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FROM POTENTIAL TO ACTUAL KNOWLEDGE: A NEW FRAMEWORK IN PROBABILISTIC EPISTEMIC LOGIC

Abstract. In this paper, we address the issue of improbable knowledge within probabilistic epistemic logic. This problem occurs when an agent possesses knowledge of a certain proposition, despite the evidential probability of that knowledge being exceedingly low. The existence of such knowledge is paradoxical and poses significant challenges for probabilistic accounts of justification. We identify that the problem arises from assumptions about the non-transitivity of the epistemic accessibility relation, which leads to the rejection of Axiom 4, according to which if one knows, one knows that one knows. By redefining the epistemic operators, we demonstrate that accepting Axiom 4 is feasible, thus preventing improbable knowledge. Furthermore, we explore the distinction between potential and actual knowledge as a solution to the problem of logical omniscience. We construct a probabilistic epistemic logic system that accurately models actual knowledge for agents with limited deductive abilities. Finally, we introduce key semantic concepts for a dynamic probabilistic epistemic logic designed for non-omniscient agents, providing a promising foundation for studying the dynamics of knowledge and its relationship with probability.

Keywords: epistemic logic, improbable knowledge, evidential probability, actual knowledge, logical omniscience, dynamic epistemic logic.

1. Introduction

Epistemic logics are formal systems that enable reasoning about knowledge, belief, and related concepts such as information and justification. The most commonly used form of these logics today is modal logic. Modal epistemic logics have proven useful for formalizing intuitions about fundamental epistemic concepts and have also gained significant attention from researchers in fields such as artificial intelligence, economics, game theory, and cognitive science.

The growing interest in epistemic logics among researchers from disciplines beyond philosophy has led to the establishment of new objectives for the logics of knowledge and belief. One consequence of this development is the attempt to integrate epistemic logic with probability theory (see e.g. Fagin & Halpern, 1994; Halpern, 2003; Halpern & Tuttle, 1993; Kooi, 2003; Williamson, 2000). However, in certain formalisms that combine logical representations of knowledge with probability theory, it becomes possible to model the emergence of so-called improbable knowledge – situations where an agent can know p even though, based on their evidence, it is almost certain that they do not know p. The existence of such knowledge seems surprising, even paradoxical, and leads to several highly undesirable consequences.

The problem of improbable knowledge is closely related to another widely discussed issue in the literature on modal epistemic logics: the problem of logical omniscience. This is the controversial assumption in epistemic logic based on possible worlds semantics, which posits that an agent knows all tautologies and all logical consequences of their knowledge. In some applications of epistemic logic, such as in game theory and philosophy, representing the knowledge of omniscient agents with unlimited deductive abilities may be accepted as a justified idealization. For instance, if we prove that even an omniscient agent cannot know a certain proposition, we can conclude that such a proposition cannot be an object of knowledge for any agent, including a human being. However, when representing the knowledge of real cognitive agents, these assumptions become highly undesirable.

In addition to conflicting with common intuitions, the concept of logical omniscience for real-world agents is supported by concrete empirical evidence that highlights the depth of the problem. Results from psychological experiments, such as the Wason selection task, the suppression task, and the Linda problem, show that agents' performance in reasoning often deviates significantly from the predictions of normative theories (see Byrne, 1989; Gigerenzer & Selten, 2002; Tversky & Kahneman, 1983; Wason, 1968). Moreover, the recent behavioral turn in economics further supports the need to revise idealized models of perfect rationality (see Thaler & Sunstein, 2008). Therefore, it is essential to reconsider the standard framework for representing agents' epistemic states.

In this paper, we start with two primary goals. First, we aim to reconstruct the problem of improbable knowledge within the framework of probabilistic epistemic logics. Second, we propose a modification of this formalism that not only prevents the emergence of such knowledge but also integrates epistemic logic with probability theory in a way that avoids the issue of logical omniscience. This approach will provide a more realistic representation of human reasoning and establish a formal framework for modeling the relationship between knowledge and probability in the context of non-omniscient agents who possess varying degrees of logical competence.

2. Modal epistemic logics

First, we will introduce the formal language of modal epistemic logics, $\mathcal{L}_{MEL}$, which serves as the common language for a wide class of epistemic systems.

Definition 1

Let Var denote the set of propositional variables, and let Ag denote the set of agents. The language of modal epistemic logics $\mathcal{L}_{MEL}$ is defined inductively as follows:

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid \varphi \to \psi \mid K_a\varphi,$$

where $p \in Var$ and $a \in Ag$. The set of all $\mathcal{L}_{MEL}$ formulas is denoted by $\Gamma_{\mathcal{L}_{MEL}}$.

$\mathcal{L}_{MEL}$ is an extension of propositional logic that includes epistemic operators K_a for every agent $a \in Ag$. The intended interpretation of $K_a\varphi$ is "agent a knows that φ". Although terms such as "agent" and "set of agents" are used here, the specific identities of the agents are not defined at this stage. The set Ag could refer to human beings, machines, programs, or even abstract players in a given game. At this point, the cardinality of the set Ag also remains undetermined. However, if Ag contains more than one element, we refer to this as a multi-agent epistemic logic. Conversely, if Ag is a singleton (i.e., $Ag = \{a\}$), then we are dealing with a single-agent epistemic logic. In this case, the index can be omitted, and the formula $K\varphi$ is interpreted as "it is known that φ" or "someone knows that φ". Single-agent epistemic logics are exact counterparts of certain alethic modal logics, because they are derived by consistently replacing the necessity operator $\Box$ with the knowledge operator K. As a result, single-agent epistemic logics inherit all the metalogical properties of alethic modal logics.

The language $\mathcal{L}_{MEL}$ can be extended by the operator $\langle K_a\rangle$ for every agent $a \in Ag$, defined for any formula $\varphi \in \Gamma_{\mathcal{L}_{MEL}}$ using the epistemic operator K_a and negation:

$\langle K_a\rangle\,\varphi \overset{\text{def}}{=} \neg K_a\neg\varphi.$

The operator $\langle K_a \rangle$ is the epistemic counterpart of the alethic possibility operator $\Diamond$. The formula $\langle K_a \rangle \varphi$ is typically interpreted as "agent a considers it possible that φ" or "φ is consistent with everything agent a knows". Other classical logical constants can be defined in the standard way:

$$\varphi \vee \psi \overset{\text{def}}{=} \neg\varphi \to \psi,\ \varphi \wedge \psi \overset{\text{def}}{=} \neg(\varphi \to \neg\psi),\ \varphi \leftrightarrow \psi \overset{\text{def}}{=} \neg((\varphi \to \psi) \to \neg(\psi \to \varphi)).$$

Hintikka (1962) was the first to propose a modal logic approach to knowledge and belief, which led to the development of semantics for epistemic logics based on the possible worlds framework established by Kripke (1959, 1963) for the logic of possibility and necessity. This approach provides a natural interpretation for the language $\mathcal{L}_{MEL}$.

