Louis Gros's Theory of the Chinese Rings

Andreas M. Hinz[1]

[1] *Mathematics Department, LMU Munich,* hinz@math.lmu.de

Abstract

After Pacioli and Cardano in the 16th, Wallis in the 17th, and Lichtenberg in the 18th century, it was the French legal professional Louis Gros who published in 1872 a *Théorie du Baguenodier*. The small book was based on the revolutionary new idea to marry the mechanical puzzle which is now known as the *Chinese Rings* with the binary number system. An unpublished, much broader manuscript, discovered recently by Lisa Rougetet, offers new insight into Gros's biography and the circumstances of his scientific achievements. Although the mathematical results were mainly empirical, we will demonstrate that they are all well-founded in modern interpretation.

MSC2020: 00A08, 01A70, 01A55, 05A15, 11B83.
Keywords: Baguenaudier/Chinese Rings, Gros/Gray code, integer sequences, binary number system.

1 Luc Agathange Louis Gros (1814–1886)

Until quite recently, virtually nothing was known about the *clerc de notaire lyonnais* (notary clerk from Lyons) who published the brief *Théorie du Baguenodier* [Gros72+1] in 1872. It contained a revolutionary mathematical interpretation of what we now call the *Chinese Rings* and which was enthusiastically taken up ten years later[1] in the first volume of the famous *Récréations Mathématiques* [Luc82] by the French number theorist Édouard Lucas (1842–1891)[2]. The book [Rou23] by Lisa Rougetet (b. 1987) now reports on her discovery of (at least) two documents that shed new light on life and work of Louis Gros. One is the unpublished hand-written manuscript [Gros72+2][3], the other is a letter [Rou23, Figure 66] which accompanied his donation of copies of [Luc82] and [Gros72+1] to the public library[4] of the city of Bourg(-en-Bresse) in the Ain department of France. Rougetet's

©2026 Andreas M. Hinz. This is an open access article licensed under the Creative Commons Attribution-NonCommercial-NoDerivs License (https://creativecommons.org/licenses/by-nc-nd/4.0/).

[1] An earlier version of the chapter on *Le jeu du baguenaudier* had been published in the *Revue scientifique de la France et de l'étranger* (2)19 (1880/1), pp. 36–42. Here we will cite Lucas's book which is easier accessible.

[2] See [Bra22, pp. 793–797].

[3] Accessible through a link in [Rou23, p. 14].

[4] Now in the Médiathèque Vailland, Bourg-en-Bresse, listed FA 115409.

book focussing on the personality of Gros and the educational impact of his life-long interest in the baguenaudier game, we will here concentrate on the scientific quality of his mathematical investigations and touch upon his biography only in connection with these studies.

Luc Agathange Louis Gros[5] was born 1814–08–09 in Frans (Ain), some 35 km north of Lyons. He developed an early interest in mathematics and built his own baguenaudier [Gros72+2, p. 134] together with his brother Claude François (Francisque) Gros (1816–1894) [Bra22, p. 574]. Practicing with the puzzle he realized its connection with the powers of 2. A couple of years later, when he was a notary clerk in Lyons, Gros decided to write a treatise on the topic, which he began during the total eclipse of the Moon on 1837–04–20, not for superstition, but for childishness ("enfantillage" [Gros72+2, p. 139]). It didn't go beyond a preface. Only ten years later, when he bought a rare-to-find baguenaudier with eleven rings [Gros72+2, pp. 140f] (see Figure 1), his interest was revived.

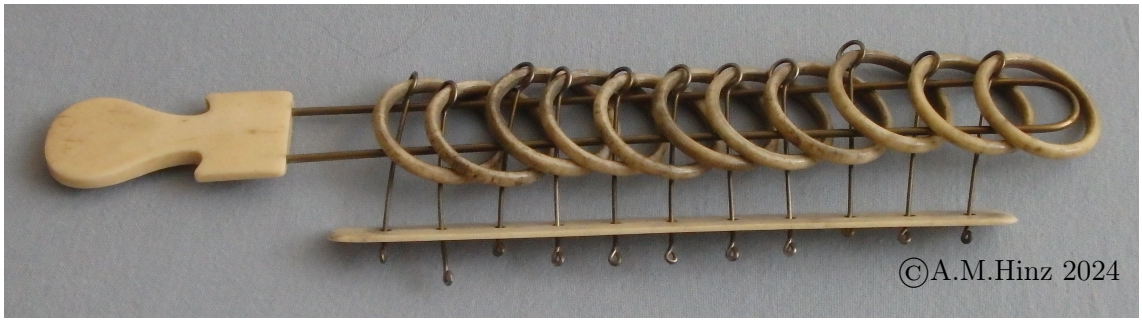


Figure 1: Baguenaudier with eleven rings (ca. 22 cm, bone and metal, France, 19th c.).

