

SOME MATHEMATICS IN THE GAME POLYGON

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1 Background on *Polygon*

Board games inspire friendly competition and promote the ability to strategize on the spot. Additionally, the study of board games is intertwined with the study of mathematics, and numerous research projects have endeavored to explore that connection. Some examples include the applications of graph theory, combinatorics, and computer science to games such as *Blokus*, *Catan*, Nim, and *Hanabi* [[Bou01], [CSD+15], [AKM19], [AC22]]. Yet again, there are contexts like the game of Sylver coinage, the game of chomp, and the postage stamp problem that mirror the formation and properties of mathematical structures such as posets, monoids, and numerical semigroups [[Guy87], [Lun69], [GK18], [EH23]].

Polygon is a strategy board game previously manufactured by Jax, Ltd. starting in 1998. Since then, the game has fallen out of production, and the only copies that are available for purchase appear to be from e-commerce vendors. Despite its obscurity, *Polygon* is an interesting game to examine through the lens of mathematics. Combinatorics, probability, and algebraic structures play key roles in gameplay and strategy.

2 Gameplay

Polygon is played on a hexagonal board subdivided into smaller hexagons with 2-4 players placing 49 hexagonal tiles on the board, with the ultimate objective of obtaining the highest total score at the game's end. The values and counts for each type of tile are listed in Table 1. The game begins with players mixing up and laying out all 49 hexagonal tiles face down so that their values are hidden. Each player draws a tile and places it face up on the board. The player with the highest-value tile starts the game, where no distinction is made between regular tiles and multiplier tiles (i.e., those containing a “ $\times$ ”), and Wild tiles (which we label here as “W”) are considered to have the highest value.

The player with the highest-value tile places it face-up on the space marked **START** on the game board, located directly in the center of the board. A black-and-white version of the game board is shown in Figure 1. For readability, we have not labeled the **START** tile as such, and we display all board spaces and tiles so that the text is right-side-up.

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Face Value of Tile	Tally
1	10
2	8
3	6
4	7
5	4
1×4	3
2×3	3
3×2	3
W	5

Table 1: An Enumeration of the Values of the 49 *Polygon* Tiles.

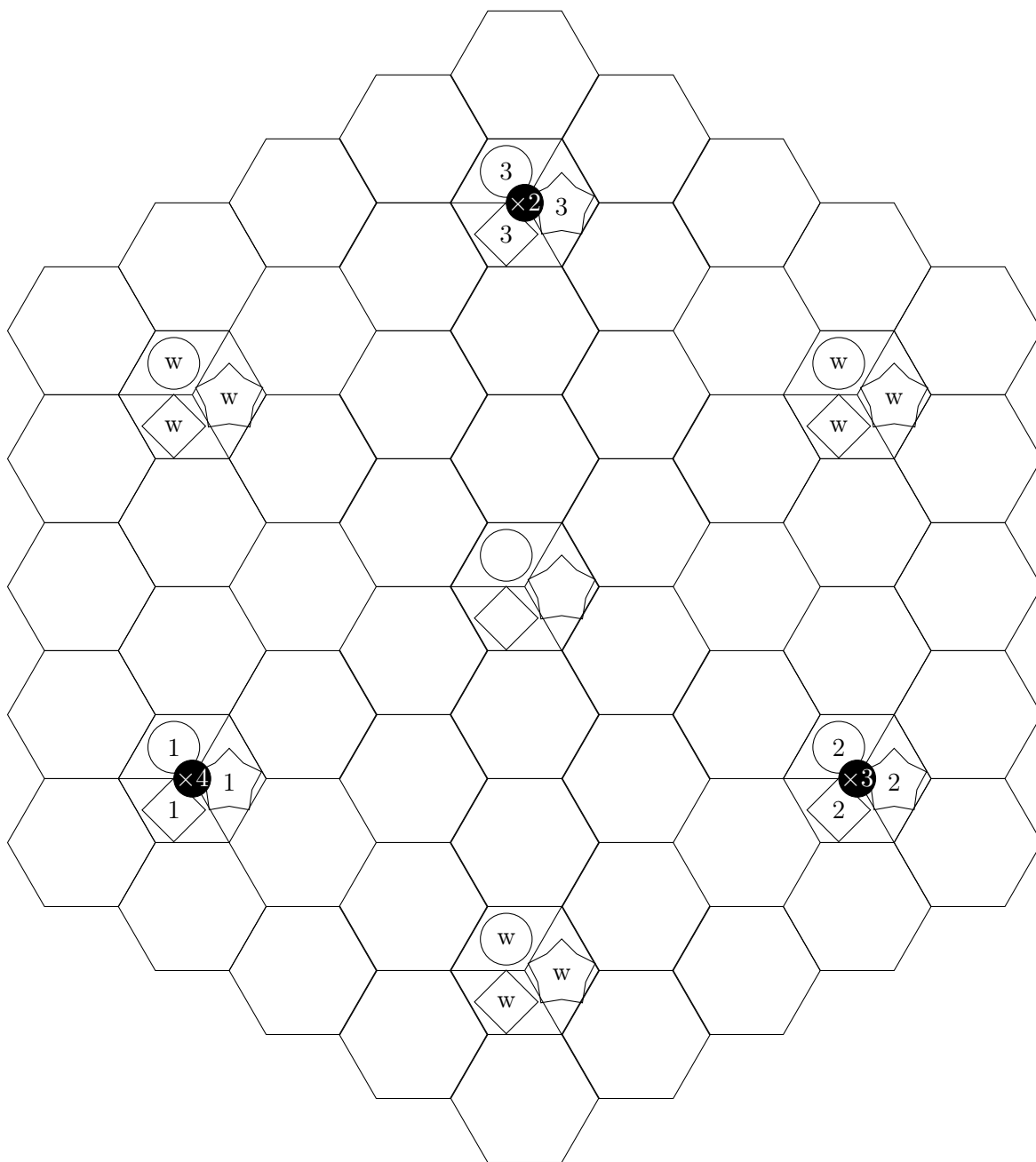
Each hexagonal tile is broken into three color/shape sections: blue/circle, red/diamond, and yellow/star. Black-and-white versions of the tiles can be found in Figure 2. The initial tile is positioned on the board so that colors and shapes match up: blue/circle on blue/circle, and so on. If there is a tie amongst players when drawing their initial tiles, each player continues picking new tiles until the tie is broken. After the initial tile is placed, the remaining players' tiles are mixed back in with the other face-down tiles, and play progresses outward from the initial tile.

