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THEORETICAL BASE FOR DETERMINING THE MODULUS OF ELASTICITY OF THIN PLATE COMPRESSED BY CONCENTRATED FORCE

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Abstract: An analytical solution to the plane problem in elasticity theory is presented for the case of a concentrated transverse force acting on a thin, sufficiently wide plate. A functional relationship is obtained which allow to determine the displacements of points on the plate located along the line of force application depending on the distance from the point of force application. Using an example, by comparing analytical and numerical calculations of the displacement of a specific point, it is shown that the formula derived from the established relationship provides quite accurate results. The proposed formula is recommended for determining the first-order elastic modulus based on measured force and displacement parameters.

KEYWORDS: thin plate, compression, stress, deformation, displacement, boundary conditions, modulus of elasticity, bone tissue, gunshot fracture, biomechanics.

1 Introduction

Rehabilitative treatment for individuals with gunshot fractures is a critical medical and socio-economic priority for the state during military operations, as its effectiveness directly impacts the risk of disability and long-term loss of working capacity among the injured. To improve treatment outcomes, scientific research is needed aimed at enhancing existing stabilizing structures and developing innovative ones, which will form the foundation of a comprehensive treatment system.

It is known [1] that due to trauma caused by a bullet impact on bone, the physical-mechanical and some other properties of the cortical layer of bone tissue adjacent to the site of injury undergo changes. For example, in [2], studies on the mechanical characteristics of the human tibial bone after a gunshot wound revealed that the elastic modulus increases with distance from the injury site, eventually reaching values typical for intact bone tissue at a certain distance. Thus, the elastic modulus can serve as an accurate indicator for determining the size of the injury zone, which is crucial for selecting appropriate fixation methods for subsequent treatment. In addition to clinical indicators, osteosynthesis systems must meet specific strength requirements. Enhancing the stiffness and stability of bone fixation in fractures reduces the likelihood of complications, optimizes the bone tissue regeneration process, positively impacts treatment outcomes, and significantly shortens its duration [3].

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The primary challenge in determining the elastic modulus of the cortical layer is caused by the difficulty of standard specimens preparing and carrying out of standardized tensile or compression tests, which are typically used to evaluate the mechanical properties of metallic and composite materials as well as their welded joints [3–6]. Therefore, in [7], a testing method was proposed that involves transmitting compressive loads to the specimen through an indenter embedded into the investigated bone layer. The applied force and the indenter's displacement are measured within the test. It is known [8] that under the loading in elastic range, the magnitude of deformations and displacements depends on the elastic modulus. Thus, the aim of this study is to establish a functional relationship between the elastic modulus and displacements under the action of a concentrated force applied to a thin elastic layer.

2 Task Description

Let us consider a sufficiently wide plate with a thickness of δ, where the lower surface is rigidly fixed, and a uniformly distributed vertical force P/δ is applied to the upper surface through a perfectly rigid indenter of diameter d (see Fig. 1). The task is to determine the vertical displacement of the point where the force is applied.

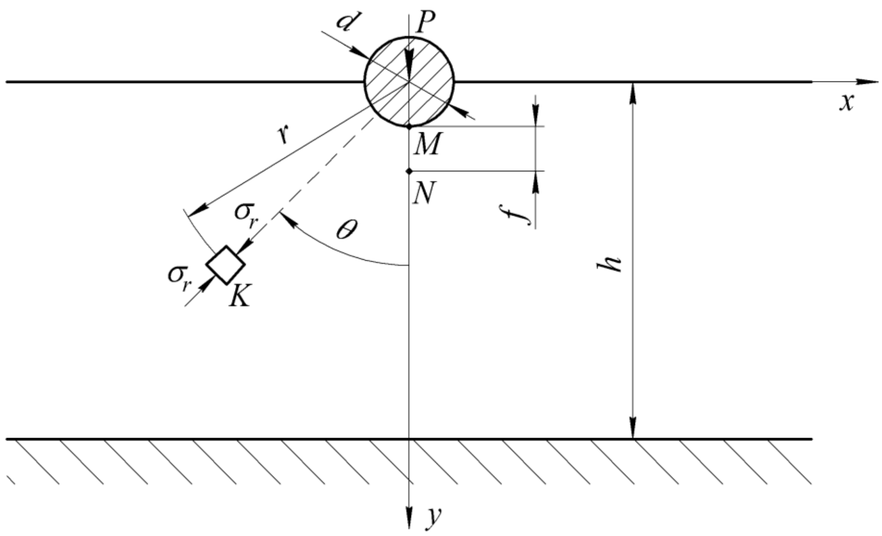


Fig. 1 Calculation scheme for the fixation and loading of the plate with a vertical force