Gros prepared an article for the *Magasin pittoresque* to publish his main idea which we call the *Gros code* (cf. infra). The letter to the editor, reproduced in its entirety in [Gros72+2, pp. 143–155], was dated 1847–12–12. Here he developed all his key concepts, including what we will call the *Gros sequence*, and provided a coding of the decimal number 1847 in binary and with a specific state of the baguenaudier with eleven rings, asserting that it is easy to write numbers with the baguenaudier. ("On peut facilement écrire des nombres avec le baguenodier.") Helas! His submission has been thrown in the basket ("a été jeté au panier" [Gros72+2, p. 154]). A second attempt ended in 1853: "Ma nouvelle élucubration est allée rejoindre la première au panier." [Gros72+2, p. 156] So we have to wait until 1871, when Gros (re-)discovered a rule he calls striking ("frappante" [Gros72+2, p. 158]), telling which ring to move in any situation, that he resumed his study of the baguenaudier by looking into two works on it by Girolamo Cardano (1501–1576) and John Wallis (1616–1703); see [HH17]. By 1872, Gros had produced three versions ("éditions") of his treatise, the first for seven rings, the second for nine and the third for eleven rings. The first went to the local printer Aimé Vingtrinier (1813–1903) for publication, the second was

[5]His full name appears in the aforementioned letter.

meant for Gros's brother Joseph Léopold (Léo) Gros (1823–1874) [Bra22, pp. 574f] and the third for the author himself or either his nephew Joseph to whom it was dedicated. Meanwhile Louis Gros had become examining magistrate in Lyons, but fearing the dismay of his colleagues for his non-professional activities he chose the job title "clerc de notaire" from the times of his first steps into the theory for his otherwise anonymous publication.

The printed first edition [Gros72+1] came out in 200 copies [Gros72+2, p. 203] and contains only the essentials of his theory. The term "Théorie" was replaced by "Traité" in the hand-written third edition which was complemented by comments also after 1872. E.g., in [Gros72+2, p. 2] appears a comment dated 1880–07–14 after he had received a copy of Lucas's article of 1880. Gros is rather critical and calls Lucas's explanations of scientific appeal more confuse ("plus embrouillées") than his. Another comment was added 1882–02–07 after he had learned about the publication in december of 1881 of Lucas's book [Luc82]. Gros considers his third edition as a memorial of all his life, occasionally adding places and people to his writing. The manuscript comes in five chapters: Arithmetical theory, Numerical table, Practical remarks, Etymological notes, and Historical chatter. In the etymological section, already quite substantial in the printed version, he derives his spelling "baguenodier", though not very convincing so that Lucas and all later authors returned to the traditional name. Rougetet, however, follows Gros, probably for reasons of compassion.

Gros's analysis of the works of Cardano and Wallis is also unfortunate, mainly because of some misinterpretations; see [HKP18, pp. 58 and 75] for a discussion. We will not pursue this any further in the present note.

Louis Gros's legal career was crowned as Councillor at the court of appeal of Lyons, and he retired in 1884.[6]

Louis Gros died 1886–08–16 in Neuville-les-Dames (Ain) [Bra22, p. 574]. In the brief obituary [Rou86, pp. 25f] it is mentioned among some other biographical details that Gros "liked to occupy his leisure time delivering himself to scientific studies".

Louis Gros was not a professional mathematician. The theory he developed was empirical and his attitude towards proof is characterized by the following passage in [Gros72+2, p. 94]: when he realizes that a certain rule led to the right result 18 434 times, he considers a person who still has doubts in a truth so well demonstrated to be worse than Thomas! He was well aware of the danger that his readers might develop the attitude of a student who, wanting to avoid mathematical proofs, will believe his teacher's word. [Gros72+2, p. 21] When it comes to deeper mathematical questions he occasionally, e.g., [Gros72+2, p. 193], turned to his brother Léo, a *professeur de mathématiques spéciales* as, by the way, Lucas was. Nevertheless, Louis Gros was confident enough in the value of his theory to speak of "ma science baguenodière" in the dedication of the third edition [Gros72+2, p. 3].

[6] Annuaire rétrospectif de la magistrature xix^{e}–xx^{e} siècles, Fiche Magistrat : Luc Agathange Louis Gros

2 The Baguenaudier

The Baguenaudier is nowadays known as the *Chinese Rings*, although its origin is uncertain; see [Hinz19]. Gros has also speculated about a Chinese invention [Gros72+2, pp. 204f], but in order to be neutral we prefer to talk about the *Linked Rings*. The contraption consists essentially of two parts: an elongated metal loop (usually with a handle on one side) and a system of rings, mutually linked in a sophisticated manner and fixed by rods in a slat. We are tempted to follow Gros in his opinion "I believe it is useless to give a more ample description of a thing that one finds on the market." [Gros72+2, p. 11] Today, this is even more applicable, if we replace "market" by "internet" and so the reader is advised to search for one of the many videos showing people who solve the puzzle at sometimes incredible speed. An even better idea is to craft one's own baguenaudier; instructions can be found in [Rou23, pp. 30–33]. The task is to separate the two components, if originally entangled as in Figure 1, or either to put them together when isolated from each other, in a sequence of individual ring moves onto or off the loop.