After the initial tile is placed, player 1 rolls a six-sided die with sides and colors corresponding to those on the game tiles: two sides each for blue/circle, red/diamond, and yellow/star. Player 1 selects two additional tiles and places them edge-adjacent to (i.e., touching) the already-placed starting tile while also matching edges according to the color/shape rolled on the die. The possible scoring combinations are shown in Figure 3. Including the use of multiplier tiles, the possible one-turn scores are 0, 10, 15, 20, 25, 30, 40, 45, 50, 60, 70, 75, and 100.

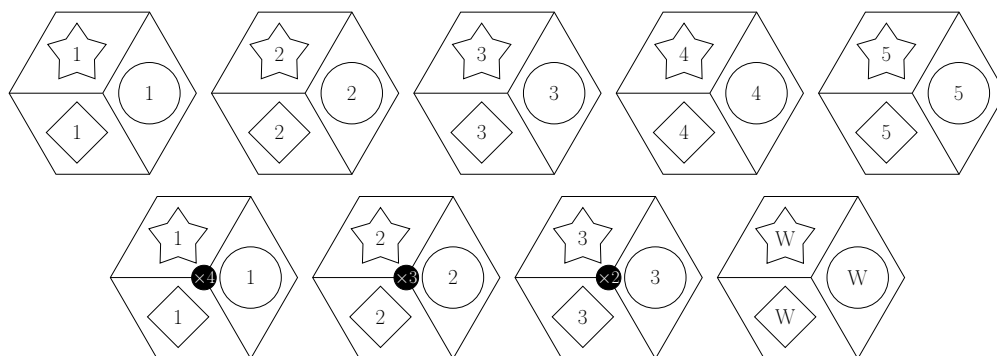
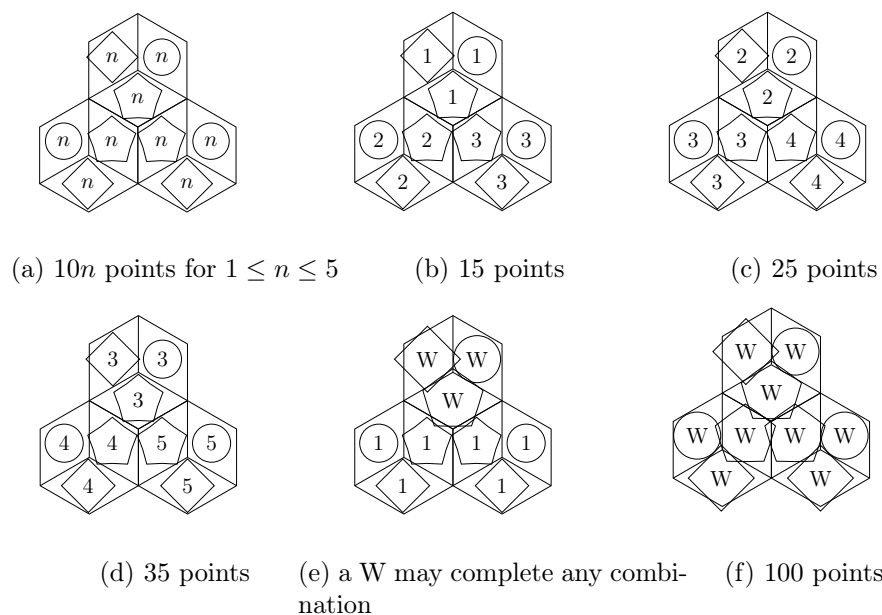
Player 2 must roll the game die until obtaining a color/shape different from player 1, as otherwise player 2's tiles will not be playable anywhere on the board. After rolling, player 2 also chooses two tiles to combine with at least one of the tiles already in play to create the best score possible. It may take two tiles to create a score, or a player may create two scores with their two tiles. However, a player is not permitted to take credit for two scores on a turn using the same grouping of tiles. Play proceeds like this for each additional player.

A few additional rules and features of the game should also be noted. A player may stack newly drawn tiles on tiles already in play in order to score. However, a player is not allowed to stack a tile on a tile with the same number. W tiles may count as any number 1 through 5, and a player may stack a W on any tile except another W. When a player creates a score with a multiplier tile, their score is multiplied by the number following the $\times$ on that tile. If more than one multiplier tile is used in a single tile grouping, the multiplier with the highest number following the $\times$ is applied to the score.

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Figure 1: The *Polygon* Game Board.

As play approaches the edges of the board, the tiles in play will reach the W and multiplier tiles printed on the board itself (as shown in Figure 1). These printed tiles create a score when a tile is placed next to a tile in play that is simultaneously next to a printed tile. A player may also stack tiles on printed tiles (following the usual rules of stacking).

Figure 2: The Nine Types of *Polygon* Tiles.Figure 3: Scoring Combinations in *Polygon*.

The game ends when the collection of face-down tiles is exhausted. Players tally the scores from all of their turns, and the player with the highest grand total wins the game. Each player will have a predetermined number of turns because, after the placement of the first tile, the 48 remaining tiles will be divided evenly amongst the players.

Number of Players	Number of Tiles per Player	Number of Turns per Player
2	24	12
3	16	8
4	12	6

Table 2: Tally of Players, Tiles per Players, and Turns per Player.

In this paper, we aim to explore mathematical topics such as combinatorics, probability, and algebra in relation to the intriguing facets of this game. To this end, we count all possible ending board configurations and examine the probability of certain events at the outset of a game. Finally, we discuss the ending scores possible for finitely many turns and generalize to a game with infinitely many turns. Appendices contain tables that supplement some of the aforementioned discussions. Additional details can be found in the appendices.