Definition 2

An epistemic model is a tuple $\mathcal{M} = (W, \{R_a : a \in Ag\}, v)$, where

- $W \neq \emptyset$ is a set of epistemic states,
- $Ag \neq \emptyset$ is a set of agents,
- $R_a \subseteq W \times W$ is an epistemic accessibility relation for any $a \in Ag$,
- $v : Var \mapsto \mathcal{P}(W)$ is a valuation function that assigns to every $p \in Var$ the set of epistemic states in which p is true.

Definition 3

Let $\mathcal{M} = (W, \{R_a : a \in Ag\}, v)$ be an epistemic model, and let $s \in W$ represent a specific state within this model. The satisfaction relation $\models$ is defined inductively in the following way:

$$\begin{array}{lll}
\mathcal{M}, s \models p & \text{iff} & s \in v(p), \\
\mathcal{M}, s \models \neg\varphi & \text{iff} & \mathcal{M}, s \not\models \varphi, \\
\mathcal{M}, s \models \varphi \to \psi & \text{iff} & \text{if } \mathcal{M}, s \models \varphi, \text{ then } \mathcal{M}, s \models \psi, \\
\mathcal{M}, s \models K_a\varphi & \text{iff} & \text{for all } t \in W\text{: if } (s,t) \in R_a, \text{ then } \mathcal{M}, t \models \varphi,
\end{array}$$

where $p \in Var$, $\varphi, \psi \in \Gamma_{\mathcal{L}_{MEL}}$, and $a \in Ag$.

Definition 4

Let $\mathcal{M} = (W, \{R_a : a \in Ag\}, v)$ be an epistemic model and let $\varphi \in \Gamma_{\mathcal{L}_{MEL}}$. The extension of a formula φ in the model $\mathcal{M}$, denoted by $[\![\varphi]\!]_{\mathcal{M}}$, is defined in the following way:

$$[\![\varphi]\!]_{\mathcal{M}} = \{s \in W : \mathcal{M}, s \models \varphi\}.$$

Other important semantic notions are defined in the standard way. We say that a formula $\varphi \in \Gamma_{\mathcal{L}_{MEL}}$ is true in an epistemic model $\mathcal{M} = (W, \{R_a : a \in Ag\}, v)$, and we write $\mathcal{M} \models \varphi$, if $\mathcal{M}, s \models \varphi$ for every $s \in W$. Additionally, we say that a formula $\varphi \in \Gamma_{\mathcal{L}_{MEL}}$ is valid, and we write $\models \varphi$, if $\mathcal{M} \models \varphi$ for every epistemic model $\mathcal{M}$.

Having established the semantics for the language $\mathcal{L}_{MEL}$, we can now introduce specific epistemic systems, starting with the minimal modal epistemic logic, which is denoted as **K**.

Definition 5

A proof system for the logic **K** consists of the following axiom schemes and inference rules:

All instantiations of propositional tautologies, (PC)

$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$, (K)

from $\varphi \to \psi$ and φ infer ψ, (modus ponens)

from $\vdash \varphi$ infer $\vdash K_a\varphi$. (Gödel's rule for K_a)

A formula is considered a **K**-theorem if it belongs to the smallest set of formulas that includes all the axioms and is closed under the inference rules. If φ is a **K**-theorem, we write $\mathbf{K} \vdash \varphi$.

The difference in notation between modus ponens and Gödel's rule for K_a in the definition above is intended to highlight the fact that these rules do not have the same strength. According to Gödel's rule, if a formula φ is a theorem, then $K_a\varphi$ is also a theorem. Modus ponens has this property as well: if both $\varphi \to \psi$ and φ are theorems, then ψ is likewise a theorem. However, modus ponens satisfies an even stronger condition: the formula $((\varphi \to \psi) \wedge \varphi) \to \psi$ is a theorem of any modal epistemic logic. An analogous condition does not hold for Gödel's rule: the formula $\varphi \to K_a\varphi$ is not a theorem of any standard modal epistemic logic.

The only axiom of the logic **K** that is not a substitution of a propositional tautology is the K axiom, which states that the knowledge operator is closed under implication. This axiom is universally accepted in all systems of modal epistemic logic. A list of other familiar epistemic axioms is provided in Table 1.

Table 1

Epistemic axioms

Name	Axiom	Property of R_a
D	$K_a\varphi \to \langle K_a\rangle \varphi$	serial
T	$K_a\varphi \to \varphi$	reflexive
4	$K_a\varphi \to K_aK_a\varphi$	transitive
5	$\neg K_a\varphi \to K_a\neg K_a\varphi$	Euclidean

One of the key advantages of Kripke's relational semantics is the precise correspondence between the epistemic axioms and the properties of accessibility relations, the latter of which can be expressed in first-order logic. To demonstrate this connection, the concept of an epistemic frame must be introduced.

Definition 6

An epistemic frame is a pair $\mathcal{F} = (W, \{R_a : a \in Ag\})$, where W and R_a for any $a \in Ag$ are interpreted according to Definition 2. An epistemic model $\mathcal{M}$ is said to be based on a frame $\mathcal{F} = (W, \{R_a : a \in Ag\})$ if $\mathcal{M} = (W, \{R_a : a \in Ag\}, v)$ for some valuation function v. We say that a formula $\varphi \in \Gamma_{\mathcal{L}_{MEL}}$ is true in an epistemic frame $\mathcal{F}$, and we write $\mathcal{F} \models \varphi$, if for every epistemic model $\mathcal{M}$ based on $\mathcal{F}$, it holds that $\mathcal{M} \models \varphi$.

The connection between epistemic axioms and the properties of accessibility relations is that the corresponding axioms define classes of frames with specific relational properties. For example, consider the condition for the axiom 4.