3 Theoretical Studies

To obtain the analytical solution of the problem, the stress function for a typical radial stress distribution, is used for a plate of unit thickness in the form [9]

$$\varphi = -\frac{P}{\pi} r \cdot \theta \sin\theta. \tag{1}$$

The stress components through the function φ in polar coordinates are determined by the following formulas [10]

$$\sigma_r = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2};\ \sigma_\theta = \frac{\partial^2\varphi}{\partial r^2};\ \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right). \tag{2}$$

Substituting the function (1) into Eq. (2), we obtain

$$\sigma_r = -\frac{2P}{\pi r}\cos\theta;\ \sigma_\theta = 0;\ \tau_{r\theta} = 0. \tag{3}$$

By analyzing the dependencies (3), it can be concluded that any element K, located at a distance r from the point of force application at an angle θ to the direction of its action, is in the state of uniaxial compression in the radial direction.

The equations of Hooke's law for the plane stress state are as follows [11]

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \mu\sigma_\theta)\,;\;\; \varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \mu\sigma_r)\,;\;\; \gamma_{r\theta} = \frac{\tau_{r\theta}}{G}, \tag{4}$$

where E is the modulus of elasticity, G is the shear modulus, and μ is the Poisson's ratio.

Substituting the stress components (3) into the equations of Hooke's law (4), we obtain the strain components

$$\varepsilon_r = -\frac{2P}{\pi r \cdot E} \cos\theta\,;\;\; \varepsilon_\theta = \frac{2\mu \cdot P}{\pi r \cdot E} \cos\theta\,;\;\; \gamma_{r\theta} = 0. \tag{5}$$

On the other hand, the strain components in polar coordinates can be expressed through the displacement components using the relationships [12]

$$\varepsilon_r = \frac{\partial u}{\partial r};\;\; \varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta};\;\; \gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \tag{6}$$

where u and v are the radial and tangential displacements, respectively.

Equating ε_r from Eqs. (5) and (6), after integration, we obtain

$$u = -\frac{2P \cdot \ln r}{\pi E} \cos\theta + f(\theta), \tag{7}$$

where $f(\theta)$ is a function of the coordinate θ only.

Substituting Eq. (7) and the expression for ε_θ from Eq. (5) into the expression for ε_θ from Eq. (6), after transformations and integration, we obtain

$$v = \frac{2P \cdot \ln r}{\pi E} \sin\theta + \frac{2\mu \cdot P}{\pi E} \sin\theta - \int f(\theta)\, d\theta + F(r), \tag{8}$$

where $F(r)$ is a function of the coordinate r only.

Substituting Eqs. (7) and (8) into the third relation (6), considering that $\gamma_{r\theta} = 0$, we obtain the integral-differential equation for determining the unknown functions $f(\theta)$ and $F(r)$

$$\frac{2(1-\mu)P}{\pi E} \sin\theta + r\frac{dF}{dr} + \int f(\theta) d\theta - F(r) + \frac{df(\theta)}{d\theta} = 0. \tag{9}$$

By differentiating Eq. (9) once on the variable r, we obtain a homogeneous second-order differential equation with constant coefficients

$$r\frac{d^2F(r)}{dr^2} = 0.$$

The solution can be written in the form [13] as follows

$$F(r) = C \cdot r + D, \tag{10}$$

where C and D are the constants of integration, which are determined from the boundary conditions.

At the same time, the constant D represents the rigid displacement of the body as a whole, so it can be assumed to be equal to zero [14].

By differentiating Eq. (9) once on the variable θ, we obtain a non-homogeneous second-order differential equation

$$\frac{d^2 f(\theta)}{d\theta^2} + f(\theta) = -\frac{2(1-\mu)P}{\pi E} \cos\theta.$$

The solution according to [15] has the form

$$f(\theta) = -\frac{(1-\mu)P \cdot \theta}{\pi E} \sin\theta + A \cdot \sin\theta + B \cdot \cos\theta, \tag{11}$$

where A and B are the constants of integration, which are determined from the boundary conditions.

