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Connecting parking functions to another classic car-related puzzle

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Abstract

Parking functions are a well-studied area of combinatorics that have connections to Cayley's tree formula, Catalan numbers, Dyck paths, and much more. We connect parking functions to another classic car-related puzzle called the gas stations puzzle. Looking at it through this lens, we give two new proofs of results in this area that are natural extensions of the original connection.

1 Introduction

The parking puzzle and the gas station puzzle are two well-known car-related mathematical puzzles with extremely elegant solutions. This work describes some of the consequences of their close connection first noticed by Reddit user /u/bobjane [/u/22].

Here are the two puzzles:

Puzzle 1 (Parking Puzzle [KW66])**.** There are n cars and n parking spaces, all initially empty. The parking spaces are in a line, left to right, along a one-way street. Each car has a preferred parking space, with the possibility that two or more cars prefer the same space. One at a time, each car will try to park it its preferred space. If its preferred space is full, it will then move to the next available space to the right of the preferred space. If there is no more spaces available to the right, then that car does not park at all. If the list of preferred spaces is random, what is the probability all the cars park? Or, equivalently, how many preference lists lead to the cars all successfully parking?

Puzzle 2 (Gas Stations puzzle [Lov79])**.** A number of gas stations are placed haphazardly around a large circular track. Altogether, the gas stations have just enough gas to traverse the circle once. Show that there is some starting point along the track (necessarily at a gas station) such that a Ford car, starting with no gas, can make it all the way around the track.

Now is a good time to stop reading an try these puzzles, or try to find the connection between them.

The parking puzzle gave rise to the combinatorial object known as parking functions. Parking functions are central to a signifcant portion of combinatorics, having connections to Cayley's tree formula, Catalan numbers, Dyck paths, and much more. See Yan's excellent survey on the subject for more information on parking functions [Yan15]. Two results in this area of interest to us are the enumeration of rational parking functions [ALW16], a generalization of parking functions, and the enumeration of labeled trees with forced subtrees [LMS13], which is a generalization of Cayley's tree formula. That gas station puzzle has not traditionally be seen as being connected to this area of combinatorics, but perhaps it should be. Using the gas station puzzle and its elegant solution, we give a new proof solving the parking puzzle. This connection between the two puzzles gives additional bonuses: first, by simply changing the amount of gas at the gas stations, we get an enumeration of rational parking functions! Second, by having different amounts of gas at different gas stations, we get the generalization of Cayley's tree formula!

If you're interested in all of this, see Wästlund for another puzzle-related take on this subject [Wä20].

2 The Parking Puzzle Solution

Note that this puzzle was original proposed and solved by Konhiem and Weiss [KW66] (Further note: we do not condone the sexism present in the original version)

Recall this puzzle involves n cars trying to park in n spaces, where each car has a preferred parking space. Notice we can identify the list of preferred spaces with a simple list of numbers $(a_1, \ldots, a_n)$ where car j prefers space a_j. For simplicity of latter results, assume each a_j comes from the set $\{0, \ldots, n-1\}$. Such lists that allow for the cars to park are call *parking functions*. For example, with $n = 3$, $(0, 2, 0)$ is a parking function, since the first car parks in spot 0, the second car parks in spot 2, and the third car tries spot 0 but is able to take spot 1. However, $(2, 1, 1)$ is not a parking function: the first car parks in spot 2, the second car in spot 1, and the last car tries spots 1 and 2 and then gives up.

After you try a few examples, you may see that the permutation of the list does not matter in determining whether it is a parking function (although the different permutations do matter when counting how many lists there are). Let σ be a permutation such that $a_{\sigma(1)} \leqslant a_{\sigma(2)} \leqslant \cdots \leqslant a_{\sigma(n)}$. To make σ unique, we will break ties via the index, so that if $a_i = a_j$ for $i < j$, we will set $\sigma(i) < \sigma(j)$. Thus, to test if $(a_1, \ldots, a_n)$ is a parking function, we could test the weakly increasing version $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ instead.

What really matters for parking functions is this: there are not too many cars trying for the same last few spaces, or equivalently we have enough cars trying to park on the earlier spaces. Notationally, we can write this as $a_{\sigma(i)} \leqslant i - 1$. We call this the *parking condition*. In other words, the $\sigma(1)$th car must prefer space 0, the $\sigma(2)$th car must prefer space 0 or 1, the $\sigma(3)$th car must prefer space 0, 1, or 2, and so on. This is a necessary and sufficient condition.