Proposition 1

Let $\mathcal{F} = (W, R_a)$ be an epistemic frame, and let $a \in Ag$. Then

$\mathcal{F} \models K_a\varphi \to K_aK_a\varphi$ iff R_a is transitive.

Analogous conditions for the other axioms can be formulated in the same manner in accordance with Table 1. It should be emphasized that the axiom K and all propositional tautologies are true in every frame.

Proof systems for logics stronger than **K** are obtained by adding other axioms to the system. Table 2 lists the most important of these logics. Generally, $\mathbf{L} \oplus \mathsf{A}$ denotes the extension of the logic $\mathbf{L}$ with the axiom A.

Table 2

Some basic epistemic systems

Name	Logic	Properties of R_a
KD	$\mathbf{K} \oplus \mathsf{D}$	serial
T	$\mathbf{K} \oplus \mathsf{T}$	reflexive
S4	$\mathbf{T} \oplus \mathsf{4}$	reflexive, transitive
S5	$\mathbf{T} \oplus \mathsf{5}$	reflexive, Euclidean
KD4	$\mathbf{KD} \oplus \mathsf{4}$	serial, transitive
KD45	$\mathbf{KD4} \oplus \mathsf{5}$	serial, transitive, Euclidean

The standard modal epistemic logics are sound and complete with respect to classes of models whose accessibility relations have the properties expressed in first-order logic. For example, the logic **S5** is sound and complete with respect to the class of all reflexive and Euclidean epistemic models. As reflexivity and Euclideanness entail both symmetry and transitivity, the logic **S5** is equivalently sound and complete with respect to the class of all equivalence epistemic models. The corresponding theorems for other epistemic logics can be formulated similarly based on the content of Table 2. The logic **K** is sound and complete with respect to the class of epistemic models with arbitrary accessibility relations.

3. Modal epistemic logics and evidential probability

Williamson (2000) introduced an extension of epistemic logics to probabilistic epistemic logics. In this section, we adopt some of the assumptions from Williamson's account and apply them to the formal framework introduced in Section 2. While this presentation may differ in several respects from Williamson's original formulation, these differences do not alter the central consequences of his approach. Moreover, our formalism is more general, because it will be applied to multi-agent logics.

The language of probabilistic epistemic logics is an extension of the language $\mathcal{L}_{MEL}$.

Definition 7

Let Var denote the set of propositional variables, and let Ag denote the set of agents. The language of probabilistic epistemic logics $\mathcal{L}_{PEL}$ is defined inductively as follows:

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid \varphi \to \psi \mid K_a\varphi \mid Pr_a(\varphi) = c$$

where $p \in Var$, $a \in Ag$ and $c \in [0,1]$. The set of all $\mathcal{L}_{PEL}$ formulas is denoted by $\Gamma_{\mathcal{L}_{PEL}}$.

The formula $Pr_a(\varphi) = c$ indicates that the evidential probability of φ for agent a takes the value c, where $c \in [0,1]$. The concept of evidential probability captures the idea of how probable a proposition is from the perspective of an agent, given their current evidence. It quantifies the agent's rational degree of belief in the proposition based on the accessible information. It should be emphasized that the language $\mathcal{L}_{PEL}$ allows for iterated epistemic and probability operators, including mixed expressions such as $K_a(Pr_b(\varphi) = c)$ or $Pr_a(K_b\varphi) = c$.

The interpretation of the language $\mathcal{L}_{PEL}$ is carried out within finite epistemic models that include a probability distribution.

Definition 8

A probabilistic epistemic model is a tuple $\mathcal{M} = (W, \{R_a : a \in Ag\}, P_{prior}, v)$, where W is finite and $P_{prior} : \mathcal{P}(W) \mapsto [0,1]$ is a uniform probability distribution, defined as follows:

$$P_{prior}(A) = \frac{|A|}{|W|}, \quad \text{for any} A \subseteq W.$$

It is worth noting that P_{prior} always takes the simple form of a uniform distribution over the subsets of a finite set W, in the sense that every epistemic state has the same probabilistic weight as every other state. Thus, for any subset $A \subseteq W$, the probability $P_{prior}(A)$ is proportional to the cardinality of A.

Because uniform prior distributions ensure that every nonempty subset of W has a nonzero probability, we can define prior conditional probabilities straightforwardly by ratios. For any probabilistic epistemic model $\mathcal{M} = (W, \{R_a : a \in Ag\}, P_{prior}, v)$, any formula $\varphi \in \Gamma_{\mathcal{L}_{PEL}}$, and any formula $\psi \in \Gamma_{\mathcal{L}_{PEL}}$ such that $[\![\psi]\!]_{\mathcal{M}} \neq \emptyset$, the conditional probability is defined as follows:

$$P_{prior}([\![\varphi]\!]_{\mathcal{M}} | [\![\psi]\!]_{\mathcal{M}}) = \frac{P_{prior}([\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}})}{P_{prior}([\![\psi]\!]_{\mathcal{M}})}.$$

This formula for conditional probability is the modal counterpart of the classical formula provided by Kolmogorov (1950).

The formula for conditional probability is also used to compute evidential probability. For any $\mathcal{M} = (W, \{R_a : a \in Ag\}, P_{prior}, v)$, any formula $\varphi \in \Gamma_{\mathcal{L}_{PEL}}$, any $s \in W$, and any $a \in Ag$, the evidential probability in the epistemic state s is calculated using the following formula:

$$P_a^s(\llbracket \varphi \rrbracket_{\mathcal{M}}) = P_{prior}(\llbracket \varphi \rrbracket_{\mathcal{M}} | R_a(s)) = \frac{P_{prior}(\llbracket \varphi \rrbracket_{\mathcal{M}} \cap R_a(s))}{P_{prior}(R_a(s))},$$

where $R_a(s) = \{x \in W : (s, x) \in R_a\}$. Thus, the evidential probability in state s is identified with its probability conditional on the total evidence available in s. We assume that the accessibility relation is at least serial, to avoid the issue where the divisor is zero.

This formula allows us to define the satisfaction condition for the probability operator.

