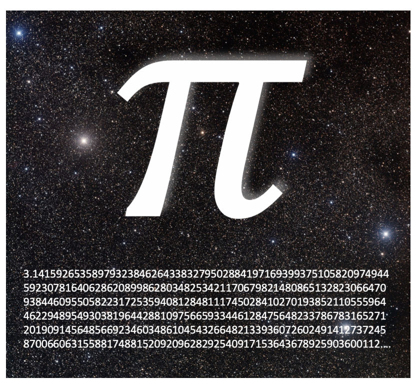
Baking star π

Sunil K. Chebolu [1]

[1] *Illinois State University, Department of Mathematics, Normal, IL 61790, USA,*
schebol@ilstu.edu

The number π is arguably the most famous number in mathematics. From the notebooks of professional mathematicians to the sweatshops of amateurs, π has been a source of wonder, fascination, and mystery since antiquity. To honor this remarkable number, every year, March 14^{th} (3/14) is celebrated in the US as the π day. Coincidentally, March 14^{th} is also the birthday of Albert Einstein.

Recall that π is the ratio of any circle's circumference to that circle's diameter. [1] It is an irrational number (a number that cannot be expressed as the ratio of two integers) whose decimal expansion is 3.141592653589793..., a non-terminating and non-repeating decimal. π is also known to be transcendental. That is, π cannot be the root of any non-zero polynomial equation with integer coefficients. Even though π has been studied since ancient times, proofs of these basic facts had to wait until the 18^{th} and 19^{th} centuries. But the main reason π is interesting and exciting is that it appears unexpectedly in many places in mathematics and physical sciences. π seems to be everywhere, also in the sky! *Did you know that you can generate the digits of π using stars?* This note explains the recipe to bake this delicious π from stars.



One of the first interesting results where π made a surprising appearance was the following remarkable formula due to Euler [Dun99]:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \frac{1}{36} + \cdots = \frac{\pi^2}{6}$$

[1] A formal proof that this ratio does not depend on the radius of the circle can be given using Calculus.

Computing the sum of this series was called the Basel Problem. When Pietro Mengoli first proposed the problem in 1644, it defeated the leading mathematicians of the day until Euler cracked it in 1735 at age 28, which brought him instant fame.

There are many avatars of Euler's formula. For instance, the fraction of lattice points on the plane visible from the origin is $6/\pi^2$, which is almost 61%. Equivalently, the probability $P(x, y)$, that two randomly chosen positive integers, x and y, are relatively prime is [Aps76]

$$P(x, y) = \frac{6}{\pi^2}$$

The last result is a basis for a probabilistic method to get the digits of π: pick random pairs of integers with replacement from a given random sample of positive integers and determine the proportion of relatively prime pairs. Equating this experimental probability with the theoretical probability of $6/\pi^2$, we get an approximation for π.

Because stars in the night sky seem to be randomly distributed on the celestial sphere, pairwise angular distances among them are an excellent source for simulating a large random sample of numbers from which we can compute π using the above-mentioned probabilistic method. In a note "Pi in the Sky," that appeared in *Nature*, Robert Matthew (2002) used this idea to compute π from the stars. Using data of 100 stars, Matthew estimated π to one decimal place. Here we use data of 1000 stars and get an estimate of π to 4 decimal places.

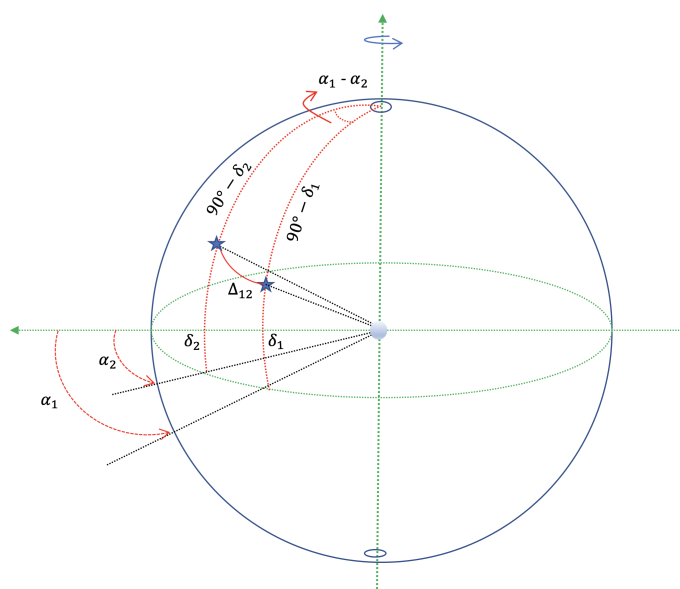


Figure 1: Angular separation Δ_{12} of two stars with equatorial coordinates (α_1, δ_1) and (α_2, δ_2) on the celestial sphere