For $n = 3$, the entire list of parking functions is

$$\begin{gathered}(0,0,0),\\(0,0,1),(0,1,0),(1,0,0),\\(0,0,2),(0,2,0),(2,0,0),\\(0,1,1),(1,0,1),(1,1,0),\\(0,1,2),(0,2,1),(1,0,2),(1,2,0),(2,0,1),(2,1,0)\end{gathered}$$

This totals 16 parking functions. Trying to count these things is not easy at first blush: it seems to be weird groups of binomial coefficients. However, the first few values are $1, 3, 16, 125$, which you may recongnize as $(n+1)^{n-1}$. Why does it work out to be such a simple formula?

Theorem 3 ([KW66]). *There are $(n+1)^{n-1}$ parking functions.*

Proof. The following elegant proof was given by Pollak (see [Rio69]). Take the same n cars, but now suppose there are $n+1$ spaces arranged in a circle (think $\{0, 1, 2, \ldots, n\}$ mod $n+1$). Thus if a car cannot park in space $n-1$, it can move onto n, and if that is full move back to space 0 and so on. Clearly now all the cars will be able to park, and there will be one empty spot. If that spot is the nth space, then the nth space was not needed at all and the cars could successfully park in the original version. Of the $(n+1)^n$ possible preference lists with the extra parking space, it is equally likely the final empty spot is any of the $n+1$ spots, and therefore in a fraction of $\frac{1}{n+1}$ of them it is the nth spot that is empty. Thus, there are $\frac{1}{n+1}(n+1)^n = (n+1)^{n-1}$ possible lists in the original puzzle where the cars can park. □

As a probability, the likelihood all the cars park is $\frac{(n+1)^{n-1}}{n^n}$, which one can see tends to $\frac{e}{n}$:

$$\begin{aligned}\frac{(n+1)^{n-1}}{n^n} &= \frac{1}{n} \cdot \left(\frac{n+1}{n}\right)^{n-1}\\ &\sim \frac{1}{n} \cdot e \qquad \text{(for large } n\text{)}\end{aligned}$$

3 The Gas Stations puzzle Solution

This puzzle can be found in Lovász [Lov79], Bollobás [Bol06] and Winkler [Win04].

Theorem 4. *There is some gas station that the Ford can start at and go all the way around the circle. Furthermore, if there is more than one place to start, each starting location is a place where the Ford will reach zero gas just as it is pulling into a gas station.*

Proof. We follow Winkler's elegant proof here.

Think of a different car, say a Buick, with way too much gas making the trip around the track. The Buick picks up all the gas along the way even though it isn't needed. If you keep track of how much gas the Buick has at all times, the point where the gas

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is minimized must be the starting point for the original puzzle. Indeed, if you take the minimum value of gas that the Buick has and subtract it from its own gas tank at every point, now the Buick will have zero gas at the starting point and nonnegative gas everywhere else. This is exactly the amount of gas the Ford will have starting at the point of minimum gas and continuing around the track, and so it will never run out of gas.

A corollary to this proof is that there may be more than one starting place for the Ford, and every possible starting place is a place where the Ford will reach zero fuel just as it is pulling into a gas station. □

Note that the proof works for an infinite number of gas stations, or even a continuous distribution of gas stations.

4 Connecting the Two Puzzles

So what is the connection between these two puzzles? We will use the gas stations puzzle insight to solve the parking puzzle. The connection between these puzzles was first noticed by Reddit user /u/bobjane [/u/22]. See Figure 1 for an illustration.