Definition 9

Let $\mathcal{M} = (W, \{R_a : a \in Ag\}, P_{prior}, v)$ be a probabilistic epistemic model, and let $s \in W$. The interpretation for propositional variables, classical logical constants, and the knowledge operator follows the same rules as in Definition 3. For the evidential probability operator, we have

$\mathcal{M}, s \models Pr_a(\varphi) = c \quad \text{iff} \quad P_a^s(\llbracket \varphi \rrbracket_{\mathcal{M}}) = c,$

where $\varphi \in \Gamma_{\mathcal{L}_{PEL}}$, $a \in Ag$ and $c \in [0, 1]$.

Note that since W is finite and P_{prior} is uniform, as introduced in Definition 8, all evidential probabilities are ratios of cardinalities and hence belong to $\mathbb{Q} \cap [0, 1]$. When $c \notin \mathbb{Q}$, the formula $Pr_a(\varphi) = c$ is therefore unsatisfiable in the class of finite uniform models.

To clarify the interaction between knowledge and evidential probability within the probabilistic epistemic framework, we present a concrete example that demonstrates how these concepts are applied in a specific model.

Example 1

Consider the following scenario. An agent a does not know whether p is true or not. In fact, p happens to be false. On the other hand, the agent knows that q is true. This example can be formalized by the following probabilistic epistemic model: $\mathcal{M} = (W, R_a, P_{prior}, v)$, where $W = \{x, y\}$, $R_a = \{(x, x), (y, y), (x, y), (y, x)\}$, $P_{prior}(\{x\}) = P_{prior}(\{y\}) = 0.5$, $v(p) = \{y\}$, and $v(q) = \{x, y\}$. Notice that this model is a faithful representation of the described scenario:

$$\mathcal{M}, x \models \neg p \wedge \neg K_a p \wedge \langle K_a \rangle p \wedge Pr_a(p) = 0.5 \wedge Pr_a(\langle K_a \rangle p) = 1 \wedge Pr_a(K_a p) = 0,$$

$$\mathcal{M}, x \models q \wedge K_a q \wedge Pr_a(q) = 1 \wedge Pr_a(K_a q) = 1.$$

4. The challenge of improbable knowledge

Now, we can address the problem of improbable knowledge, which arises from the interaction between knowledge and evidential probability as previously formalized. This issue occurs when an agent possesses knowledge of a proposition, despite the evidential probability of that knowledge being exceedingly low.

Let us consider the following example, originally introduced by Williamson (2011), which demonstrates how improbable knowledge can arise within a probabilistic epistemic framework.

Example 2

Consider the probabilistic epistemic model $\mathcal{M} = (W, R_a, P_{prior}, v)$, where $W = \{x_1, \ldots, x_m, y_{m+1}, \ldots, y_n, z\}$. For any i, i', j, j' such that $1 \leq i$, $i' \leq m < j$, $j' \leq n$, the following conditions hold: $(x_i, x_{i'}) \in R_a$, $(x_i, y_j) \in R_a$, $(y_j, y_{j'}) \in R_a$ and $(y_j, z) \in R_a$, but $(x_i, z) \notin R_a$. Additionally, let $v(p) = \{x_1, \ldots, x_m, y_{m+1}, \ldots, y_n\}$. Notice that, due to the structure of R_a, for any i such that $1 \leq i \leq m$, we have $R_a(x_i) = v(p)$, where $R_a(x_i) = \{y \in W : (x_i, y) \in R_a\}$. Thus, for any i such that $1 \leq i \leq m$, it holds that $\mathcal{M}, x_i \models K_a p$. Because the prior distribution P_{prior} is uniform, the evidential probability that agent a knows p in a state x_i is given by

$$P_a^{x_i}(\llbracket K_a p \rrbracket_{\mathcal{M}}) = m/n.$$

For $m = 1$, this yields $P_a^{x_i}(\llbracket K_a p \rrbracket_{\mathcal{M}}) = 1/n$. So, for sufficiently large n, the evidential probability will be close to zero.

Given the adequacy of the presented model for representing knowledge and evidential probability, one must acknowledge the possibility of improbable knowledge arising. This situation is not only surprising but also appears paradoxical, raising significant philosophical issues.

The first problem concerns the relationship between knowledge, justification, and probability. According to the classical definition, dating back to Plato's *Theaetetus*, knowledge is justified true belief. While the famous examples provided by Gettier (1963) suggest that fulfilling these three conditions may not be sufficient for knowledge, the traditional definition is still understood as establishing the necessary conditions. Thus, knowledge entails justification. But what justifies belief? Probabilistic accounts of justification (see e.g. Conee & Feldman, 2004; Moser, 1989; Plantinga, 1993; Swinburne, 2001) assert that epistemic justification is fundamentally a matter of high probability or likelihood. However, if improbable knowledge were to occur, an agent could know that p, and thus have a justification for believing that p, even though the probability of knowing p is extremely low. This scenario challenges probabilistic accounts of justification and might even force us to reconsider the classical conception of knowledge[1].

The second problem is, in a sense, the inverse of the first. The possibility of improbable knowledge poses a significant issue to those authors who argue that a sufficient condition for an agent to have justification for a certain proposition is that the probability of that proposition, based on the agent's data, exceeds a certain threshold value $k \in \mathbb{R}$. Consider the Moorean sentence of the form $p \wedge \neg K_a p$[2]. The existence of improbable knowledge could lead to a scenario where an agent a has justification for believing $p \wedge \neg K_a p$. For instance, let $\mathcal{M} = (W, R_a, P_{prior}, v)$ and $s \in W$ be such that $\mathcal{M}, s \models K_a p$, but $P^s_a(\llbracket K_a p \rrbracket_{\mathcal{M}}) < 1/n$. In this case, $P^s_a(\llbracket \neg K_a p \rrbracket_{\mathcal{M}}) > (n-1)/n$. However, given $\mathcal{M}, s \models K_a p$, it follows that $P^s_a(\llbracket p \rrbracket_{\mathcal{M}}) = 1$. Thus, $P^s_a(\llbracket p \wedge \neg K_a p \rrbracket_{\mathcal{M}}) > (n-1)/n$, which can exceed the threshold k for sufficiently large n. This suggest that there can be cases where the probability of a Moorean conjunction, based on one's evidence, surpasses the given threshold, thus satisfying the supposed sufficient condition for justification. However, it is implausible to assert that one could be justified in believing a statement like "Sara passed the exam, but I don't know that Sara passed the exam".