A little practise reveals that only two types of moves are possible:

- Ring 1, the one at the free end of the loop, i.e. the rightmost in Figure 1, can move up onto the loop or down from it at any time.
- The first ring to the left of the rightmost ring on the loop can move up or down.

Under the implicit assumption that one wants to make as few moves as possible, it is immediate that no two moves of the same type should be performed consecutively. Starting from the situation where all rings are down, as Gros did after a friend had lent him a commercial baguenaudier, and where, lacking a ring on the loop, only a move of ring 1 is possible, it is therefore clear how to proceed, namely with a sequence of moves of alternating types. Surprisingly, this sequence does not end with all rings on the loop, but in a state where the leftmost ring, i.e. the one closest to the handle, is on the loop and all others are off, simply because there is no ring to the left of the leftmost ring. Gros produced a list of these moves and analyzed it for mathematical regularities.

He realized that powers of 2 played a role in the number of moves of individual rings and assumed a connection to the binary number system: "Mettons en parallèle ce système [de numération binaire] et le baguenodier." [Gros72+2, p. 15] (Let us put this system and the baguenaudier in parallel.) For the states of the baguenaudier he chose a symbolic representation consisting of a horizontal line standing for the loop and dots above or below that line to depict the rings on or off the loop, respectively. We will replace this sybolic characterization of a state by a binary string of length n, the number of rings. In this string, a 0 in the rth position (from the right) will stand for ring r to be off the loop, whereas 1 means an elevated ring. This is just for mathematical convenience, but Gros's revolutionary idea was another binary representation of a state, namely its distance, i.e. the minimal number of moves, from the starting point, expressed as a binary number. It is remarkable that Gros also considered the converse, namely to code a (binary) number by its equivalent on the baguenaudier, as, e.g., on the title page of his *Théorie* [Gros72+1], where the year 1872 is given in binary as 11101010000 and symbolic as 10011111000 on his

baguenaudier with $n = 11$ rings, thereby anticipating what will later be called the *Gray code*.

Before we will make precise Gros's theory, let us mention why he thought nobody had developed it in the past. In the aforementioned letter of 1847 he pointed out three reasons [Gros72+2, pp. 145–147]:

- Mathematicians rarely considered the game.
- The starting configuration was traditionally with all rings *on* the loop.
- For the number of moves the *accelerated count* was employed.

The latter is caused by the fact that in the physical realization of the baguenaudier every move of ring 2 is accompanied by a move of ring 1, i.e. they move up or down together, which causes some authors to count this as one move. As a consequence, the relation between move numbers and powers of 2 does not become evident. We will address the accelerated count (see, e.g., [Hinz17, Section 6]) only briefly, but not the question of how long, with respect to performance, it will take for a person to free the loop from the rings, because this is just a matter of scaling.

3 The Theory

We define $B := \{0, 1\}$ equipped with the addition $+ = \dot\vee$, the *binary exclusive or*. For $n \in \mathbb{N}$ we put[7] $B^n = \{[n] \to B\}$, $B_0^n = \{[n]_0 \to B\}$ for the sets of bitstrings of length n with offset 1 or 0, respectively. We write $s = s_n \dots s_1 \in B^n$ and $d = d_{n-1} \dots d_0 \in B_0^n$, where s will stand for a state of the n Linked Rings, and the *Gros code* d of s will lead to the binary representation of the *Gros weight* of s, i.e. the distance $\mathrm{d}(\alpha^{(n)}, s)$ of s to[8] $\alpha^{(n)} = 0^n$, which is the number of moves necessary and sufficient to get all rings down starting in state s.

3.1 The Gros code

The *Gros code* [Gros72+1, p. 13][Gros72+2, emphasized on p. 18] is defined as the mapping

$$G : B^n \to B_0^n,\ s \mapsto d$$

where d is constructed by $d_{r-1} = d_r + s_r$ for $r = n, \dots, 1$ with the understanding that $d_n = 0$. Here are a couple of examples ($\omega^{(n)} = 10^{n-1}$ is, besides $\alpha^{(n)}$, the other *extreme state* [Gros72+1, p. 14], where only a move of ring 1 is possible, $\mathrm{o}^{(n)} = 1^n$ is the state with all rings on the loop, M is the *Mersenne sequence* (A000225[9]), and ℓ is the *Lichtenberg*

[7] $[\nu] := \{1, \dots, \nu\}$, $[\nu]_0 := \{0, \dots, \nu - 1\}$, $\nu \in \mathbb{N}_0$.

[8] The bitstring 0^n is composed of n zeros; similarly for other concatenations.