3 Ending Board Configurations

As mentioned, there are 49 tiles that players place on the board during gameplay. If we disallow stacking tiles, counting all the hexagons on the game board, one obtains the number 58; however, six of those hexagons are actually valued tiles printed on the board, leaving only 52 blank hexagons to fill during the gameplay. It is also always guaranteed that the **START** space is covered by a playable tile and that tile cannot have a value of 1. So, there are $\binom{36}{1} = 36$ ways to select that particular tile. Next, there are $\binom{51}{48} = 20,825$ ways to cover the board with the remaining 48 tiles. Each tile can only be oriented one way on the game board according to how the circle, star, and diamond for each respective hexagon are printed. So, the only other factor in the construction of an ending game board is to select where each type of tile will be placed on the board. This depends on which type was put on the **START** hexagon. This is calculated by:

$$\frac{48!}{10!7!7!6!5!4!3!3!3!} + \frac{48!}{10!8!6!6!5!4!3!3!3!} + \frac{48!}{10!8!7!5!5!4!3!3!3!} + \frac{48!}{10!8!7!6!4!4!3!3!3!} + \frac{48!}{10!8!7!6!5!3!3!3!3!} + \frac{48!}{10!8!7!6!5!4!2!3!3!} + \frac{48!}{10!8!7!6!5!4!3!2!3!} = 1.35306438 \times 10^{39}.$$
 It should also be noted that the game board itself can be oriented in 6 different ways corresponding to the sides of the hexagonal shape of the board itself. So, there are

$$36 \cdot 20,825 \cdot 6 \cdot (1.35306438 \times 10^{39}) = 6.08635419 \times 10^{45}$$

possible ending game board configurations if no stacking is allowed. If stacking is allowed, counting the ending board configurations becomes much more complicated. This amounts to the following:

1. Select the tile that will cover **START**;
2. Select n , where $5 \leq n \leq 48$ (the first player will need to use 3 hexagons on the board and the second player will need to use 2 more since they must roll a different color/shape than the first player) of the remaining 51 hexagons to be covered;
3. Select n of the remaining 48 tiles to cover the n chosen hexagons;
4. Select m , where $1 \leq m \leq n$ of the tiles on the board on which to stack;
5. Select the heights [starting from the second tile and above] $(m_1, \dots, m_m)$ of each of the m stacks where $1 \leq m_i \leq 48 - n$ ($48 - n$ is the upper bound in the event that each player gets the same color on each subsequent roll after the first and second players first turns) and $m_1 + \dots + m_m = 48 - n$;

6. Select the $m_1, \dots, m_m$ tiles from the 48 remaining tiles.

We leave this as a project to the reader to fill in the details.

4 Chance

There is a bit of chance involved in playing *Polygon*, in the form of (1) rolling the game die and (2) drawing tiles. The perhaps more interesting applications of probability appear in drawing tiles since previous draws affect subsequent ones. After the placement of the initial tile to start the game, tile drawing becomes much more complicated, as not only is each player's draw affected by their first draw but also by the draws of the other players. Specific probability explorations are relevant for thinking about early gameplay: the probability of a tie when choosing player 1 and the probabilities of obtaining each achievable score for player 1 on their first turn.

4.1 Probability there is a tie for choosing player 1

We start by considering a 2-player game. If both players choose a tile of the same value, they must replace the tiles and start over. For choosing player 1, since multiplier and regular tiles function in the same way, we may say there are thirteen 1 tiles, eleven 2 tiles, nine 3 tiles, seven 4 tiles, four 5 tiles, and five W tiles. Then, the probability of a tie (which we could also express using binomial coefficients) is

$$\frac{13 \cdot 12}{49 \cdot 48} + \frac{11 \cdot 10}{49 \cdot 48} + \frac{9 \cdot 8}{49 \cdot 48} + \frac{7 \cdot 6}{49 \cdot 48} + \frac{4 \cdot 3}{49 \cdot 48} + \frac{5 \cdot 4}{49 \cdot 48} = \frac{412}{2352} \approx 0.175$$

Similarly, for a 3-player game, the probability of a 3-way tie is

$$\frac{\binom{13}{3}}{\binom{49}{3}} + \frac{\binom{11}{3}}{\binom{49}{3}} + \frac{\binom{9}{3}}{\binom{49}{3}} + \frac{\binom{7}{3}}{\binom{49}{3}} + \frac{\binom{4}{3}}{\binom{49}{3}} + \frac{\binom{5}{3}}{\binom{49}{3}} = \frac{73}{2303} \approx 0.032$$

In a 3-player game, there could also be a tie for starting player between only two players. This case requires more care since the non-tying player must have a tile with a lower value. One way to handle this is to use cases based on what the highest-value drawn tile is. The possible values for the highest value are 2, 3, 4, 5, or W, which leads to

$$P(\text{2-way Tie}) = \frac{\binom{11}{2}\binom{13}{1}}{\binom{49}{3}} + \frac{\binom{9}{2}\binom{24}{1}}{\binom{49}{3}} + \frac{\binom{7}{2}\binom{33}{1}}{\binom{49}{3}} + \frac{\binom{4}{2}\binom{40}{1}}{\binom{49}{3}} + \frac{\binom{5}{2}\binom{44}{1}}{\binom{49}{3}} = \frac{369}{2303} \approx 0.16$$

Then, the overall probability of a tie for a starting player in a 3-player game is $\frac{442}{2303}$ or approximately 0.19. Similarly, for a 4-player game, we have three possibilities: a 4-way tie, a 3-way tie, or a 2-way tie. We leave calculating the probabilities for each of these to the reader; the overall probability of a tie in this case is $\frac{23291}{105938}$ or approximately 0.22.

Since players would likely become bored breaking ties for player 1 over and over, it is encouraging that all of these probabilities are relatively low. Considering the number of tries needed as a geometric random variable, the expected number of tries it will take

to establish player 1 is $\frac{1}{1-p}$, where p is the probability of a tie on any one attempt. This expected value is largest in the 4-player case but is still only 1.28 tries, suggesting that *Polygon's* mechanic for choosing player 1 is a good one.