Another difficulty, as noted by Williamson (2011), arises within the framework of decision theory and involves a relatively simple decision-making scenario. Suppose agent a is in a situation where it is rational for a to take action D if and only if a knows that p; otherwise, taking action D would be irrational. Now, let us assume that p represents a case of improbable knowledge for agent a. In this case, taking action D would be rational, because a knows that p. However, from the perspective of a's available evidence, it is almost certain that a does not know that p. Consequently, it is nearly certain, based on a's evidence, that taking action D would be irrational. In this scenario, it seems indeterminate whether a should take action D or not.

The problems presented above appear sufficient to conclude that the existence of improbable knowledge would be highly undesirable. But what is the source of this problem, and how might it be prevented? Note that the adopted assumptions about probability are not inherently controversial. While Williamson assumed that the probability distribution is uniform, which could be contested, denying this assumption would not undermine the current analysis. After all, this assumption serves to demonstrate that a particular phenomenon can occur, not that it must occur.

We believe the source of the problem does not lie in adopted assumptions, but rather in what was not assumed. In the model illustrating the occurrence of improbable knowledge in Example 2, the accessibility relation is not transitive. However, if we were to impose the condition of transitivity – along with the assumption of reflexivity, which was indeed adopted in Williamson's example – it would no longer be possible to construct an analogous model for improbable knowledge.

Theorem 1

Let $\mathcal{M} = (W, \{R_a : a \in Ag\}, P_{prior}, v)$ be a reflexive and transitive probabilistic epistemic model, let $s \in W$, $a \in Ag$, and $\varphi \in \Gamma_{\mathcal{L}_{PEL}}$. Then the following equivalence holds:

$\mathcal{M}, s \models K_a\varphi$ iff $\mathcal{M}, s \models Pr_a(K_a\varphi) = 1$.

Proof.

We will prove both directions of the equivalence.

($\Longrightarrow$) Assume that $\mathcal{M}, s \models K_a\varphi$. In order to prove that $\mathcal{M}, s \models Pr_a(K_a\varphi) = 1$, it suffices to prove $R_a(s) \subseteq [\![K_a\varphi]\!]_{\mathcal{M}}$, as it is known that:

$$P_a^s([\![K_a\varphi]\!]_{\mathcal{M}}) = P_{prior}([\![K_a\varphi]\!]_{\mathcal{M}} | R_a(s)) = \frac{P_{prior}([\![K_a\varphi]\!]_{\mathcal{M}} \cap R_a(s))}{P_{prior}(R_a(s))}.$$

Hence, let $x \in R_a(s)$. We aim to show that $x \in [\![K_a\varphi]\!]_{\mathcal{M}}$. Let $y \in W$ be such that $(x, y) \in R_a$. Since $(s, x) \in R_a$ and $(x, y) \in R_a$, by the transitivity of the relation R_a, we obtain $(s, y) \in R_a$. Given that $\mathcal{M}, s \models K_a\varphi$ and $(s, y) \in R_a$, we have $\mathcal{M}, y \models \varphi$. Therefore, $\mathcal{M}, x \models K_a\varphi$, i.e., $x \in [\![K_a\varphi]\!]_{\mathcal{M}}$. This establishes that $R_a(s) \subseteq [\![K_a\varphi]\!]_{\mathcal{M}}$. In this case, $[\![K_a\varphi]\!]_{\mathcal{M}} \cap R_a(s) = R_a(s)$. Thus, we finally have:

$$P_a^s([\![K_a\varphi]\!]_{\mathcal{M}}) = \frac{P_{prior}(R_a(s))}{P_{prior}(R_a(s))} = 1.$$

This concludes the proof that $\mathcal{M}, s \models Pr_a(K_a\varphi) = 1$.

($\Longleftarrow$) Assume that $\mathcal{M}, s \models Pr_a(K_a\varphi) = 1$. Then:

$$P_a^s(\llbracket K_a\varphi \rrbracket_{\mathcal{M}}) = \frac{P_{prior}(\llbracket K_a\varphi \rrbracket_{\mathcal{M}} \cap R_a(s))}{P_{prior}(R_a(s))} = 1.$$

Therefore, $P_{prior}(\llbracket K_a\varphi \rrbracket_{\mathcal{M}} \cap R_a(s)) = P_{prior}(R_a(s))$. Since P_{prior} is uniform over W, this equality of probabilities implies:

$$\frac{|\llbracket K_a\varphi \rrbracket_{\mathcal{M}} \cap R_a(s)|}{|W|} = \frac{|R_a(s)|}{|W|}.$$

By Definition 8, W is finite, and therefore $R_a(s) \subseteq W$ is also finite. From $|\llbracket K_a\varphi \rrbracket_{\mathcal{M}} \cap R_a(s)| = |R_a(s)|$, $\llbracket K_a\varphi \rrbracket_{\mathcal{M}} \cap R_a(s) \subseteq R_a(s)$, and finiteness of $R_a(s)$, we obtain $\llbracket K_a\varphi \rrbracket_{\mathcal{M}} \cap R_a(s) = R_a(s)$. By reflexivity of R_a, $s \in R_a(s)$, and thus $s \in \llbracket K_a\varphi \rrbracket_{\mathcal{M}}$. This concludes the proof that $\mathcal{M}, s \models K_a\varphi$. $\square$

Of course, ensuring that the accessibility relation in probabilistic models is transitive has significant consequences. According to Proposition 1, the axiom 4 will hold in the resulting logic. This consequence may at first glance seem concerning, and for many, even unacceptable.

The axiom 4, also known as the Positive Introspection Axiom (or the KK Principle), states that if one knows, one knows that one knows. This has been the subject of intense epistemological debate (see e.g. Hendricks, 2007, pp. 80–114). Williamson himself has repeatedly argued against adopting this axiom (see Williamson, 1992, 2000). Indeed, if the operator K_a, for any $a \in Ag$, is interpreted as an operator of actual knowledge, then rejecting Axiom 4 appears fully justified. However, if the operator K_a is interpreted as an operator of potential knowledge, the acceptance of the axiom 4 seems far less problematic. If an agent potentially knows that φ – because, for example, it is within his cognitive abilities to carry out a procedure that would lead him to discover that φ – then that agent also potentially knows that he potentially knows that φ. Under this interpretation, axiom 4 can be accepted, thereby preventing the emergence of improbable knowledge.