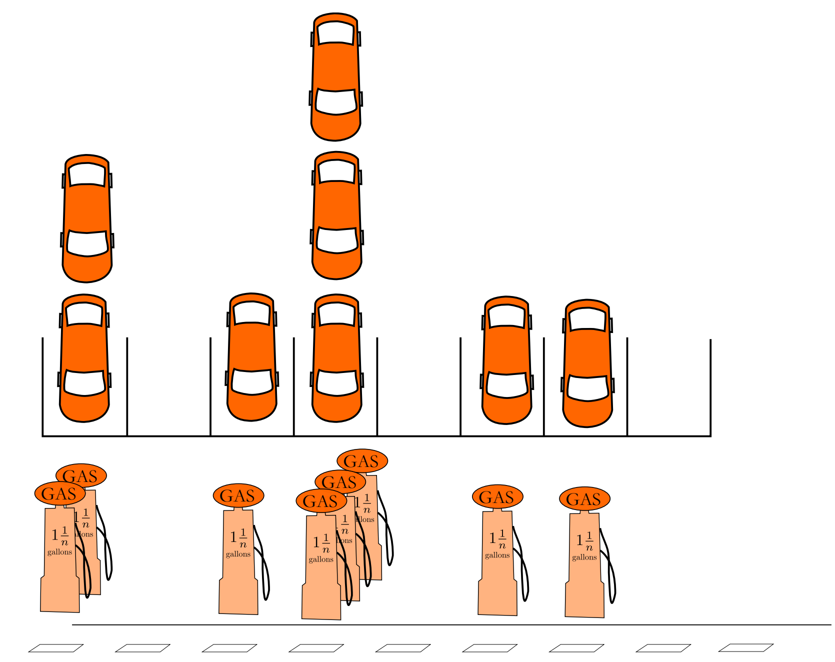


Figure 1: This figure illustrates how solutions to the parking puzzle correspond to certain scenarios in the gas stations puzzle. Note that for the gas stations puzzle, the road is circular but here is shown as straight. Feel free to glue one side of this paper to the other for a more accurate representation.

(Reproving Theorem 3, showing there are $(n+1)^{n-1}$ parking functions)

Proof. Thus, we are trying to solve the parking puzzle, which is counting the number of parking functions $(a_1, \ldots, a_n)$, where car j prefers space a_j. Just as Pollak did, we will think of these a_j as potentially taking values from 0 to n, for a total of $n+1$ possible values. Let us now bring it into the realm of the gas stations puzzle. Consider a track of length $n+1$ miles, where we place a gas station with one gallon at position a_j for each $j = 1, \ldots, n$, and for simplicity we will assume the rather poor gas mileage of one mile per gallon. Multiple gas stations are allowed at the same location. Issue: we don't have enough gas to get around the track. Okay, so let's just scale up how much gas is at each location: instead of one gallon, place $\frac{n+1}{n}$ gallons of gas at each station at position a_j for each $j = 1, \ldots, n$. By the gas stations puzzle, there must be a place along the track where a Ford can start with no gas and go all the way around the track. Cruicially, *there cannot be two places.* Why not? As we discussed, both locations must be places were the Ford reaches zero gallons. However, with fewer than n gas stations between the two stations, we see the Ford has gone an integer amount of miles but collected a non-integer amount of gas, and thus must have a non-integer amount of gas at the second location. Since 0 is an integer (see [Sei00]), it can't have zero gas at the second location, contradicting our assumption there were two locations with zero.

Now the proof proceeds much like Pollak's proof. Of all $(n+1)^n$ lists, the starting point for the Ford along the circular track is equally likely to be any of them. Thus there are $(n+1)^{n-1}$ possible lists where the starting point is position 0.

Why does this correspond to parking functions? Recall parking functions are precisely those list that satisfy the parking condition: if we reorder the list in weakly increasing order $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$, we have $a_{\sigma(i)} \leqslant i-1$. For the Ford to start at position zero and make it all the way around the track, we need it to reach every gas station. To ensure it reaches the gas station located at $a_{\sigma(i)}$, we need there to be enough gas to reach this location, and thus the Ford will need to collect at least $a_{\sigma(i)}$ gallons of fuel from earlier gas stations. Before reaching location $a_{\sigma(i)}$, the Ford will reach locations $a_{\sigma(1)}$, …, $a_{\sigma(i-1)}$, and thus it will have reached $i-1$ gas stations and collected $\frac{n+1}{n}(i-1)$ gallons of fuel. So the Ford will have enough fuel exactly when $a_{\sigma(i)} \leqslant \frac{n+1}{n}(i-1)$ for $i = 1, 2, 3, \ldots, n$. Since $i-1$ is at most $n-1$, the fraction $\frac{n+1}{n}$ may as well be rounded to 1, and hence we exactly satisfy the parking condition $a_{\sigma(i)} \leqslant i-1$. □

5 Rational Parking Functions

A natural question you may ask is what if we change the amount of gas at each station from $\frac{n+1}{n}$ to another number? Is there an interesting result? Yes, we get a ennumeration of rational parking functions!