[9] This refers to the *Online Encyclopedia of Integer Sequences*®; offsets may be different.

sequence (A000975)):[10]

$$G(\alpha^{(n)}) = 0^n \qquad \Big(G(\alpha^{(n)})\Big)_2 = 0$$
$$G(\omega^{(n)}) = 1^n \qquad \Big(G(\omega^{(n)})\Big)_2 = M_n = 2^n - 1$$
$$G(\mathrm{o}^{(2n-1)}) = (10)^{n-1}1 \qquad \Big(G(\mathrm{o}^{(2n-1)})\Big)_2 = \ell_{2n-1} = \frac{1}{3}M_{2n}$$
$$G(\mathrm{o}^{(2n)}) = (10)^n \qquad \Big(G(\mathrm{o}^{(2n)})\Big)_2 = \ell_{2n} = \frac{2}{3}M_{2n}$$

A useful tool is

Lemma 1. $\forall\, n \in \mathbb{N}\, \forall\, r \in [n]_0 :\ d_r = \sum_{\tau=r+1}^{n} s_\tau = |\{\tau \in [n] \setminus [r] \mid s_\tau = 1\}| \mod 2\,.$

Proof. The last identity is obvious (cf. [Gros72+1, p. 15]). Moreover, $\sum_{\tau=r+1}^{n} s_\tau \in B$ fulfills the recurrence for d w.r.t. r. □

An immediate consequence of Lemma 1 is

Lemma 2. $\forall\, n \in \mathbb{N}\, \forall\, s \in B^n :\ G(0s) = 0G(s),\ G(1s) = 1\overline{G(s)}$,
where $\overline{d} = (1 - d_{n-1}) \ldots (1 - d_0) \in B_0^n$ *for* $d \in B_0^n$*; note that* $(d)_2 + (\overline{d})_2 = M_n$*, e.g.,* $\ell_n + \ell_{n-1} = M_n$.

We chose to set off the finite sequence s with the index 1 because traditionally the rings are labelled beginning with ring 1. If we define $\sigma \in B_0^n$ by $\forall\, \rho \in [n]_0 :\ \sigma_\rho = s_{\rho+1}$, the Gros coding $d_n = 0$, $d_\rho = d_{\rho+1} + \sigma_\rho$ can be viewed as $d = \int \sigma$ in the sense of finite binary sequences (cf. [HKP18, Section 0.4.4]). It is a bijection on B_0^n with the inverse, called the *Gray code*, given by $\sigma_\rho = d_{\rho+1} + d_\rho = d_{\rho+1} - d_\rho$, i.e. $\sigma = d'$.

3.2 The Gros weight

The main result is

Theorem 3. $\forall\, n \in \mathbb{N}\, \forall\, s \in B^n :\ (G(s))_2 = \mathrm{d}(\alpha^{(n)}, s)\,.$

With this theorem the classical question, namely how many moves are necessary and sufficient to assemble n rings on the loop, or either to take them off, is answered by the Lichtenberg numbers (see the examples above). Gros recognized these numbers [Gros72+1, p. 92], but they had been known already a hundred years before by G. C. Lichtenberg (1742–1799), who also noticed "a similarity of this machine to a calculating machine for Leibnitz's dyadic system" [Hinz17]. Gros was also aware [Gros72+2, pp. 185–189] of the explicit formula for the Lichtenberg numbers $\ell_n = \lfloor \frac{2}{3} 2^n \rfloor$; cf. [Hinz17, Proposition 5.1]. His

[10] We write $(d)_2$ for the number represented by the binary string $d \in B_0^n$, i.e. $(d)_2 = \sum_{\rho=0}^{n-1} d_\rho \cdot 2^\rho \in [2^n]_0\,.$

argument is that $\dfrac{\ell_n}{2^n}$ is the binary fraction $(0.(10)^\infty)_2 = \displaystyle\sum_{\nu=0}^{\infty} \frac{1}{2^{2\nu+1}} = \frac{2}{3}$ truncated. With the same method he then [Gros72+2, pp. 191–193] solves the problem to find the sequence ℓ' of numbers representing the lengths of minimal solutions for the task to bring down an alternating distribution of rings, i.e. $s = 1010\ldots \in B^n$. The reader is advised to do this exercise with the result $\ell'_n = \lfloor \frac{4}{5} 2^n \rfloor = 1, 3, 6, 12, 25, 51, 102, 204, 409, \ldots$ (A077854).

Proof of Theorem 3. We show by induction on n that

$$\mathrm{d}(\alpha^{(n)}, s) = (G(s))_2 \text{ and } \left(\overline{G(s)}\right)_2 = \mathrm{d}(s, \omega^{(n)}) \qquad (*)$$

For $n = 1$ we have

$$\mathrm{d}(\alpha^{(1)}, s) = \mathrm{d}(0, s_1) = s_1 = d_1 + s_1 = d_0 = d_0 \cdot 2^0 = (G(s))_2$$

and

$$\mathrm{d}(s, \omega^{(1)}) = \mathrm{d}(s_1, 1) = 1 - s_1 = 1 - (d_1 + s_1) = 1 - d_0 = (1 - d_0) \cdot 2^0 = \left(\overline{G(s)}\right)_2$$

Let the claim be true for $n \in \mathbb{N}$. Then, for $s \in B^n$, a possible move of ring $n+1$ can be left out in a shortest path from $\alpha^{(n+1)}$ to $0s$, so that with Lemma 2 and the induction assumption:

$$\mathrm{d}(\alpha^{(n+1)}, 0s) = \mathrm{d}(\alpha^{(n)}, s) = (G(s))_2 = (0G(s))_2 = (G(0s))_2$$