4.2 Probability that player 1 scores zero points on turn one

A perhaps more interesting consideration is the probability that player 1 scores zero points on turn 1. We could approach this situation by considering cases, such as the case that the initial tile is W. Since we are concerned with counting tile groupings that score 0 points, it again suffices to treat the multiplier tiles as regular number tiles. The reader may verify that only three pairs of subsequent tiles give a score of 0 when the initial tile is W: 1 and 4, 1 and 5, and 2 and 5. So, given that the initial tile is W, the probability of scoring zero points on move 1 is

$$P(1, 4) + P(1, 5) + P(2, 5) = \frac{\binom{13}{1}\binom{7}{1}}{\binom{48}{2}} + \frac{\binom{13}{1}\binom{4}{1}}{\binom{48}{2}} + \frac{\binom{11}{1}\binom{4}{1}}{\binom{48}{2}} = \frac{187}{1128} \approx 0.166$$

We can proceed in this way to calculate the probability that no points are scored on move 1 given each other possible value of the initial tile (2, 3, 4, or 5). Since ties for player 1 aren't allowed, the initial tile cannot have a value of 1. As another example, consider the case when the initial tile is 2. This time, we will start by counting the tile groupings that score points. To score points when the initial tile is 2, the other two tiles must be of the form 22, 13, 34, or Wn , where $n \neq 5$. Thus, the probability of not scoring points in this case is

$$1 - \left(\frac{\binom{10}{2}}{\binom{48}{2}} + \frac{\binom{13}{1}\binom{9}{1}}{\binom{48}{2}} + \frac{\binom{9}{1}\binom{7}{1}}{\binom{48}{2}} + \frac{\binom{5}{1}\binom{39}{1}}{\binom{48}{2}} \right) = \frac{59}{94} \approx 0.628$$

Using this approach, we can find the overall probability that player 1 scores 0 points on their first turn, being careful not to overcount possibilities we have already accounted for. Alternatively, we may list all possible 3-tile groupings (see the Appendix) and find the probabilities of obtaining each of the 30 groupings that result in a score of zero. We must treat the initial tile separately as, once again, it cannot have a value of 1. As one example, the probability that player 1 obtains a 1, 2, and 4 is

$$\frac{11}{36} \left(\frac{\binom{13}{1}\binom{7}{1}}{\binom{48}{2}} \right) + \frac{7}{36} \left(\frac{\binom{13}{1}\binom{11}{1}}{\binom{48}{2}} \right)$$

where the first term is the probability when the first tile is a 2 and the second the probability when the first tile is a 4. Proceeding in this way, we obtain that the overall probability of scoring zero points on turn one is approximately 0.586. Notice that we do not have to consider the die roll on move one since the first player may place their second and third tiles as needed.

We can similarly calculate the probabilities of every possible nonzero turn-one score, again accounting for the fact that the first tile cannot be a 1. Using the probabilities in Table 3, we may also obtain that the expected number of points for the first player on turn 1 is approximately 21.03 points.

Score	Probability	Score	Probability
15	0.024	45	0.048
20	0.004	50	0.036
25	0.025	60	0.110
30	0.033	70	0.024
35	0.021	75	0.069
40	0.020	100	0.0007

Table 3: The Probabilities of Obtaining Nonzero Scores on Turn One.

5 Ending Scores in *Polygon*

Another question that arises is the following: What are the possible total scores a player can have at the end of the game? Any one-turn score must be a multiple of five, so every total ending score must end in either 0 or 5. Of course, zero is possible, and Table 4 below shows each multiple of five (not including 5) up to 100 is also a possible ending score. Note that the column headings in Table 4 do not actually specify that the given tiling must be created on the indicated turn, but rather, each specified tiling could occur on any turn.

Total	Turn 1 Tiling	Score	Turn 2 Tiling	Score
10	111	10		
15	123	15		
20	222	20		
25	234	25		
30	333	30		
35	345	35		
40	444	40		
45	1 2x3 3	45		
50	2 3x2 4	50		
55	222	20	345	35
60	1x4 2 3	60		
65	222	20	1 2x3 3	45
70	3x2 4 5	70		
75	2x3 3 4	75		
80	111	10	3x2 45	70
85	444	40	1 2x3 3	45
90	333	30	2x3 2 2	60
95	555	50	1 2x3 3	45
100	WWW	100		

Table 4: Possible Ending Scores Up to 100.

What other multiples of 5 are possible total ending scores? Because a player is given

only finitely many turns, there must be an upper bound on how large their ending score can be. If we generalize our question to a game where infinitely many moves are possible, the question is still interesting. Therefore, we break this section into two subsections. In the first, we will consider the possible multiples of five that create ending scores for a player. In the second, we remove the restriction of finitely many turns and explore mathematical structures called numerical semigroups.

5.1 Ending Scores with Finitely Many Turns

Each player has a predetermined total number of moves based on the number of players (see Table 2). Because there are only finitely many tiles afforded to each player, in addition to the 6 printed tiles on the board, and because we know the score any three tiles will create, it should be possible to determine precisely what ending scores are possible for a player.

We now consider possible ending scores by restricting ourselves to a game involving four players, with each player scoring with only one tile grouping per turn for brevity. However, it is possible to extend our reasoning to fewer players or scoring twice per turn. In a four-player game, 10 is the lowest nonzero score possible and is achieved via a single 3-tile grouping of 1 tiles, while 500 is the largest. Note that zero is a possible total score if a player rolls inconveniently on every single turn. The largest possible score is 500 because 100 is the largest possible score on a single turn, and it is only possible to score 100 twice overall in the game. Then, 75 is the second largest possible score to achieve in a tiling on a turn, so a player scores 75 on the remaining four turns. It's possible to obtain any other multiple of 5 in between 10 and 500 as well. We summarize the ways to obtain these scores in the Appendix.