Although the language $\mathcal{L}_{PEL}$ interpreted in probabilistic epistemic models allows for the description of potential knowledge in the context of evidential probability, the formalization of actual knowledge within a probabilistic framework remains an open question. The following section presents a formalism designed to address this issue.

5. Integrating awareness into probabilistic epistemic logic

Many authors have pointed out that the failure to distinguish between actual knowledge (explicit knowledge) and potential knowledge (implicit knowledge) while interpreting the operator K_a for all $a \in Ag$, leads to the problem of logical omniscience (see van Benthem, 2006; Fagin & Halpern, 1988; Fagin, Halpern, Moses, & Vardi, 1995; Konolige, 1986; Levesque, 1984). The problem of logical omniscience is connected to two rules of inference. This is the Gödel's rule for K_a according to which the agent knows all theorems of a given epistemic logic, including all propositional tautologies, and the monotonicity rule which may be formulated in the following way:

if $\vdash \varphi \to \psi$, then $\vdash K_a\varphi \to K_a\psi$.

This rule is a consequence of the application of the Gödel's rule for K_a and the rule of modus ponens to the axiom K. It implies that agents know all logical consequences of their knowledge. This assumption is highly problematic when we aim to accurately represent the knowledge of non-ideal agents who possess a certain degree of logical competence but are not logically omniscient.

The distinction between implicit and explicit knowledge is further supported by findings in psychology and cognitive science. For instance, studies have shown (see Schacter, 1987, 1994) that implicit memory influences our performance by allowing us to rely on accumulated experiences without conscious recall, whereas explicit memory typically involves a deliberate and intentional act of recalling information. Solaki (2017) observed that there are various levels or types of knowledge, corresponding to different ways it is acquired.

Referring to the idea proposed by Fagin and Halpern (1988), we introduce a model of knowledge representation that distinguishes between actual and potential knowledge, as well as representing evidential probability. To achieve this, it will be necessary to add a new operator to the language of probabilistic epistemic logic: the awareness operator.

Definition 10

Let Var denote the set of propositional variables, and let Ag denote the set of agents. The language of probabilistic epistemic logics with the awareness operator $\mathcal{L}_{PEL-A}$ is defined inductively as follows:

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid \varphi \to \psi \mid K_a\varphi \mid A_a\varphi \mid Pr_a(\varphi) = c$$

where $p \in Var$, $a \in Ag$ and $c \in [0,1]$. The set of all $\mathcal{L}_{PEL-A}$ formulas is denoted by $\Gamma_{\mathcal{L}_{PEL-A}}$.

The intended interpretation of $A_a\varphi$ is that "the agent a is aware that φ" or "the agent a is informed that φ". The awareness operator can be applied to a formula independently of the K_a operator.

To obtain a semantics for the language $\mathcal{L}_{PEL-A}$, it is necessary to modify the semantics for $\mathcal{L}_{PEL}$ in such a way that the awareness operator can be interpreted.

Definition 11

A probabilistic epistemic model with the awareness function is a tuple $\mathcal{M} = (W, \{R_a : a \in Ag\}, \{\mathcal{A}_a : a \in Ag\}, P_{prior}, v)$, where $\mathcal{A}_a : W \mapsto \mathcal{P}\left(\Gamma_{\mathcal{L}_{PEL-A}}\right)$ is the awareness function for any $a \in Ag$, and the remaining elements are understood in accordance with Definition 8.

Definition 12

Let $\mathcal{M} = (W, \{R_a : a \in Ag\}, \{\mathcal{A}_a : a \in Ag\}, P_{prior}, v)$ be a probabilistic epistemic model with the awareness function, and let $s \in W$. Then, Definition 9 is extended with the following condition:

$\mathcal{M}, s \models A_a\varphi \quad \text{iff} \quad \varphi \in \mathcal{A}_a(s).$

where $\varphi \in \Gamma_{\mathcal{L}_{PEL-A}}$, and $a \in Ag$,

The awareness function represents the set of formulas that agent a is consciously aware of in each epistemic state. It determines the scope of the agent's potential knowledge by specifying which propositions the agent can consider or reason about in a given state.

It seems natural that if an agent knows something, then the agent cannot be completely unaware of it. In this context, it is justified to introduce a distinction between potential and actual knowledge. Potential knowledge is modeled by the operator K_a for any $a \in Ag$, while the actual knowledge that φ holds when φ is a subject of potential knowledge and the agent is aware that φ. Thus, a new epistemic operator – the actual knowledge operator – is defined in the language $\mathcal{L}_{PEL-A}$ for any formula φ and any agent $a \in Ag$ as follows:

$E_a\varphi \overset{\text{def}}{=} A_a\varphi \wedge K_a\varphi.$

Let $\mathcal{M} = (W, \{R_a : a \in Ag\}, \{\mathcal{A}_a : a \in Ag\}, P_{prior}, v)$ be a probabilistic epistemic model with the awareness function, let $s \in W$, and let

$\varphi \in \Gamma_{\mathcal{L}_{PEL-A}}$. The following condition for the actual knowledge operator is obtained according to Definition 12:

$\mathcal{M}, s \models E_a\varphi$ iff $\varphi \in \mathcal{A}_a(s)$ and for all $t \in W$: if $(s,t) \in R_a$, then $\mathcal{M}, t \models \varphi$.

It is important to note that the operator E does not function like a standard modal operator. For example, the formula $E_a(p \rightarrow p)$ is not universally valid, because agents might not be aware of that $p \rightarrow p$. Furthermore, there is no equivalence between $E_a(p \rightarrow q)$ and $E_a(\neg q \rightarrow \neg p)$, because it may be necessary to model an agent's knowledge without assuming that the agent recognizes the equivalence between $p \rightarrow q$ and $\neg q \rightarrow \neg p$. The counterparts of standard epistemic axioms, such as T, $\mathsf{4}$, or $\mathsf{5}$, do not necessarily apply to actual knowledge, although they can be added during the construction of the proof system for a given logic. However, some properties of explicit knowledge correspond to those of implicit knowledge. The rule corresponding to Gödel's rule for K_a now takes the following form: if $\vdash \varphi$, then $\vdash A_a\varphi \rightarrow E_a\varphi$. The schema for explicit knowledge that is analogous to the K axiom is $(E_a\varphi \wedge E_a(\varphi \rightarrow \psi) \wedge A_a\psi) \rightarrow E_a\psi$. Since E_a is defined as the conjunction of awareness and potential knowledge, from $E_a\varphi$ and $E_a(\varphi \rightarrow \psi)$ we obtain $K_a\varphi$ and $K_a(\varphi \rightarrow \psi)$, which by the axiom K entail $K_a\psi$, but not necessarily $A_a\psi$. The additional clause $A_a\psi$ is therefore needed, as awareness is not assumed to be closed under logical consequence.