Since the only legal move of ring $n+1$ is between $0\omega^{(n)}$ and $1\omega^{(n)}$, we get

$$\begin{aligned} \mathrm{d}(\alpha^{(n+1)}, 1s) &= \mathrm{d}(\alpha^{(n+1)}, 0\omega^{(n)}) + 1 + \mathrm{d}(1\omega^{(n)}, 1s) = \mathrm{d}(\alpha^{(n)}, \omega^{(n)}) + 1 + \mathrm{d}(\omega^{(n)}, s) \\ &= M_n + 1 + \left(\overline{G(s)}\right)_2 = 2^n + \left(\overline{G(s)}\right)_2 = \left(1\overline{G(s)}\right)_2 = (G(1s))_2 \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{d}(0s, \omega^{(n+1)}) &= \mathrm{d}(0s, 0\omega^{(n)}) + 1 + \mathrm{d}(1\omega^{(n)}, \omega^{(n+1)}) = \mathrm{d}(s, \omega^{(n)}) + 1 + \mathrm{d}(\omega^{(n)}, \alpha^{(n)}) \\ &= \left(\overline{G(s)}\right)_2 + 1 + M_n = \left(\overline{G(s)}\right)_2 + 2^n = \left(1\overline{G(s)}\right)_2 = \left(\overline{G(0s)}\right)_2 \end{aligned}$$

and

$$\mathrm{d}(1s, \omega^{(n+1)}) = \mathrm{d}(s, \alpha^{(n)}) = (G(s))_2 = (0G(s))_2 = (G(0s))_2 = \left(\overline{G(1s)}\right)_2 \qquad \square$$

Remark. From Lemma 2 we get $G(0s) = \overline{G(1s)}$ for $s \in B^n$. With $(*)$ this means $\mathrm{d}(\alpha^{(n+1)}, 0s) = \mathrm{d}(1s, \omega^{(n+1)})$. Therefore, the Gray code is also called the *reflected binary code*. $\square$

The proof of Theorem 3 implicitly shows that for every state s of the Linked Rings there are solutions to get to the extreme states and consequently one can reach any other state. This question of existence (cf. the discussion in [Hinz19, pp. 86ff]) has not been

addressed in a satisfactory manner by any author[11] before 2013 (in Chapter 1 of the first edition of [HKP18], based on graph theory)! A naive approach employed by Gros is to produce an exhaustive list of all states on the (shortest) path from $\alpha^{(n)}$ to $\omega^{(n)}$ for $n = 7$ in [Gros72+1, between pp. 12 and 13] and $n = 11$ in [Gros72+2, pp. 26–88]. To make this argument sound for general n, we have to show that the next state in the list ordered by $(d)_2$ from 0 to M_n, is obtained by a legal move of one ring. Let us write the states as $\sigma \in B_0^n$, i.e. with $\sigma_\rho = s_{\rho+1}$ for $\rho \in [n]_0$, and define the bijective $\gamma : B_0^n \to B_0^n$ by $\gamma(\sigma) = G(s)$. Then we have to prove

Lemma 4. *For $d \in B_0^n \setminus \{0^n\}$ let $\widetilde{d} \in B_0^n \setminus \{1^n\}$ be defined by $(\widetilde{d})_2 = (d)_2 - 1$. Then $\widetilde{\sigma} = \gamma^{-1}(\widetilde{d})$ to $\gamma^{-1}(d) = \sigma$ is a legal move for the n Linked Rings. It will be called "move number $(d)_2 \in [M_n]$" in the canonical path from $\gamma^{-1}(0^n) = \alpha^{(n)}$ to $\gamma^{-1}(1^n) = \omega^{(n)}$.*

Proof. We may, of course, assume that $n \geqslant 2$. Every $d \in B_0^n \setminus \{0^n, 1^n\}$ is of the form $d = \underline{d}d_\rho(1 - d_\rho)^\rho$ with a unique $\rho \in [n-1]$ and $\underline{d} = d_{n-1} \dots d_{\rho+1}$ (d contains both bits and bit d_ρ appears for the first time after ρ opposite bits from the right); in particular, $d_\rho \neq d_0$. Then $\sigma = \underline{\sigma}\sigma_\rho\omega^{(\rho)}$ with $\underline{\sigma} = \sigma_{n-1} \dots \sigma_{\rho+1}$, i.e. the (only) movable rings are 1 and $\rho + 1 \in [n] \setminus \{1\}$.

If $(d)_2$ is odd, i.e. $d_\rho = 0$ and $\sigma_\rho = d_{\rho+1}$, then $\widetilde{d} = \underline{d}01^{\rho-1}0$. For $\rho = 1$ we get $\widetilde{\sigma} = \underline{\sigma}\sigma_1 0$; for $\rho > 1$ we have $\widetilde{\sigma} = \underline{\sigma}\sigma_\rho 10^{\rho-2}1$. In all cases $\widetilde{\sigma}$ to σ corresponds to a legal move of ring 1 (from d_1 to $1 - d_1$).