5.2 Ending Scores with Infinitely Many Turns

When considering the possible ending scores in *Polygon* with infinitely many turns possible, it is impossible not to notice similarities to other problems such as the Postage Stamp Problem [Lun69], the Coin Problem [Alf05], or the Knapsack Problem [Mat96]. These problems and ours are all centered on the same idea: Given a set of k objects with specified values $n_1, n_2, n_3, \dots, n_k$ what possible values of n can be formed from sums, combinations, or groupings of these objects? For our problem, if n can be obtained, there is a k -tuple of non-negative integers $(x_1, \dots, x_k)$ such that the linear Diophantine equation $n = x_1n_1 + x_2n_2 + \dots + x_kn_k$ is satisfied. We can view this as an algebraic structure as follows: Given integers $n_1, \dots, n_k > 0$, set

$$\langle n_1, \dots, n_k \rangle = \{x_1n_1 + x_2n_2 + \dots + x_kn_k \mid x_1, \dots, x_k \in \mathbb{N}_0\}$$

where $\mathbb{N}_0$ denotes the non-negative integers. This structure is called a *numerical semigroup*. Formally, a numerical semigroup S is defined as a subset of non-negative integers such that (i) S is closed under addition, (ii) $0 \in S$, and (iii) S has a finite complement in $\mathbb{N}_0$. Writing $S = \langle n_1, \dots, n_k \rangle$ is the most common and clear way to specify a numerical semigroup that provides a list of *generators*. This algebraic structure is the set of all

non-negative integers obtained by adding copies of $n_1, \dots$, and n_k together. With this definition in mind, the authors of [CGO20] note that the smallest nontrivial numerical semigroup is $S_1 = \langle 2, 3 \rangle$. Based on the possible one-turn scores, we know that all possible scores belong to the numerical semigroup $\langle 2, 3 \rangle$.

There are several properties that are studied amongst semigroups. One such parameter is the *Frobenius number* of a numerical semigroup $\langle n_1, \dots, n_k \rangle$, denoted $F(\langle n_1, \dots, n_k \rangle)$, and is defined as the largest positive integer n so that $n \notin \langle n_1, \dots, n_k \rangle$. For example, since the numerical semigroup S_1 is generated by consecutive integers 2 and 3, $F(\langle 2, 3 \rangle) = 1$. Note that any even integer $n = 2m = 2m + 3(0)$ for some positive integer m while for any odd integer n excluding 1, $n = 2m + 1 = 2(m - 1) + 2 + 1 = 2(m - 1) + 3(1)$ for some integer $m \geq 1$.

Other properties that are commonly examined pertain to the ways in which elements of the numerical semigroup can be broken down in terms of the generators. Assume that $n_1, \dots, n_k$ is a set of generators for a numerical semigroup S . For $n \in S$, we let $Z(n) = \left\{ (x_1, \dots, x_k) \mid n = \sum_{i=1}^k x_i n_i \right\}$ denote the *set of factorizations* of $n \in S$ and $L(n) = \left\{ \sum_{i=1}^k x_k \mid (x_1, \dots, x_k) \in Z(n) \right\}$ denote the *set of factorization lengths* of $n \in S$. For example, for $8 \in \langle 2, 3 \rangle$, $Z(8) = \{(4, 0), (1, 2)\}$ where the first coordinate specifies the number of 2's and the second coordinate specifies the number of 3's. Also, $L(8) = \{3, 4\}$ since $4 + 0 = 4$ and $1 + 2 = 3$. Each element of $Z(n)$ represents a distinct *factorization* of element n . Such an expression is $n = x_1 n_1 + \dots + x_k n_k$ of the form which presents n as a linear combination of the generators $n_1, \dots, n_k$. We denote the maximum and minimum factorization lengths of $n \in S$ by $\ell(n) = \min L(n)$ and $L(n) = \max L(n)$. By the previous example, then, $\ell(8) = 3$ and $L(8) = 4$. The *elasticity* of a nonzero element $n \in S$, $\rho(n)$ measures a deviation between $\ell(n)$ and $L(n)$ and is defined as $\rho(n) = \frac{L(n)}{\ell(n)}$. So, once again using S_1 as an example and $n = 8$, $\rho(8) = \frac{4}{3}$. The *elasticity* of S is defined as $\rho(S) = \sup\{\rho(n) \mid n \in S\}$.

Still other invariants studied in the theory of non-unique factorizations yield more information on the structure of the factorization set. One such measure is known as the *delta set*. More explicitly, for a numerical semigroup $\langle n_1, \dots, n_k \rangle$, if $L(n) = \{x_1, \dots, x_t\}$ with the x_i 's listed in increasing order, then the set $\Delta(n) = \{x_i - x_{i-1} \mid 2 \leq i \leq t\}$ and $\Delta(S) = \bigcup_{0 < n \in S} \Delta(n)$ are called the *delta set of element n* and the *delta set of numerical semigroup S* respectively. For example, with $8 \in S_1$, $\Delta(8) = \{4 - 3\} = \{1\}$.

Finally, we turn to the notion of “primality” within a numerical semigroup, which is analogous to the usual definition of prime taken from number theory. A nonzero element $x \in \langle n_1, \dots, n_k \rangle$ is *prime* if whenever x divides a sum $y + z$, then either x divides y or x divides z . It is the case that no element of a numerical semigroup is prime. However, we can define a parameter within a semigroup that measures how far an element is from being prime. For any nonzero $x \in \langle n_1, \dots, n_k \rangle$, define $\omega(x) = m$ if m is the smallest positive integer so that whenever x divides $x_1 + \dots + x_t$ for $x_i \in \langle n_1, \dots, n_k \rangle$, then there is a set $T \subset \{1, 2, \dots, t\}$ of indices with $|T| \leq m$ so that x divides $\sum_{i \in T} x_i$. A prime element

would have *omega*-value equal to 1. Using Sage, we obtain that $\omega(8) = 4$.

In [CGO20], the authors create a table displaying the set of factorizations and the set of factorization lengths for values 2 through 19 of the semigroup $S_1 = \langle 2, 3 \rangle$. We have extended those values to 25, only considered values that are multiples of 5 in line with game-ending scores, and added columns for the minimum factorization length, the set of factorizations, the maximum factorization length, the delta set, and omega primality. Next, in [CGO20], the authors determined that $\ell(n) = \lceil \frac{n}{3} \rceil$ and $L(n) = \lfloor \frac{n}{2} \rfloor$ for any $n \in S_1 = \langle 2, 3 \rangle$; the set of lengths of factorizations for $n \geq 4$, $L(n) = \left\{ \lceil \frac{n}{3} \rceil, \lceil \frac{n}{3} \rceil + 1, \dots, \lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor \right\}$, $\rho(n) = \frac{3}{2}$, and $\Delta(n) = \{1\}$. A closed formula for $\omega(n)$ has not been determined.