To address the issue of improbable knowledge within our probabilistic epistemic framework that incorporates awareness, we introduce an additional assumption concerning the agents' awareness functions. Specifically, we assume that agents' awareness is monotonic with respect to their epistemic accessibility relations.

Definition 13

Let $\mathcal{M} = (W, \{R_a : a \in Ag\}, \{\mathcal{A}_a : a \in Ag\}, P_{prior}, v)$ be a probabilistic epistemic model with the awareness function, and let $a \in Ag$. The awareness function $\mathcal{A}_a$ is monotonic if for all $s, t \in W$:

if $(s,t) \in R_a$, then $\mathcal{A}_a(s) \subseteq \mathcal{A}_a(t)$.

The monotonicity assumption is crucial for blocking the occurrence of improbable knowledge. Without this assumption, it is possible for an agent to be aware of a proposition in the current state but unaware of it in some accessible states, leading to situations where the evidential probability of $E_a\varphi$ is less than 1, despite the agent actually knowing φ.

With the monotonicity of awareness established, we can formalize the relationship between actual knowledge and evidential probability in the following theorem.

Theorem 2

Let $\mathcal{M} = (W, \{R_a : a \in Ag\}, \{\mathcal{A}_a : a \in Ag\}, P_{prior}, v)$ be a reflexive and transitive probabilistic epistemic model with the monotonic awareness function, let $s \in W$, $a \in Ag$, and $\varphi \in \Gamma_{\mathcal{L}_{PEL-A}}$. Then the following equivalence holds:

$\mathcal{M}, s \models E_a\varphi$ iff $\mathcal{M}, s \models Pr_a(E_a\varphi) = 1$.

Proof.

We will prove ($\Longrightarrow$). The proof of the reverse implication follows in a manner entirely analogous to the proof of Theorem 1.

Assume that $\mathcal{M}, s \models E_a\varphi$. In order to prove that $\mathcal{M}, s \models Pr_a(E_a\varphi) = 1$, it suffices to prove $R_a(s) \subseteq [\![E_a\varphi]\!]_{\mathcal{M}}$, as it is known that:

$$P_a^s([\![E_a\varphi]\!]_{\mathcal{M}}) = P_{prior}([\![E_a\varphi]\!]_{\mathcal{M}} | R_a(s)) = \frac{P_{prior}([\![E_a\varphi]\!]_{\mathcal{M}} \cap R_a(s))}{P_{prior}(R_a(s))}.$$

Hence, let $x \in R_a(s)$. To show that $x \in [\![E_a\varphi]\!]_{\mathcal{M}}$, we need to prove that $\mathcal{M}, x \models K_a\varphi$ and $\mathcal{M}, x \models A_a\varphi$.
Let $y \in W$ be such that $(x, y) \in R_a$. Since $(s, x) \in R_a$ and $(x, y) \in R_a$, by the transitivity of the relation R_a, it follows that $(s, y) \in R_a$. Given that $\mathcal{M}, s \models E_a\varphi$ and $(s, y) \in R_a$, we have $\mathcal{M}, y \models \varphi$. Therefore, $\mathcal{M}, x \models K_a\varphi$. Since $\mathcal{M}, s \models E_a\varphi$, we have $\mathcal{M}, s \models A_a\varphi$. Thus, $\varphi \in \mathcal{A}_a(s)$. Since $(s, x) \in R_a$, by the monotonicity of awareness, it follows that $\mathcal{A}_a(s) \subseteq \mathcal{A}_a(x)$. Consequently, given that $\varphi \in \mathcal{A}_a(s)$, we have $\varphi \in \mathcal{A}_a(x)$. Therefore, $\mathcal{M}, x \models A_a\varphi$. Since we have established that $\mathcal{M}, x \models K_a\varphi$ and $\mathcal{M}, x \models A_a\varphi$, it follows that $x \in [\![E_a\varphi]\!]_{\mathcal{M}}$. This concludes the proof that $R_a(s) \subseteq [\![E_a\varphi]\!]_{\mathcal{M}}$. In this case, $[\![E_a\varphi]\!]_{\mathcal{M}} \cap R_a(s) = R_a(s)$. Thus, we finally have:

$$P_a^s([\![E_a\varphi]\!]_{\mathcal{M}}) = \frac{P_{prior}(R_a(s))}{P_{prior}(R_a(s))} = 1.$$

This concludes the proof that $\mathcal{M}, s \models Pr_a(E_a\varphi) = 1$. □

Theorem 2 demonstrates that under the monotonicity assumption, actual knowledge E_a is characterized by maximal evidential probability, thereby avoiding the problem of improbable knowledge.

It should be emphasized that monotonicity of awareness is expressed by the axiom $A_a\varphi \rightarrow K_aA_a\varphi$. However, the assumption of the monotonicity is a reasonable and natural condition that does not lead to logical omniscience or other problematic idealizations in epistemic logic. Monotonicity does not require agents to be aware of all logical consequences of their knowledge or to possess infinite deductive capabilities. The awareness function is not assumed to be closed under logical implication, thereby avoiding the attribution of logical omniscience[3].

The flexibility of this formalism allows for the generation of different logics based on the specific assumptions we impose – whether at the syntactic level, by constructing an appropriate proof system, or at the semantic level, by placing conditions and constraints on the accessibility relation, the awareness function, or the probability distribution function. By introducing restrictions on these components, we can derive logics of varying strength and applicability. This allows us to model, among other things, the degree of rationality the agent should possess and the extent of knowledge the given logic is meant to represent. Moreover, further extensions of the language $\mathcal{L}_{PEL-A}$ are possible, allowing for the modeling of additional epistemic phenomena. In the following section, we introduce the concept of an epistemic update, which plays a crucial role in the process of dynamizing the given logic.