Rational parking functions, introduced by Armstrong, Loehr, and Warrington [ALW16], are a generalization of parking functions. For q and n relatively prime, rational parking functions are sequences $(a_1, \ldots, a_n)$ of non-negative integers such that the weakly increasing rearrangement $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ satisfies $a_{\sigma(i)} \leqslant \frac{q}{n}(i-1)$ for $i = 1, \ldots, n$. Armstrong et al. went very deep into the theory of rational parking functions, and the starting point was a result that such sequences were still counted by a simple power function: q^{n-1}.

This is very surprising given that again the sequences seem a bit weird. Consider the example of $q = 3$ and $n = 5$. Here, $(2, 0, 0, 1, 1)$ would be an example of such a function, since if we rearrange in weakly increasing order $(0, 0, 1, 1, 2)$, we have $0 \leqslant \frac{3}{5} \cdot 0$, $0 \leqslant \frac{3}{5} \cdot 1$, $1 \leqslant \frac{3}{5} \cdot 2$, $1 \leqslant \frac{3}{5} \cdot 3$, and $2 \leqslant \frac{3}{5} \cdot 4$. Here is a partial list of such parking functions:

$$\begin{gathered}(0,0,0,0,0),\\(0,0,0,0,1),(0,0,0,1,0),(0,0,1,0,0),(0,1,0,0,0),(1,0,0,0,0),\\(0,0,0,0,2),(0,0,0,2,0),(0,0,2,0,0),(0,2,0,0,0),(2,0,0,0,0),\\\vdots\\(0,0,1,1,2),(0,1,0,1,2),(1,0,0,1,2),\ldots,(2,1,1,0,0)\end{gathered}$$

We can count these if we just list the ones in weakly increasing order along with the number of permutations:

$$\begin{gathered}(0,0,0,0,0)\times 1,\\(0,0,0,0,1)\times 5,\\(0,0,0,0,2)\times 5,\\(0,0,0,1,1)\times 10,\\(0,0,0,1,2)\times 20,\\(0,0,1,1,1)\times 10,\\(0,0,1,1,2)\times 30\end{gathered}$$

This gives a total of 81 rational parking functions with $q = 3$ and $n = 5$, verifying the formula q^{n-1}. Why, despite this modification, do these parking functions still work out to have such a nice formula? From the point of view of the gas stations puzzle, it's obvious: we are just using $\frac{q}{n}$ gallons of gas for each gas station instead of $\frac{n+1}{n}$! Here is our new proof:

Theorem 5 (Armstrong et. al. [ALW16]). *Consider two relatively prime positive integers q and n. Let $(a_1, \ldots, a_n)$ be a sequence of non-negative integers such the weakly increasing rearrangement $(b_1, \ldots, b_n)$ satisfies $b_i \leqslant \frac{q}{n}(i-1)$ for $i = 1, \ldots, n$. Then there are q^{n-1} such sequences.*

Proof. For the n-tuple $(a_1, \ldots, a_n)$ consider each a_i as a location along a circular track of lenth q where q/n gallons of gas will be placed. Notice that the condition of the Ford being able to start at 0 and go around the track is exactly the condition $a_{\sigma(i)} \leqslant \frac{q}{n}(i-1)$, since that means the Ford will have enough gas to reach the gas station represented by $a_{\sigma(i)}$. So how many such sequences are there? Since q and n are relatively prime, we see that the Ford cannot reach gas zero more than once along the track, and therefore the Ford has a unique starting point. Therefore, of all q^n possible sequences, exactly $1/q$ start at position 0, and therefore there are q^{n-1} such sequences. $\square$

6 Trees with forced subtrees

Now that we have a nice result where we change the amount of gas at all the gas stations, you might next wonder the following: what if different gas stations have different amounts of gas? Do we get another result from this? The answer is yes, but it will take a detour to discuss how these parking functions relate to counting trees.

One of the most beautiful results in the combinatorics of graph theory is Cayley's formula which states there are $(n+1)^{n-1}$ labeled trees on $n+1$ vertices. This is well known to be connected to parking functions; as we will see, one can construct an explicit bijection between parking functions of length n and trees on $n+1$ vertices. An interesting but less-well known result is that we still have a nice formula even if we "force" the tree to have a haphazzard collection of subtrees. Forcing a subtree will be equivalent to adding extra gas to a particular gas station.