The formulas previously found for $\ell(n)$, $L(n)$, $\rho(n)$, $\Delta(n)$, as well as the list of factorizations in $L(n)$ for any $n \in S_1$, therefore apply to all of the possible total scores in this variation of *Polygon*. We have included these values for a few possible *Polygon* scores in Table 5. It should be noted that the number of factorizations for a given score does not necessarily match the number of ways in which this score can be obtained during gameplay. For example, a score of 10 can only be obtained by scoring 10 on a single turn. Given the list of possible one-turn scores, it may, therefore, seem more natural to consider an algebraic structure with 10 and 15 as generators. However, “ $\langle 10, 15 \rangle$ ” is not a numerical subgroup as it does not have a finite complement in $\mathbb{N}_0$.

n	$Z(n)$	$\ell(n)$	$L(n)$	$L(n)$	$\Delta(n)$	$\omega(n)$
10	$\{(5, 0), (2, 2)\}$	$\{4\}$	$\{4, 5\}$	$\{5\}$	$\{1\}$	5
15	$\{(6, 1), (3, 3), (0, 5)\}$	$\{5\}$	$\{5, 6, 7\}$	$\{7\}$	$\{1\}$	9
20	$\{(10, 0), (7, 2), (4, 4), (1, 6)\}$	$\{7\}$	$\{7, 8, 9, 10\}$	$\{10\}$	$\{1\}$	10
25	$\{(11, 1), (8, 3), (5, 5), (2, 7)\}$	$\{9\}$	$\{9, 10, 11, 12\}$	$\{12\}$	$\{1\}$	14

Table 5: The set of factorizations $Z(n)$, minimum factorization $\ell(n)$, set of factorization lengths $L(n)$, maximum factorization $L(n)$, delta set $\Delta(n)$, and omega primality $\omega(n)$ for each $10 \leq n \leq 25$ such that $n \in S_1$ and $n = 5m$ for some positive integer m .

It should also be noted that the computations referenced in Table 5 can be performed using the `numericalsgps` package for the computer algebra system `GAP` [see [CGO20], [CO18], [DGM15]].

6 Topics for Further Exploration

There are several possible extensions of this work for the interested reader. In our discussion, we have simplified the game rules so that players can only score with one group of three tiles per turn. The possibility of scoring with two groups in a turn leads to more complex scoring possibilities while also potentially putting player 1 at a slight initial disadvantage, as on the game’s very first turn, it is only possible to score with one set of three tiles. The interested reader may also wish to further explore board configurations in the context of tile stacking, as we have only scratched the surface of what stacking

configurations are possible. More examination of the role of the game die is also an area of further study, as we explored possible ending scores but did not address the role of the die in how probable each possible score is. Finally, one may pursue a more detailed exploration of the correspondence of the connections between $\langle 2, 3 \rangle$ and the range of possible ending scores in the infinite-turns case of the game.

References

- [CGO20] S. Chapman, R. Garcia, and C. O'Neill, Beyond coins, stamps, and chicken McNuggets: An invitation to numerical semigroups, in: *A Project-Based Guide to Undergraduate Research in Mathematics*, 2020, 177–202.
- [CO18] S. T. Chapman and C. O'Neill, Factoring in the Chicken McNugget monoid, *Mathematics Magazine* **91** (2018), 323–336.
- [AKM19] J. Austin, B. G. Kronenthal, and S. M. Molitoris, The settlers of “Catanbinatorics”, *Mathematics Magazine* **92** (2019), 187–198.
- [AC22] J. Austin and E. Curl, Exploring combinatorics and graph theory with simple Blokus, *College Mathematics Journal* **53** (2022), 273–281.
- [EH23] R. Eaton and K. Herzinger, A further investigation of positions in Sylver coinage for which four has been chosen, *Integers* **23** (2023).
- [GK18] I. García-Marco and K. Knauer, Chomp on numerical semigroups, *Algebraic Combinatorics* **1** (2018), 371–394.
- [CSD+15] C. Cox, J. de Silva, P. Deorsey, F. H. J. Kenter, T. Retter, and J. Tobin, How to make the perfect fireworks display: Two strategies for *Hanabi*, *Mathematics Magazine* **88** (2015), 323–336.
- [Bou01] C. L. Bouton, Nim, a game with a complete mathematical theory, *Ann. Math.* **3**(1/4) (1901), 35–39.
- [Lun69] W. F. Lunnon, A postage stamp problem, *Comput. J.* **12** (1969), 377–380.
- [Guy87] R. K. Guy, Twenty questions concerning Conway’s Sylver coinage, *Amer. Math. Monthly* **83** (1987), 634–637.
- [DGM15] M. Delgado, P. A. García-Sánchez, and J. Morais, NumericalSgps: A GAP package for numerical semigroups, 2015, available at <http://www.gap-system.org>.
- [Alf05] J. L. R. Alfonsín, *The Diophantine Frobenius Problem*, OUP Oxford, 2005.
- [Mat96] G. B. Mathews, On the partition of numbers, *Proc. London Math. Soc.* **28** (1896), 486–490.

7 Appendix A: Tiling and Scores in *Polygon*

Below are the tables which list all possible combinations of three tiles and the official score in the game *Polygon*.

Tiles	Score	Tiles	Score	Tiles	Score	Tiles	Scores	Tiles	Scores	Tiles	Scores	Tiles	Scores
111	10	121	0	131	0	141	0	151	0	11x41	40	12x31	0
112	0	122	0	132	15	142	0	152	0	11x42	0	12x32	0
113	0	123	15	133	0	143	0	153	0	11x43	0	12x33	45
114	0	124	0	134	0	144	0	154	0	11x44	0	12x34	0
115	0	125	0	135	0	145	0	155	0	11x45	0	12x35	0
111x4	40	121x4	0	131x4	0	141x4	0	151x4	0	11x41x4	40	12x31x4	0
112x3	0	122x3	0	132x3	45	142x3	0	152x3	0	11x42x3	0	12x32x3	0
113x2	0	123x2	30	133x2	0	143x2	0	153x2	0	11x43x2	0	12x33x2	45
11W	40	12W	30	13W	45	14W	0	15W	0	11x4W	40	12x3W	45