6. Probability, actual knowledge, and model update

Dynamic epistemic logics, starting from the logic of public announcements presented by Plaza (1989) and Gerbrandy and Groeneveld (1997), enable us to model the process of knowledge change. A dynamic epistemic logic for actual knowledge based on public announcements and the awareness framework was proposed in Wójcik (2020); in the present paper, this line of research is extended by incorporating evidential probability. In what follows, we extend the language $\mathcal{L}_{PEL-A}$ by augmenting it with a dynamic operator.

Definition 14

Let Var denote the set of propositional variables, and let Ag denote the set of agents. The language of probabilistic dynamic epistemic logics with the awareness operator $\mathcal{L}_{PDEL-A}$ is defined inductively as follows:

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid \varphi \rightarrow \psi \mid K_a\varphi \mid A_a\varphi \mid [\varphi]\varphi \mid Pr_a(\varphi) = c$$

where $p \in Var$, $a \in Ag$ and $c \in [0, 1]$. The set of all $\mathcal{L}_{PDEL-A}$ formulas is denoted by $\Gamma_{\mathcal{L}_{PDEL-A}}$.

The standard interpretation of the formula $[\varphi]\psi$ is “after a public announcement φ, it holds that ψ”. However, the updated epistemic model can result not only from public announcements but also from other actions, such as observation or positive verification. Depending on the context, the interpretation of the dynamic operator can be broader. For example, it may be understood as “after the observation that φ, it holds that ψ” or “after publicly verifying the truth of φ, it holds that ψ”.

Because the language $\mathcal{L}_{PDEL-A}$ contains the dynamic operator, it enables us to express the phenomenon of knowledge change. The main idea behind modeling updated knowledge is that whenever an agent receives new information, all epistemic states that contradict this information are removed from the epistemic model representing the agent’s knowledge prior to receiving the new information. Therefore, the new information causes a transition from an initial epistemic model to its sub-model bounded by this new information.

Definition 15

Let $\mathcal{M} = (W, \{R_a : a \in Ag\}, \{\mathcal{A}_a : a \in Ag\}, P_{prior}, v)$ be a probabilistic epistemic model with the awareness function. The updated model by the new information φ is defined as a tuple $\mathcal{M}|\varphi = (W', \{R'_a : a \in Ag\}, \{\mathcal{A}'_a : a \in Ag\}, P'_{prior}, v')$, where

- $W' = [\![\varphi]\!]_{\mathcal{M}}$,
- $R'_i = R_i \cap (W' \times W')$, for any $i \in Ag$,
- $\mathcal{A}'_i = \mathcal{A}|_{W'}$, for any $i \in Ag$,
- P'_{prior} is defined as

$$P'_{prior}(A) = \frac{|A|}{|W'|}, \quad \text{for any } A \subseteq W',$$

- $v'(p) = v(p) \cap W'$, for any $p \in Var$.

Definition 16

Let $\mathcal{M} = (W, \{R_a : a \in Ag\}, \{\mathcal{A}_a : a \in Ag\}, P_{prior}, v)$ be a probabilistic epistemic model with the awareness function, and let $s \in W$. Then, Definition 12 is extended with the following condition:

$\mathcal{M}, s \models [\varphi]\psi$ iff if $\mathcal{M}, s \models \varphi$, then $\mathcal{M}|\varphi, s \models \psi$,

where $\varphi, \psi \in \Gamma_{\mathcal{L}_{PDEL-A}}$, $a \in Ag$, and $\mathcal{M}|\varphi$ is a model $\mathcal{M}$ restricted to all those epistemic states where φ holds.

In defining the updated prior probability distribution P'_{prior}, we cannot simply set it as the restriction of the original prior probability P_{prior} to W'. This is because P_{prior} is a probability measure over the original set of states W, and its restriction to W' does not sum up to 1 over W', unless $W' = W$. This would violate the requirement that a probability distribution over W' must have a total probability of 1.

Static and dynamic epistemic logics take a qualitative perspective on information, whereas probability theory adopts a quantitative perspective on information and utilizes numerical probability functions. It is important to note that both dynamic epistemic logic and probability theory study the phenomenon of information change. Probability theory typically employs Bayesian updating to represent this change, whereas dynamic epistemic logic models a transition from an initial epistemic model to its submodel, which is constrained by the new information. The main difference is that dynamic epistemic logic concerns itself not only with knowledge about the world but also with knowledge about knowledge, or higher-order information, whereas probability theory does not. As noted by Demey and Kooi (2014), this distinction makes the project of probabilistic dynamic epistemic logic particularly interesting. Such systems inherit the detailed perspective on information from probability theory while integrating the representation of higher-order information change from dynamic epistemic logic. Although there are numerous works in this field (see Van Benthem, 2003; van Benthem, Gerbrandy & Kooi, 2009; Demey & Kooi, 2014; Kooi, 2003; Sack, 2009), developing dynamic epistemic logic for non-omniscient agents that incorporates probability remains an open question. However, the framework outlined above brings us closer to achieving this goal.

7. Summary

This paper introduces a probabilistic extension to the modal epistemic logics. We apply this formalism to present an example of improbable knowledge. The existence of such knowledge is paradoxical in nature and poses significant challenges for probabilistic accounts of justification.

The key takeaway from our analysis is the recognition that the problem arises from assumptions about the non-transitivity of the accessibility relation, leading to the rejection of Axiom 4. By redefining the epistemic operators, we demonstrate that accepting Axiom 4 is feasible, preventing improbable knowledge and enhancing the accuracy of modeling agents' actual knowledge.

We further explore the distinction between potential and actual knowledge as a solution to the problem of logical omniscience. This distinction is fundamental to constructing a system of probabilistic epistemic logic that accurately represents actual knowledge for agents with limited deductive abilities.

Finally, we introduce key semantic concepts for a dynamic probabilistic epistemic logic designed for non-omniscient agents. This formalism provides a promising foundation for studying the dynamics of knowledge and its relationship with probability, offering a more realistic framework for modeling the epistemic states of agents with bounded cognitive abilities.

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NOTES

[1] A more detailed analysis of the problems associated with probabilistic accounts of justification falls outside the scope of this paper. For a thorough discussion, see Backes (2023).

[2] Moore (1942) was the first to identify the paradoxical nature of sentences like ‘It is true that p, but I don’t believe that p’, which may be true but would be absurd for any subject to utter in the first person.

[3] For a more detailed discussion on the monotonicity of the awareness function, see Thijsse (1993).

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