For our purposes, it is easier to discuss rooted labeled trees: the vertex set will be $V = \{0, 1, \dots n\}$, where 0 is considered the root. There are just as many rooted trees as unrooted trees: consider the "forgetful" bijection where we just forget the tree has a root! Thus, we don't lose anything by considering rooted trees.

Before we describe a bijection between rooted labeled trees and parking functions, we will state the result with forced subtrees. Let $\{F_i\}_{i=1}^k$ be a collection of vertex-disjoint rooted trees on the vertex set $V \setminus \{0\} = \{1, 2, \dots, n\}$. How many ways are there to complete this forest into a tree T rooted at 0? We assume that the root of each subtree F_i must be the closest vertex to the root of the entire tree, and we call this *respecting the root of each subtree*. Despite these forced subtrees, we still get a power function as the result.

Theorem 6. *If there are ℓ total edges within the forest $\{F_i\}_{i=1}^k$ on vertex set $\{1, 2, \dots, n\}$ where each tree F_i has a root, then there are $(n+1)^{n-1-\ell}$ ways of connecting them into a rooted tree rooted at 0 respecting the roots of each subtree.*

This result is previously known; for example, it is equivalent to Lemma 6 from Lu, Mohr, and Székely [LMS13] where they prove this result and then apply it to matchings in hypergraphs and the lopsided version of the Lovász local lemma.

As a nice example of this result, consider how many labeled trees are there with an edge between vertices 1 and 2? If we consider this edge as a subtree rooted at 1, we have there are $(n+1)^{n-2}$ such trees. We can also root this at 2 and get another $(n+1)^{n-2}$ such trees, for a total of $2(n+1)^{n-2}$ trees containing the edge betweeen 1 and 2.

How does all this relate to the gas stations puzzle? As I mentioned before, forcing subtrees is a natural result of the gas stations having different amounts of gas. But before we can see this explicitly, let's see a way to find Cayley's formula using the gas stations puzzle.

7 Connecting parking functions to rooted, labeled trees

Consider n gas stations represented by the n-tuple $(a_1, \dots, a_n)$. Each gas station has two numbers associated with it: its *index* i, and its *location* a_i. We also make use of

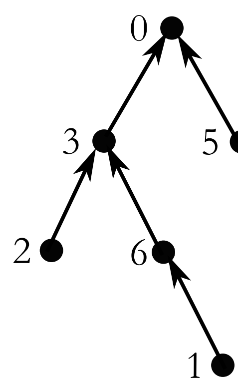
σ, the same permutation from before causing $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ to be weakly increasing (with ties broken by the index). We will extend this permutation to include 0 by simply declaring $\sigma(0) = 0$.

Given $(a_1, \ldots, a_n)$ such that $a_{\sigma(i)} \leqslant i - 1$, we need to create a rooted tree on vertex set $V = \{0, 1, \ldots, n\}$. **Here is the rule: we connect vertex i to $\sigma(a_i)$.**

For example, consider the n-tuple $(4, 1, 0, 2, 0, 1)$. First we determine what σ is for this n-tuple:

location a_i	4	1	0	2	0	1
index i	1	2	3	4	5	6
$\sigma(i)$	3	5	2	6	4	1
$a_{\sigma(i)}$	0	0	1	1	2	4

We will think about constructing this graph in order of σ, so we will start with vertices 3 and 5. Since $\sigma(a_3) = \sigma(a_5) = \sigma(0) = 0$, these are attached to the root zero, and we add edges $3 \to 0$ and $5 \to 0$. Next we attach vertices 2 and 6: since $\sigma(a_2) = \sigma(a_6) = \sigma(1) = 3$, we add the edge $2 \to 3$ and $6 \to 3$. Next we add vertex 4, and since $\sigma(a_4) = \sigma(2) = 5$, we add the edge $4 \to 5$. Finally, we see $\sigma(a_1) = \sigma(4) = 6$, so we add the edge $1 \to 6$. Our final graph looks like this:



Because we attach vertices in σ order, the parking condition $a_{\sigma(i)} \leqslant i - 1$ guarantees that every vertex is attached to an earlier vertex. Since each vertex is attached to exactly one earlier vertex, the process must yield a tree.