Substituting Eq. (11) into Eq. (7), we obtain the general expression for the radial displacement

$$u = -\frac{2P \cdot \ln r}{\pi E} \cos\theta - \frac{(1-\mu)P \cdot \theta}{\pi E} \sin\theta + A \cdot \sin\theta + B \cdot \cos\theta, \tag{12}$$

while substituting Eqs. (10) and (11) into Eq. (8), we obtain the general expression for the tangential displacement

$$v = \frac{2P \cdot \ln r}{\pi E} \sin\theta + \frac{2\mu \cdot P}{\pi E} \sin\theta - \frac{(1-\mu)P \cdot \theta}{\pi E} \cos\theta + \frac{(1-\mu)P}{\pi E} \sin\theta + A \cdot \cos\theta - B \cdot \sin\theta + \\ + C \cdot r. \tag{13}$$

Since points located on the y-axis, displace vertically downward under the action of the vertical force P, and they do not displace horizontally, we can state that $v = 0$ at $\theta = 0$. Then, according to Eq. (13), the constants A and C must be equal zero. Therefore, the vertical displacements of the points on this axis will be described by the following formula

$$u|_{\theta=0} = -\frac{2P \cdot \ln r}{\pi E} + B. \tag{14}$$

The constant B is determined from the condition that any displacements of points at the boundary are impossible

$$u|_{\substack{\theta=0 \\ r=h}} = -\frac{2P \cdot \ln h}{\pi E} + B = 0.$$

This allows us to determine it

$$B = \frac{2P \cdot \ln h}{\pi E}. \tag{15}$$

Substituting Eq. (15) into Eq. (14), we obtain the final formula for the vertical displacements of points located on the y-axis

$$u|_{\theta=0} = \frac{2P}{\pi E} \ln\frac{h}{r}. \tag{16}$$

By considering the Eqs. (3) and (16), we can observe that the stress and displacement at the point of application of the force ($r = 0$) tend to infinity, which is physically impossible. This is due to the assumption that the entire applied force is concentrated on an infinitely small area. In reality, however, the load transfer to the specimen occurs through an indenter, which has finite dimensions.

Since the indenter is absolutely rigid with respect to the plate, the vertical displacements of all its points are the same. Therefore, to determine the displacement of the point of application of the load, we will find the vertical displacement of point M, which is also located on the y-axis at a distance of $d/2$ from the center of the indenter cross-section, into position N (Fig. 1)

$$u|_{\substack{\theta=0 \\ r=\frac{d}{2}}} = \frac{2P}{\pi E} ln \frac{2h}{d}.$$

Considering that this solution was obtained for a plate of unit thickness, taking into account the actual thickness of the studied plate δ, the vertical displacement of the indenter f will be determined by the formula

$$f = \frac{2P}{\pi E \cdot \delta} ln \frac{2h}{d}. \tag{17}$$

4 Numerical calculations

Let us consider a plate made of cortical bone layer, for which $E = 17GPa$, $\mu = 0.3$ [16]. Let the diameter of the indenter be $d = 5mm$, the distance from the center of the indenter cross-section to the fixed edge of the plate be $h = 40mm$, the applied load be $P = 60N$. The thickness of the cortical bone layer is typically $\delta = 3mm$ [17]. With the given dimensions and load, the vertical displacement of the indenter, calculated using formula (17), is $f = 2.077 \cdot 10^{-6}m$.

Since the load in the finite element model must be applied as distributed over a certain area, to reproduce the conditions of the considered problem, a semi-circular cut with a diameter of $2\rho = 1mm$ (Fig. 2) was made in the plate model at the point of the force application.