Tiles	Score	Tiles	Score	Tiles	Score	Tiles	Scores	Tiles	Scores	Tiles	Scores	Tiles	Scores
13x21	0	1W1	40	221	0	231	15	241	0	251	0	21x41	0
13x22	30	1W2	30	222	20	232	0	242	0	252	0	21x42	0
13x23	0	1W3	45	223	0	233	0	243	25	253	0	21x43	60
13x24	0	1W4	0	224	0	234	25	244	0	254	0	21x44	0
13x25	0	1W5	0	225	0	235	0	245	0	255	0	21x45	0
13x21x4	0	1W1x4	40	221x4	0	231x4	60	241x4	0	251x4	0	21x41x4	0
13x22x3	45	1W2x3	45	222x3	60	232x3	0	242x3	0	252x3	0	21x42x3	0
13x23x2	0	1W3x2	45	223x2	0	233x2	0	243x2	50	253x2	0	21x43x2	60
13x2W	45	1WW	45	22W	60	23W	60	24W	50	25W	0	21x4W	60

Tiles	Score	Tiles	Score	Tiles	Score	Tiles	Scores	Tiles	Scores	Tiles	Scores	Tiles	Scores
22x31	0	23x21	30	2W1	30	331	0	341	0	351	0	31x41	0
22x32	60	23x22	0	2W2	60	332	0	342	25	352	0	31x42	60
22x33	0	23x23	0	2W3	60	333	30	343	0	353	0	31x43	0
22x34	0	23x24	50	2W4	50	334	0	344	0	354	35	31x44	0
22x35	0	23x25	0	2W5	0	335	0	345	35	355	0	31x45	0
22x31x4	0	23x21x4	60	2W1x4	60	331x4	0	341x4	0	351x4	0	31x41x4	0
22x32x3	60	23x22x3	0	2W2x3	60	332x3	0	342x3	75	352x3	0	31x42x3	60
22x33x2	0	23x23x2	0	2W3x2	60	333x2	60	343x2	0	353x2	0	31x43x2	0
22x3W	60	23x2W	60	2WW	60	33W	60	34W	75	35W	35	31x4W	60

Tiles	Score	Tiles	Score	Tiles	Score	Tiles	Scores	Tiles	Scores	Tiles	Scores	Tiles	Scores
32x31	45	33x21	0	3W1	45	441	0	451	0	41x41	0	42x31	0
32x32	0	33x22	0	3W2	60	442	0	452	0	41x42	0	42x32	0
32x33	0	33x23	60	3W3	60	443	0	453	35	41x43	0	42x33	75
32x34	75	33x24	0	3W4	75	444	40	454	0	41x44	0	42x34	0
32x35	0	33x25	0	3W5	35	445	0	455	0	41x45	0	42x35	0
32x31x4	60	33x21x4	0	3W1x4	60	441x4	0	451x4	0	41x41x4	0	42x31x4	0
32x32x3	0	33x22x3	0	3W2x3	75	442x3	0	452x3	0	41x42x3	0	42x32x3	0
32x33x2	0	33x23x2	60	3W3x2	60	443x2	0	453x2	70	41x43x2	0	42x33x2	75
32x3W	75	33x2W	60	3WW	75	44W	40	45W	70	41x4W	0	42x3W	75

Tiles	Score	Tiles	Score	Tiles	Score	Tiles	Scores	Tiles	Scores	Tiles	Scores	Tiles	Scores
43x21	0	4W1	0	551	0	51x41	0	52x31	0	53x21	0	5W1	0
43x22	50	4W2	50	552	0	51x42	0	52x32	0	53x22	0	5W2	0
43x23	0	4W3	75	553	0	51x43	0	52x33	0	53x23	0	5W3	35
43x24	0	4W4	40	554	0	51x44	0	52x34	0	53x24	70	5W4	70
43x25	70	4W5	70	555	50	51x45	0	52x35	0	53x25	0	5W5	50
43x21x4	0	4W1x4	0	551x4	0	51x41x4	0	52x31x4	0	53x21x4	0	5W1x4	0
43x22x3	75	4W2x3	75	552x3	0	51x42x3	0	52x32x3	0	53x22x3	0	5W2x3	0
43x23x2	0	4W3x2	75	5523x2	0	51x43x2	0	52x33x2	0	53x23x2	0	5W3x2	0
43x2W	75	4WW	75	55W	50	51x4W	0	52x3W	0	53x2W	70	5WW	70

Tiles	Score	Tiles	Score	Tiles	Score	Tiles	Scores	Tiles	Scores	Tiles	Scores
1x41x41	40	1x42x31	0	1x43x21	0	1x4W1	40	2x32x31	0	2x33x21	45
1x41x42	0	1x42x32	0	1x43x22	60	1x4W2	60	2x32x32	60	2x33x22	0
1x41x43	0	1x42x33	60	1x43x23	0	1x4W3	60	2x32x33	0	2x33x23	0
1x41x44	0	1x42x34	0	1x43x24	0	1x4W4	0	2x32x34	0	2x33x24	75
1x41x45	0	1x42x35	0	1x43x25	0	1x4W5	0	2x32x35	0	2x33x25	0
1x41x41x4	40	1x42x31x4	0	1x43x21x4	0	1x4W1x4	40	2x32x31x4	0	2x33x21x4	60
1x41x42x3	0	1x42x32x3	0	1x43x22x3	60	1x4W2x3	60	2x32x32x3	60	2x33x22x3	0
1x41x43x2	0	1x42x33x2	60	1x43x23x2	0	1x4W3x2	60	2x32x33x2	0	2x33x23x2	0
1x41x4W	40	1x42x3W	60	1x43x2W	60	1x4WW	60	2x32x3W	60	2x33x2W	75

Tiles	Score	Tiles	Score	Tiles	Score	Tiles	Scores
2x3W1	45	3x23x21	0	3x2W1	45	WW1	45
2x3W2	60	3x23x22	0	3x2W2	70	WW2	60
2x3W3	75	3x23x23	60	3x2W3	60	WW3	75
2x3W4	75	3x23x24	0	3x2W4	75	WW4	75
2x3W5	0	3x23x25	0	3x2W5	70	WW5	70
2x3W1x4	60	3x23x21x4	0	3x2W1x4	60	WW1x4	60
2x3W2x3	60	3x23x22x3	0	3x2W2x3	60	WW2x3	75
2x3W3x2	75	3x23x23x2	60	3x2W3x2	60	WW3x2	75
2x3WW	75	3x23x2W	60	3x2WW	75	WWW	100

8 Appendix B: Additional Ending Scores (in a game of Four Players)

The column headings “Turn 1”, etc. do not actually specify that the indicated tiling must be created on that indicated turn, rather each specified tiling could occur on any turn.