We can also reverse this process, showing it is a bijection. Given a tree on vertex set $V = \{0, \ldots, n\}$ rooted at zero, we first note that any vertex i attached to 0 forces $a_i = 0$, and these i values, however many there are, will be $\sigma(1), \sigma(2), \ldots$. For any vertex i attached to $\sigma(1)$, we know $a_i = 1$, and this gives the next values of σ. Any vertex i attached to $\sigma(2)$ will be such that $a_i = 2$. Continuing this process, at no point is there abiguity, and we evetually have the completed n-tuple.

Thus, we can see that parking functions are in exact coorespondance with labeled trees on $n + 1$ vertices. Thus there are $(n + 1)^{n-1}$ such labeled trees, which is Cayley's formula!

8 Relating gas to forced subtrees

For simplicity we will prove just a small portion of Theorem 6. Here, the forest we will be using is very simple: it will consist of node 2 being the child of node 1, and all other nodes are single vertices. Because the $2 \to 1$ subtree has two nodes, this corresponds in the gas stations puzzle to a gas station with twice the gas.

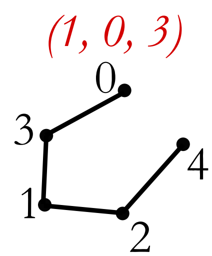
Theorem 7. *The number of trees on vertex set $V = \{0, 1, \ldots, n\}$, rooted at 0, where we force node 2 to be the child of node 1, is given by $(n+1)^{n-2}$.*

Proof. Consider $n-1$-tuples $(a_1, a_3, a_4, \ldots, a_n)$, with each $a_i \in \{0, \ldots, n\}$. Each a_i will represent the location where $\frac{n+1}{n}$ gallons of gas will be placed on a circular track of length $n+1$. However, a_1 is special in that it represents where $2\left(\frac{n+1}{n}\right)$ gallons of gas will be placed. Once again, we can see that no combination of gas stations, except all of them, lead to a integer amount of gas, and therefore the Ford's starting point must be unique. The number of sequences starting at 0 is all the sequences $(n+1)^{n-1}$ divided by $n+1$, which gives $(n+1)^{n-2}$. The sequence then must satisfy the parking condition $a_{\sigma(i)} \leqslant i-1$ just as before, with the caveat that a_2 does not exist and is "included" with a_1.

What remains is to show the correspondance of such sequences with rooted trees where 1 is the parent node of node 2. Again, we will root the tree at 0, and we again follow the rule: we connect vertex i to $\sigma(a_i)$. We will proceed as before, with two modifications: first, in creating the permutation σ, if $\sigma(i) = 1$, then we will force $\sigma(i+1) = 2$. Second, when we attach 1 to $\sigma(a_1)$, we add two edges $1 \to \sigma(a_1)$ and $2 \to 1$.

Just as before, this process creates a bijection between valid $n-1$-tuples where the car can start at location 0 and rooted labeled trees with the edge $2 \to 1$. But since a_1 gives a double helping of gas, this allows for the situation where we never add a later edge involving vertex 1 but do add an edge going towards vertex 2.

For example, consider the case of $n = 4$, and the $n-1$-tuple $(a_1, a_3, a_4) = (1, 0, 3)$.



For this $n-1$-tuple, we have the following values:

location a_i	1	–	0	3
index i	1	2	3	4
$\sigma(i)$	3	1	2	4
$a_{\sigma(i)}$	0	1	–	3

Notice the dashes are because index 2 does not have a location, since we are considering it being "lumped in" with the double helping of gas at index 1. In creating the graph,

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we start with $3 \to 0$ since $\sigma(a_3) = \sigma(0) = 0$. Next we add the edge $1 \to 3$, since $\sigma(a_1) = \sigma(1) = 3$. At this point we automatically add $2 \to 1$ since that is forced. Finally, since $\sigma(a_4) = \sigma(3) = 2$, we connect $4 \to 2$. Notice that $(1, 0, 3)$ is not a normal parking function, since normally we would not have enough gas to reach position 3 just having gas stations at locations 0 and 1. However, since there is a double helping of gas at position 1, we can reach position 3 and everything works. While permutations of normal parking functions are still parking functions, that does not work here. In particular, the $n-1$-tuple $(3, 0, 1)$ is not allowed, since the double helping of gas is in the wrong place to make it around the track. □

See Figure 2 to see an example of Theorem 7 with $n = 4$. If we want to generalize this theorem to Theorem 6, we simply add gas stations with the amount of gas in proportion to whatever the order of the the subtree is we want to force.