Consider a large random sample of N stars on the celestial sphere with equatorial coordinates. Let (α_k, δ_k) denote the right ascension and declination of the kth star in this sample. Using the fundamental cosine formula in spherical trigonometry [Wen, Section 3], the angular distance Δ_{ij}

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between the ith and jth star is given by

$$\cos\Delta_{ij} = \sin\delta_i \sin\delta_j + \cos\delta_i \cos\delta_j \cos(\alpha_j - \alpha_i), \quad 1 \leqslant i < j \leqslant N$$

In this formula, right ascension (RA) takes values between 0 and 360 degrees [2] and declination has a value between −90 and +90 degrees.

Figure 2: A random distribution of stars in the constellation of Saggita. Image Credit: Tim Stone.

The set $\{\cos\Delta_{ij},\ 1 \leqslant i < j \leqslant N\}$ can be viewed as a random sample from the uniform distribution on $[-1,1]$. These random numbers are converted into a random sample of positive integers in $[0,10^n]$ using the following sequence of transformations:

$$[-1,1] \xrightarrow{1+x} [0,2] \xrightarrow{\frac{x}{2}} [0,1] \xrightarrow{10^n x} [0,10^n] \xrightarrow{[x]} \{k \in \mathbb{Z} : 0 \leqslant k \leqslant 10^n\}$$

($\mathbb{Z}$ is the set of integers and n is a positive integer that depends on the size of sample. For a sample of $N = 1000$ stars, we took $n = 8$. $[x]$ denotes the integer part of x.)

Thus, we get a random sample of $\binom{N}{2}$ positive integers in $[0,10^n]$ from the stars:

$$r_{ij} = \left[10^n\left(\frac{1+\cos(\Delta_{ij})}{2}\right)\right], \quad 1 \leqslant i < j \leqslant N$$

We then pick random pairs of integers from this sample with replacement and determine the proportion p of pairs that are relatively prime using the Euclidean algorithm. If the sample is sufficiently random and large, this empirical value would be close to the theoretical value of $6/\pi^2$. Equating these two quantities gives the following approximation for π.

$$\pi \approx \sqrt{\frac{6}{p}}$$

Applying this recipe to the 100 brightest stars in the night sky, Matthew got $p \approx 0.613333$. Therefore,

$$\pi \approx \sqrt{\frac{6}{0.613333}} = 3.12772$$

[2]Technically, RA is expressed in hours, minutes, and seconds. That can be converted into degrees using the equation 1 hour = 15 degrees.

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Remarkably, with just 100 stars, this method gave an approximation for π that is within 0.5% of its correct value. What if we increase the sample size? Using the data of right ascension and declination of 1000 random stars from [Git] we applied the above method using SageMath software. The relevant code can be seen here [Sage]. Each iteration of this program picks one million pairs of integers at random in a random sample (generated from 1000 random stars) in $[0, 10^8]$. This being a probabilistic method, we get a slightly different answer every time we run the program. Ten iterations of this computation gave an average estimate of 3.141540567 for π, an error that is less than 0.002%.

The above method can also be viewed as a statistical hypothesis test. Since the error in the estimate for π obtained using this method is pretty low, one may also argue that it supports the hypothesis that the bright stars are randomly distributed on the celestial sphere.

In summary, we have seen how one can calculate the digits of π from the stars. This is interesting because, a priori, these two seem unrelated. It shows an inextricable link between π, an entity from the abstract universe of mathematics, and the stars, which belong to our physical universe. It reinforces Galileo's spirit: *Mathematics is the language in which God has written the universe.*

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References

[Aps76] Apostol, Tom M. (1976) Introduction to Analytic Number Theory, *Springer-Verlag.*, xii+ 338.

[Dun99] Dunham, William. (1999) Euler: The Master of Us All, *Mathematical Association of America*, **22**, xxviii+185.

[Mat02] Matthews, Robert A. J. (2002) Pi in the Sky, *Nature* **374**, 681-682.

[Wen] Wenning, Carl J., Spherical Trigonometry, TCAA Guide 10.
`http://tcaa.us/TCAAGuides.aspx`

[Git] `https://github.com/astronexus/HYG-Database`

[Sage] `https://sagecell.sagemath.org/?q=davfuu`