If $(d)_2$ is even, i.e. $d_\rho = 1$ and $\sigma_\rho = 1 - d_{\rho+1}$, then $\widetilde{d} = \underline{d}01^\rho$ and $\widetilde{\sigma} = \underline{\sigma}(1 - \sigma_\rho)\omega^{(\rho)}$, i.e. ring $\rho + 1$ is moved regularly (from $d_{\rho+1}$ to $1 - d_{\rho+1}$).

Finally, if $d = 1^n$, i.e. $\sigma = \omega^{(n)}$, then $\widetilde{d} = 1^{n-1}0$ and $\widetilde{\sigma} = 10^{n-2}1$ which gives a move of ring 1 (from 1 to 0). □

Instead of Lemma 4 we could, of course, have considered the two states $\widetilde{\sigma}$ obtained by a legal move from a non-extreme $\sigma = \underline{\sigma}\sigma_\rho\omega^{(\rho)}$, $\rho \in [n-1]$, to show that $\gamma(\widetilde{\sigma}) = \gamma(\sigma) \pm 1$, as was suggested by Lucas in [Luc82, pp. 178f]. We have thus filled his argument [Luc82, p. 179] to run through all states σ by following $(d)_2$ in natural order. Every state is passed precisely once because γ is bijective. This sequence of moves provides actually the *unique* shortest path from $\alpha^{(n)}$ to $\omega^{(n)}$ because, as Gros points out [Gros72+1, p. 15], in a minimal solution any ring must not move twice in a row since the second move would undo the first one, and therefore ring 1 moves in every odd numbered move and the even numbered moves are forced by the rules.

3.3 The Gros sequence

Let us write the move number $(d)_2 \in [M_n]$ with $d = \underline{d}d_{\rho+1}10^\rho$, $\underline{d} = d_n \dots d_{\rho+2}$ and $d_n = 0$, as $2^\rho(2k + 1)$, where $\rho \in [n]_0$ and $k \in [2^{n-1}]_0$. Then $\rho = 0$ if $(d)_2$ is odd and it is the old $\rho \in [n-1]$ otherwise. Moreover, $2k + 1 = (\underline{d}d_{\rho+1}1)_2$, whence $k = (\underline{d}d_{\rho+1})_2$. Assuming an unlimited supply of rings and noting that ρ and k do not depend on n, the sequence ρ with respect to $\delta := (d)_2$ is known as the *binary carry sequence* (A007814) with $\rho(2\delta - 1) = 0$

[11] See, however, Lichtenberg in [Hinz17, pp. 4f] and S. N. Afriat (1925–2021) [Afr82, pp. 45ff] for the $o^{(n)}$ to $\alpha^{(n)}$ task.

and $\rho(2\delta) = \rho(\delta) + 1$, and k is given by $k(2\delta - 1) = \delta - 1$ and $k(2\delta) = k(\delta)$ (A025480). The proof of Lemma 4 tells us the following:

Proposition 5. *In move number* $(\underline{d}d_{\rho+1}10^\rho)_2 = 2^\rho(2k+1)$ *of the minimal path from* $\alpha^{(n)}$ *to* $\omega^{(n)}$ *ring* $\rho+1$ *is moved and it goes up iff* $d_{\rho+1} = 0$*, i.e. for even* k*.*

Proposition 5 leads to the *Gros sequence* (A001511)

$$g = 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1,\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1,\ \ldots$$

for the rings moved [Gros72+1, p. 12] and to the *paper folding sequence* (A014577)

$$\varphi = 1,\ 1,\ 0,\ 1,\ 1,\ 0,\ 0,\ 1,\ 1,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ \ldots$$

for the move directions, up for 1 and down for 0. To explain the latter name, we note that $\varphi(2^{n-1}) = 1$, $\varphi(3 \cdot 2^{n-1}) = 0$ and $\varphi(2^n - (d)_2) = d_{\rho+1}$, $\varphi(2^n + (d)_2) = 1 - d_{\rho+1}$, if $\rho < n-1$, so that $\varphi(2^n + (d)_2) = 1 - \varphi(2^n - (d)_2)$ for all $(d)_2 \in [M_n]$, which is the defining characteristic of the sequence of turns in a strip of paper folded repeatedly in the same direction; cf. [HKP18, pp. 81ff]. A similar recurrence holds for the Gros sequence, namely $g(2^{n-1}) = n$ and $g(2^n + (d)_2) = g((d)_2) = g(2^n - (d)_2)$ (see [HKP18, (1.6)]). We can also conclude that ring $\rho+1$ is moved for the first time in move 2^ρ, that before this a transfer $\alpha^{[\rho)}$ to $\omega^{(\rho)}$ is performed and thereafter a transfer $\omega^{(\rho)}$ to $\alpha^{(\rho)}$. More generally, in the minimal path from $\alpha^{(n)}$ to $s \in B^n$ ring $\rho+1$ moves in every move numbered $2^\rho(2k+1) \in [(d)_2]$, where again $d = G(s)$. The condition on k translates to