9 Conclusion

We will end by mentioning a connection to Fermat's last theorem [Wil95]. Since Theorem 6 involves arbitrary power functions, does that mean there is a combinatorial interpretation involving trees or parking functions of the famous equation $a^n + b^n \neq c^n$ for $n \geqslant 3$? There is in fact a connection here. We don't have enough space to get into the details, but we encourage the reader to work it out in the margin of this page.

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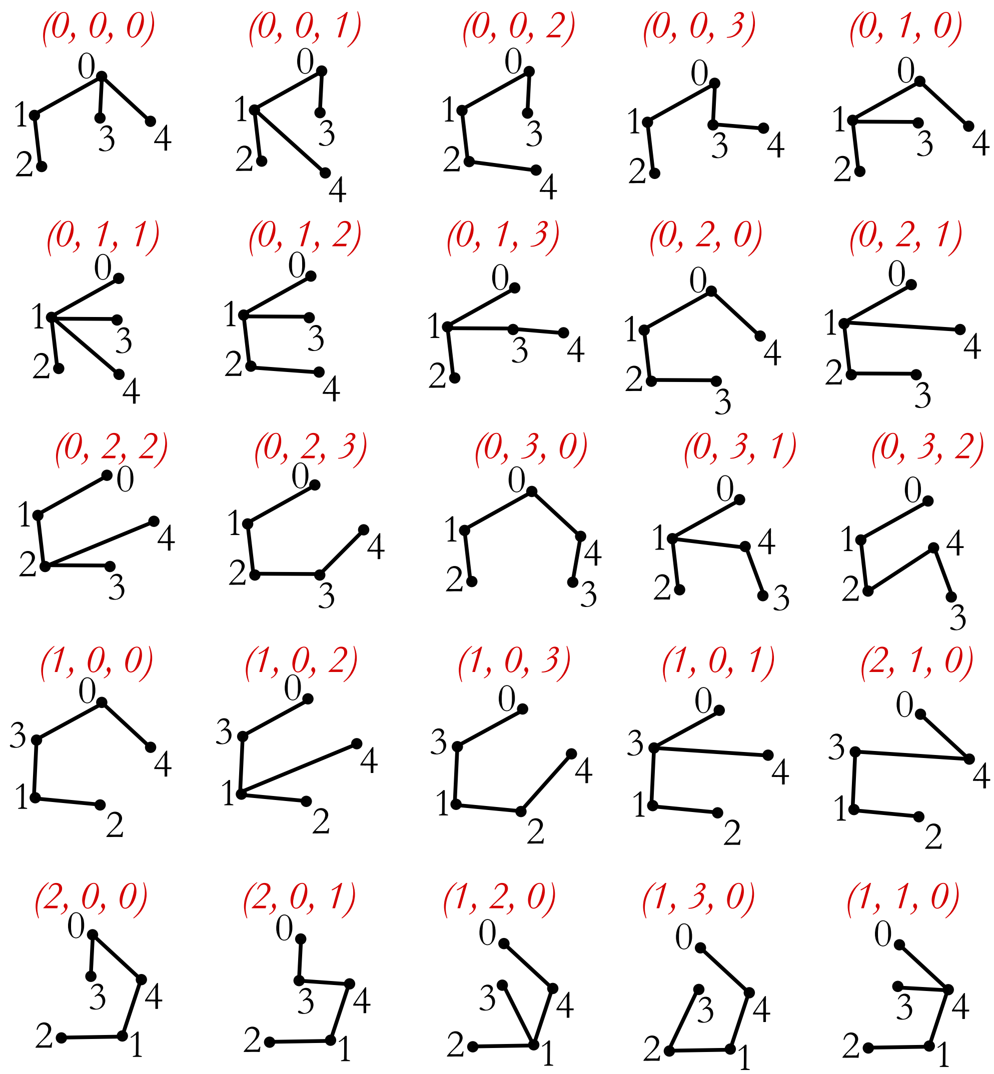

Figure 2: The 25 labeled rooted trees with on $n+1=5$ vertices with node 2 the child of node 1. Above each tree is the corresponding $n-1$-tuple.

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