Total	Turn 1	Score	Turn 2	Score	Turn 3	Score
105	555	50	1 2x3 3	45	111	10
110	WWW	100	111	10		
115	WWW	100	123	15		
120	WWW	100	222	20		
125	WWW	100	234	25		
130	WWW	100	333	30		
135	WWW	100	345	35		
140	WWW	100	444	40		
145	WWW	100	1 2x3 3	45		
150	WWW	100	2 3x2 4	50		
155	WWW	100	222	20	345	35
160	WWW	100	1x4 2 3	60		
165	WWW	100	222	20	1 2x3 3	45
170	WWW	100	3 x 2 4 5	70		
175	WWW	100	2x3 3 4	75		
180	WWW	100	3x2 45	70	111	10
185	WWW	100	1 2x3 3	45	444	40
190	WWW	100	2x3 2 2	60	333	30
195	WWW	100	1 2x3 3	45	555	50
200	WWW	100	WWW	100		

Total	Turn 1	Score	Turn 2	Score	Turn 3	Score	Turn 4	Score
205	WWW	100	222	20	345	35	2 3x2 4	50
210	WWW	100	333	30	3x2 4 5	70	111	10
215	WWW	100	333	30	3x2 4 5	70	123	15
220	WWW	100	333	30	3x2 4 5	70	222	20
225	WWW	100	333	30	3x2 4 5	70	234	25
230	WWW	100	333	30	3x2 4 5	70	333	30
235	WWW	100	333	30	3x2 4 5	70	345	35
240	WWW	100	333	30	3x2 4 5	70	444	40
245	WWW	100	333	30	3x2 4 5	70	1 3 2x3	45
250	WWW	100	333	30	3x2 4 5	70	2 4 3x2	50
255	WWW	100	345	35	3x2 4 5	70	2 4 3x2	50
260	WWW	100	333	30	3x2 4 5	70	2 3 1x4	60
265	WWW	100	345	35	3x2 4 5	70	2 3 1x4	60
270	WWW	100	WWW	100	3x2 4 5	70		
275	WWW	100	WWW	100	4 2x3 3	75		
280	WWW	100	WWW	100	333	30	2 4 3x2	50
285	WWW	100	WWW	100	345	35	2 4 3x2	50
290	WWW	100	WWW	100	1 3 2x3	45	1 3 2x3	45
295	WWW	100	WWW	100	1 3 2x3	45	2 4 3x2	50
300	WWW	100	WWW	100	333	30	3x2 4 5	70

Total	Turn 1	Score	Turn 2	Score	Turn 3	Score	Turn 4	Score	Turn 5	Score
305	WWW	100	WWW	100	4 2x3 3	75	333	30		
310	WWW	100	WWW	100	4 2x3 3	75	345	35		
315	WWW	100	WWW	100	4 2x3 3	75	444	40		
320	WWW	100	WWW	100	4 2x3 3	75	1 3 2x3	45		
325	WWW	100	WWW	100	4 2x3 3	75	2 4 3x2	50		
330	WWW	100	WWW	100	3x2 4 5	70	2 3 1x4	60		
335	WWW	100	WWW	100	4 2x3 3	75	2 3 1x4	60		
340	WWW	100	WWW	100	3x2 4 5	70	3x2 4 5	70		
345	WWW	100	WWW	100	4 2x3 3	75	3x2 4 5	70		
350	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75		
355	WWW	100	WWW	100	4 2x3 3	75	2 4 3x2	50	333	30
360	WWW	100	WWW	100	4 2x3 3	75	2 4 3x2	50	345	35
365	WWW	100	WWW	100	4 2x3 3	75	2 3 1x4	60	444	40
370	WWW	100	WWW	100	4 2x3 3	75	2 3 1x4	60	1 3 2x3	45
375	WWW	100	WWW	100	4 2x3 3	75	3x2 4 5	70	333	30
380	WWW	100	WWW	100	4 2x3 3	75	3x2 4 5	70	345	35
385	WWW	100	WWW	100	4 2x3 3	75	3x2 4 5	70	444	40
390	WWW	100	WWW	100	4 2x3 3	75	3x2 4 5	70	1 3 2x3	45
395	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	1 3 2x3	45
400	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	555	50

Total	Turn 1	Score	Turn 2	Score	Turn 3	Score	Turn 4	Score	Turn 5	Score	Turn 6	Score
405	WWW	100	WWW	100	4 2x3 3	75	3x2 4 5	70	2 3 1x4	60		
410	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	2 3 1x4	60		
415	WWW	100	WWW	100	4 2x3 3	75	3x2 4 5	70	3x2 4 5	70		
420	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	3x2 4 5	70		
425	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	4 2x3 3	75		
430	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	333	30	555	50
435	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	345	35	555	50
440	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	4 2x3 3	75	123	15
445	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	4 2x3 3	75	222	20
450	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	4 2x3 3	75	234	25
455	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	4 2x3 3	75	1 3x2 2	30
460	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	4 2x3 3	75	345	35
465	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	4 2x3 3	75	1 1 1x4	40
470	WWW	100	WWW	100	3x2 4 5	70	3x2 4 5	70	3x2 4 5	70	2 3 1x4	60
475	WWW	100	WWW	100	4 2x3 3	75	3x2 4 5	70	3x2 4 5	70	2 3 1x4	60
480	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	3x2 4 5	70	2 3 1x4	60
485	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	4 2x3 3	75	2 3 1x4	60
490	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	3x2 4 5	70	3x2 4 5	70
495	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	4 2x3 3	75	3x2 4 5	70
500	WWW	100	WWW	100	4 2x3 3	75	4 2x3 3	75	4 2x3 3	75	4 2x3 3	75