$$\begin{aligned} 2^\rho(2k+1) &\leqslant \sum_{\beta=0}^{n-1} d_\beta \cdot 2^\beta \\ 2k+1 &\leqslant \sum_{\beta=0}^{\rho-1} d_\beta \cdot 2^{\beta-\rho} + d_\rho + \sum_{\beta=\rho+1}^{n-1} d_\beta \cdot 2^{\beta-\rho} \\ k &\leqslant \sum_{\beta=\rho+1}^{n-1} d_\beta \cdot 2^{\beta-\rho-1} + d_\rho - 1 \end{aligned}$$

so that (recall that k can be 0) the number of moves of ring $r = \rho + 1$ is

$$\sum_{\tau=0}^{n-\rho-2} d_{\tau+\rho+1} \cdot 2^\tau + d_\rho = \left\lfloor \frac{(d)_2}{2^r} \right\rfloor + d_{r-1}$$

In [Gros72+2, p. 95] Gros gives two recipes to find this number from the possibly non-integer ratio $\dfrac{(d)_2}{2^r}$. One is based on the classical rounding function $\lfloor x \rceil = \lfloor x + \frac{1}{2} \rfloor$, the other uses a clever parity argument:

Proposition 6. *In the shortest path from* $\alpha^{(n)}$ *to* $s \in B^n$*,* $n \in \mathbb{N}$*, ring* $r \in [n]$ *is moved* $\left\lfloor \dfrac{(d)_2}{2^r} \right\rfloor$ *times, where* $d = G(s)$*. The same number is obtained by choosing among* $\left\lfloor \dfrac{(d)_2}{2^r} \right\rfloor$ *and* $\left\lfloor \dfrac{(d)_2}{2^r} \right\rfloor + 1$ *the even value, if* $s_r = 0$*, the odd value, if* $s_r = 1$*.*

As an example, let us consider the classical problem of getting nine rings onto (or off) the loop, i.e. $n = 9$ and $s = \mathrm{o}^{(9)}$. Then $(d)_2 = \ell_9 = 341$ and we can calculate $\dfrac{(d)_2}{2^r}$ for $r = 9, \ldots, 1$ to obtain 1, 1, 3, 5, 11, 21, 43, 85, 171 from Proposition 6. This list was known by the Japanese mathematician Yoriyuki Arima (1714–1783) in the 18th century, cf. [Döl18, Table 2]. It can also be deduced from the observation that ring r plays the same role as ring 1 in the minimal path from $\alpha^{(n-r+1)}$ to $\mathrm{o}^{(n-r+1)}$ so that we can also consider $\frac{1}{2}\ell_{10-r}$ instead of $\dfrac{(d)_2}{2^r}$. The list is the beginning of the sequence of differences of ℓ, i.e. the *Jacobsthal sequence* (A001045), see [Hinz17, Section 7].

We get another immediate consequence of Proposition 6 if we subtract the number of moves of ring 2 in the classical task for $n \geqslant 2$ rings:

Corollary 7. *The number of moves between $\alpha^{(n)}$ and $\mathrm{o}^{(n)}$ reduces to $\ell_n - \lfloor \frac{1}{4}\ell_n \rceil = 3\ell_{n-2}+1$, the* Purkiss sequence (A051049)*, see [Hinz17, Section 6], if we apply the accelerated count. The whole path from $\alpha^{(n)}$ to $\omega^{(n)}$ reduces for $n \geqslant 2$ to $M_n - \lfloor \frac{1}{4}M_n \rceil = 3\cdot 2^{n-2}-1$ (A083329) moves [Luc82, p. 184]. Note that the first sequence oscillates between 2^{n-1}, for n odd, and M_{n-1} for even n, while the second sequence has binary representation 101^{n-2}.*

Contrary to his customs, Gros even produces a proof for our Proposition 6 in [Gros72+2, pp. 97–99], albeit a little obscure. Here is our approach.

Proof of Proposition 6. We have

$$\left\lfloor \frac{(d)_2}{2^r} \right\rceil = \left\lfloor \frac{(d)_2}{2^r} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{1}{2}(d_{r-1}+1) + \sum_{\beta=r}^{n-1} d_\beta \cdot 2^{\beta-r} \right\rfloor = \left\lfloor \frac{(d)_2}{2^r} \right\rfloor + d_{r-1}$$

This number can also be written as $\displaystyle\sum_{\beta=r+1}^{n-1} d_\beta \cdot 2^{\beta-r} + d_r + d_{r-1}$, where the big sum represents an even number, so that the whole number is even iff $d_r = d_{r-1}$, which in turn is equivalent to $s_r = 0$. But the latter means that the number of moves performed on ring r must be even (because it started with 0). □