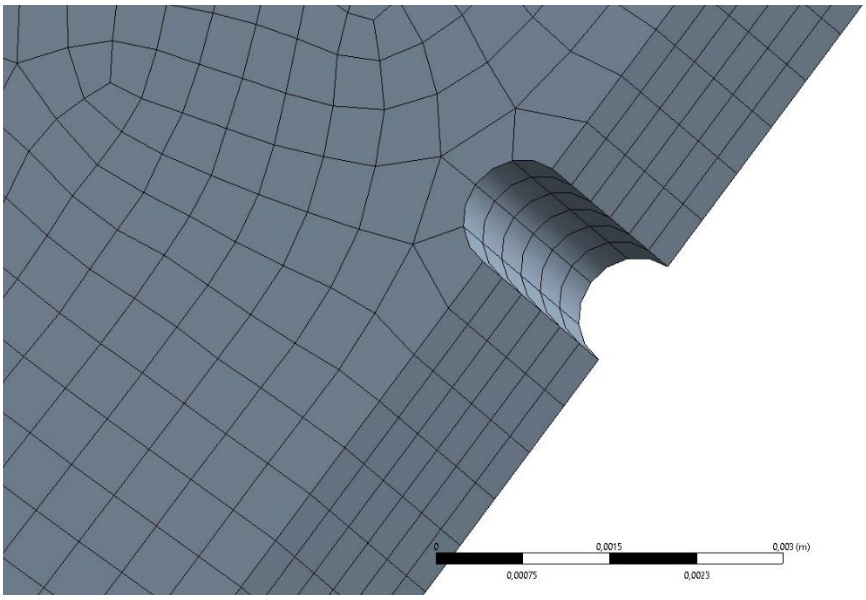


Fig. 2 The cut in the plate model to simulate the application of the force distributed over the thickness

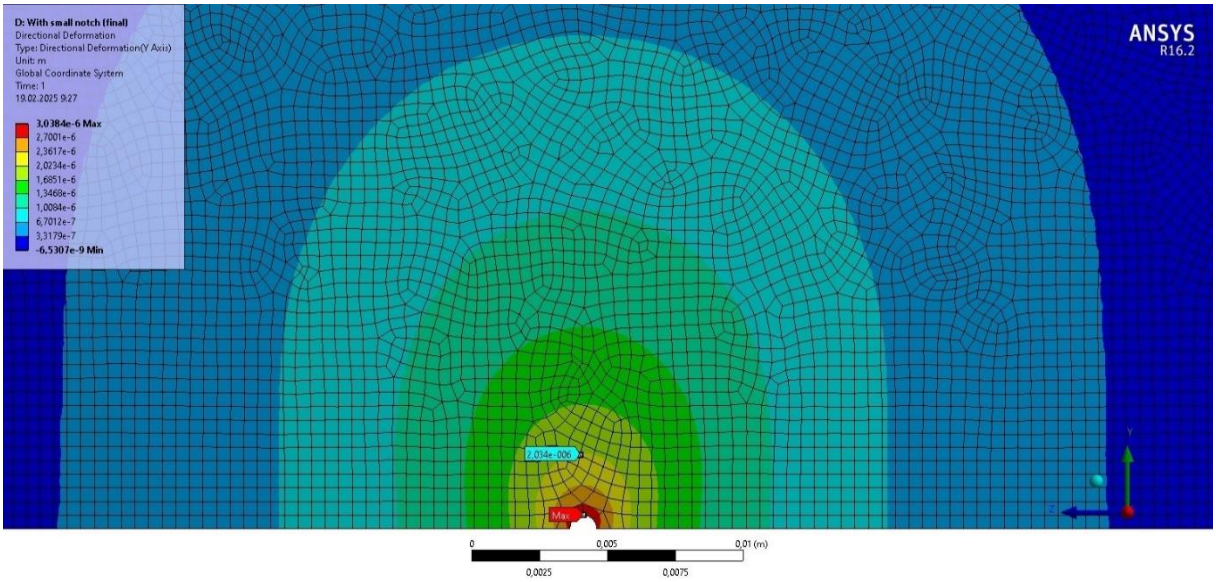
This cut was positioned so that the distance from its lower point to the fixed edge of the plate is $h = 40mm$. The pressure p, uniformly distributed over the formed cylindrical surface, will be equivalent to the application of a concentrated force P if the integral sum of its vertical components acting on each element $\delta \cdot \rho d\theta$ of this surface satisfies the equilibrium condition

$$P = 2 \int_0^{\frac{\pi}{2}} p \cdot \delta \cdot \rho \cos \theta \, d\theta.$$

