \Delta w \right) \Delta s < \infty. \end{aligned}$$

Using Lemma 3.3 once more (in case of W_2 and U_2), we obtain

$$\begin{aligned} \infty &> \int_0^\infty U_2(r) \left(\int_0^r W_2(s) \Delta s \right) \Delta r = \int_0^\infty \left(\int_r^\infty |u(z)| \Delta z \right) \left(\int_0^r \frac{s}{a(s)} \Delta s \right) \Delta r \\ &= \int_0^\infty \left(\int_0^r \frac{s}{a(s)} \Delta s \right) \left(\int_r^\infty |u(z)| \Delta z \right) \Delta r. \end{aligned} \quad (12)$$

Define functions W_3, U_3 by

$$W_3(t) = \int_0^t \frac{s}{a(s)} \Delta s, \quad U_3(t) = |u(t)|.$$

Then, by (12), we have

$$\int_0^\infty W_3(r) \left(\int_r^\infty U_3(z) \Delta z \right) \Delta r < \infty.$$

By Lemma 3.3 (in case of W_3 and U_3), we obtain

$$\begin{aligned} \infty &> \int_0^\infty U_3(z) \left(\int_0^z W_3(r) \Delta r \right) \Delta z = \int_0^\infty \left(|u(z)| \int_0^z \left(\int_0^r \frac{w}{a(w)} \Delta w \right) \Delta r \right) \Delta z \\ &= \int_0^\infty |u(z)| Q(z) \Delta z. \end{aligned}$$

For $t \in \mathbb{T}$, we get

$$\begin{aligned} &\int_t^\infty \left(\int_s^\infty \frac{1}{a(w)} \left(\int_w^\infty \left(\int_r^\infty |u(z)| \Delta z \right) \Delta r \right) \Delta w \right) \Delta s \\ &= \int_t^\infty \left(\int_s^\infty U_1(w) \Delta w \right) \Delta s = \int_t^\infty \left(W_1(s) \int_s^\infty U_1(w) \Delta w \right) \Delta s \\ &\stackrel{\text{by (4)}}{\leq} \int_t^\infty U_1(w) \left(\int_0^w W_1(s) \Delta s \right) \Delta w = \int_t^\infty w U_1(w) \Delta w \\ &= \int_t^\infty \left(\frac{w}{a(w)} \int_w^\infty \left(\int_r^\infty |u(z)| \Delta z \right) \Delta r \right) \Delta w = \int_t^\infty \left(W_2(w) \int_w^\infty U_2(r) \Delta r \right) \Delta w \\ &\stackrel{\text{by (4)}}{\leq} \int_t^\infty \left(U_2(r) \int_0^r W_2(w) \Delta w \right) \Delta r = \int_t^\infty \left(\int_r^\infty |u(z)| \Delta z \right) \left(\int_0^r \frac{w}{a(w)} \Delta w \right) \Delta r \\ &= \int_t^\infty \left(\int_0^r \frac{w}{a(w)} \Delta w \right) \left(\int_r^\infty |u(z)| \Delta z \right) \Delta r = \int_t^\infty W_3(r) \left(\int_r^\infty U_3(z) \Delta z \right) \Delta r \\ &\stackrel{\text{by (4)}}{\leq} \int_t^\infty U_3(z) \left(\int_0^z W_3(r) \Delta r \right) \Delta z = \int_t^\infty \left(|u(z)| \int_0^z \left(\int_0^r \frac{w}{a(w)} \Delta w \right) \Delta r \right) \Delta z \\ &= \int_t^\infty |u(z)| Q(z) \Delta z. \end{aligned}$$

□

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