3.4 Which ring to move first?

Gros noticed [Gros72+1, p. 14] that for every non-extreme state with movable rings 1 and $\rho + 1 > 1$ (recall that rings are numbered w.r.t. s) we have $d_0 \neq d_\rho$ and that one has to move the ring r with $d_{r-1} = 0$ in order to approach $\omega^{(n)}$, the one with $d_{r-1} = 1$ to approach $\alpha^{(n)}$ [Gros72+1, p. 15]. This can be read from the proof of Lemma 4, where we used the 0 bit with the goal to increase the distance to $\alpha^{(n)}$. Recall also that the move with number $(d)_2$ was from $(d)_2 - 1$ to $(d)_2$, so to get from the state corresponding to d to $\alpha^{(n)}$, we have to use the same ring, but the opposite direction in that move. Here is a simple algorithm:

If we know $d \neq 0^n$, we proceed as follows in order to approach $\alpha^{(n)}$ (cf. [Gros72+2, pp. 112–115]):

1. If $d_0 = 1$, then move ring 1 from $1 - d_1$ to d_1.

2. Otherwise let ρ be the number of trailing 0s in d and move ring $\rho + 1$ from $1 - d_{\rho+1}$ to $d_{\rho+1}$.

If we know s (or either σ) different from $\alpha^{(n)}$, we make use of Lemma 1 and proceed as follows:

1. If the number of 1s in s (or σ) is odd, then move ring 1 from s_1 to $1 - s_1$ (or from σ_0 to $1 - \sigma_0$).

2. Otherwise let $r - 2$ be the number of trailing 0s in s (or $\rho - 1$ be the number of trailing 0s of σ) and move ring $r(= \rho + 1)$ from s_r to $1 - s_r$ (from σ_ρ to $1 - \sigma_\rho$).

The rest of the solution is governed by the rule never to move the same ring twice in a row.

The original task, namely to get all rings off the loop, is now solved: the starting state $\mathrm{o}^{(n)}$ has n bits 1; therefore, if the number of rings is odd, we move ring 1 down from the loop[12], if the number of rings is even, we move ring 2 down; then we continue in the aforementioned manner.

Let us finally point out that there is no way to decide directly which ring to move first if we want to go from some non-extreme s to some non-extreme $t \neq s$. We have to determine the sign of the difference $(G(t))_2 - (G(s))_2$ in order to deduce whether we have to increase or decrease $(G(s))_2$; see [Luc82, p. 179]. In particular, this applies when either s or t is $\mathrm{o}^{(n)}$. If, e.g., in the spirit of Gros we want to get from the situation in Figure 1, i.e. $s = \mathrm{o}^{(11)}$ with $(G(s))_2 = \big((10)^5 1\big)_2 = \ell_{11} = 1365$, to the state $t \in B^{11}$ representing the current year on the baguenaudier with 11 rings, i.e. $(G(t))_2 = 2026$, we calculate $(G(t))_2 - (G(s))_2 = 661$. As this is positive, we have to approach $\omega^{(11)}$, which is achieved, since $\rho = 1$ and $d_0 = 1 \neq 0 = d_1$, by moving ring $r = 2$ first; see the beginning of the current section. The last move (by ring 2) then leads us to $t = 10000011111$. Note that our 11-ring baguenaudier will expire by the end of 2047!

References

[Afr82] S. N. Afriat, *The Ring of Linked Rings*, Duckworth, London, 1982.

[Bra22] R. Brasseur, *Dictionnaire des professeurs de mathématiques en classe de mathématiques spéciales entre 1852 et 1914*, pdf, 2022.

[Döl18] S. Döll and A. M. Hinz, Kyū-renkan—the Arima sequence, *Adv. Stud. Pure Math.* **79** (2018), 321–335.

[Gros72+1] L. Gros, *Théorie du Baguenodier par un clerc de notaire lyonnais*, Aimé Vingtrinier, Lyon, 1872.

[12]This is the reason why most Linked Rings games on the market have an odd number of rings.

[Gros72+2] L. Gros, *Traité du Baguenodier*, 3rd edition, manuscript, 1872.

[HH17] A. Heeffer and A. M. Hinz, "A difficult case": Pacioli and Cardano on the Chinese Rings, *Recreat. Math. Mag.* **8** (2017), 5–23.

[Hinz17] A. M. Hinz, The Lichtenberg sequence, *Fibonacci Quart.* **55**(5) (2017), 2–12.

[Hinz19] A. M. Hinz, Are the Chinese Rings Chinese?, in: J. N. Silva (ed.), *Proceedings of the Recreational Mathematics Colloquium VI—G4G Europe*, Ludus, Lisboa, 2019, 83–101.

[HKP18] A. M. Hinz, S. Klavžar, and C. Petr, *The Tower of Hanoi—Myths and Maths*, 2nd edition, Springer/Birkhäuser, Cham, 2018.

[Luc82] É. Lucas, *Récréations Mathématiques*, Gauthier-Villars, Paris, 1882.

[Rou23] L. Rougetet, *Le binaire au bout des doigts*, EDP Sciences/Université Grenoble Alpes, Les Ulis/Grenoble, 2023.

[Rou86] E. Roullet, Des devoirs des magistrats, *Procès-verbal de l'audience solennelle de rentrée de la Cour d'Appel de Lyon*, Mougin-Rusand, Lyon, 1886.