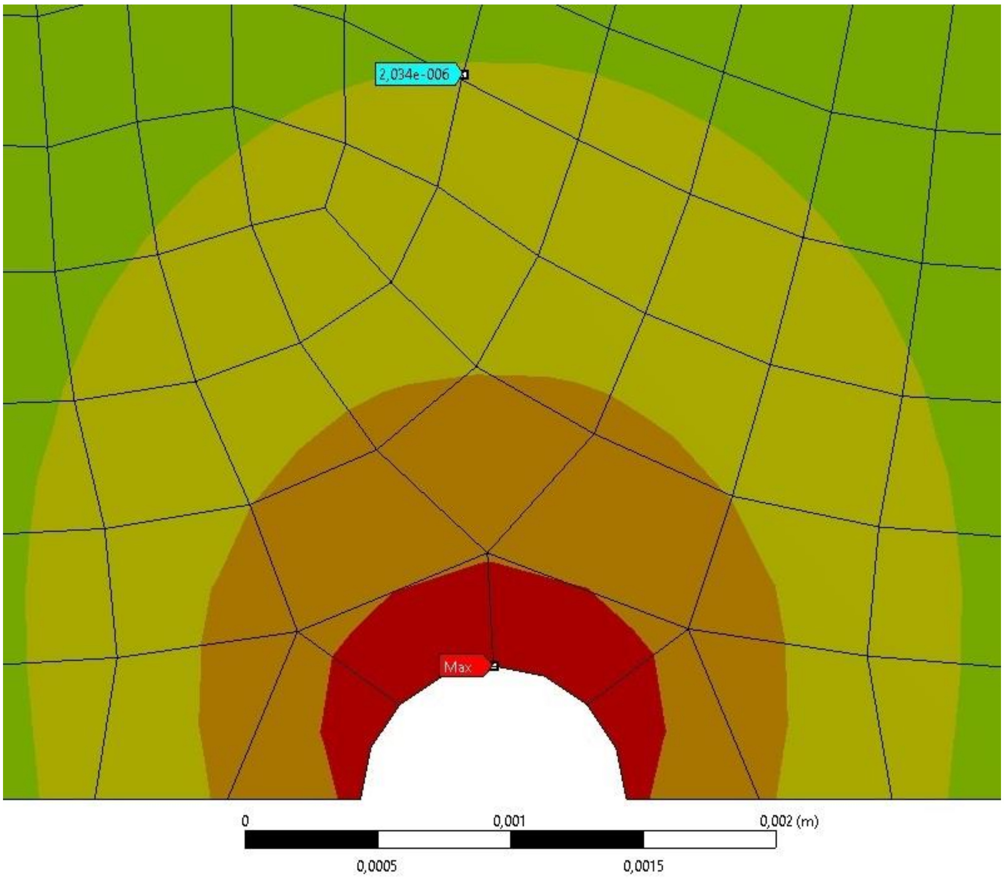
After integrating, we will obtain

$$p = \frac{P}{2\rho \cdot \delta}. \tag{18}$$

Thus, according to formula (18), the application of the concentrated load $P = 60N$ will be equivalent to the application of the pressure $p = 20MPa$. In the calculations using the finite element method, the geometric model consisted of 80304 elements with 365941 nodes was used. It was determined that the vertical displacement of the node located at the distance of $d/2 = 2.5mm$ from the lower point of the cut along the y-axis is $2.034 \cdot 10^{-6}m$ (Fig. 3).



a



b

Fig. 3 Calculation results: a) the fields of vertical displacements in the studied plate; b) the value of displacement at the distance of $d/2 = 2.5mm$ from the lower point of the cut

Thus, the difference between the results of the analytical and numerical calculations was slightly more than 2%. This is explained by the fact that the position of the node, where the displacement was fixed in the model, depends on the automatic mesh division, so its distance from the bottom of the cut cannot exactly be $2.5mm$, and slight deviations from the y-axis are also possible. Therefore, formula (17) establishes a relationship between the modulus of normal elasticity of the plate and the vertical displacement of the indenter with high accuracy. Hence, for determining the modulus of elasticity based on experimental data for the applied force and indenter displacement, it is recommended to use its value calculated from formula (17)

$$E = \frac{2P}{\pi\delta \cdot f} ln \frac{2h}{d}. \tag{19}$$

CONCLUSIONS

Using the methods of elasticity theory, a plane problem of the compression of a thin, wide plate by a force transmitted through an absolutely rigid indenter is solved. Analytical expressions for stresses and strains determining at all points of the plate is obtained. The functional relationship between the displacements of points along the force application line and the distance from the point of its application is established under certain boundary conditions corresponding to the clamping and loading conditions of the samples defined by the experimental procedure. The formula for determining the indenter displacement depending on the applied load is substantiated. The following conclusions can be made:

1. The results obtained using the proposed formula closely match the numerical calculations performed with the finite element method, with displacement values differing by slightly more than 2%. The discrepancy arises because the finite element model cannot precisely pinpoint the location where the displacement was determined analytically.

2. The proposed formula incorporates the material's normal elasticity modulus, making it suitable for determining this modulus from experimental data on the applied force and indenter displacement.

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