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Article

Bit Symmetry Entails the Symmetry of the Quantum Transition Probability

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Abstract: It is quite common to use the generalized probabilistic theories (GPTs) as generic models to reconstruct quantum theory from a few basic principles and to gain a better understanding of the probabilistic or information theoretic foundations of quantum physics and quantum computing. A variety of symmetry postulates was introduced and studied in this framework, including the transitivity of the automorphism group (1) on the pure states, (2) on the pairs of orthogonal pure states [these pairs are called 2-frames] and (3) on any frames of the same size. The second postulate is Müller and Ududec's *bit symmetry*, which they motivate by quantum computational needs. Here we explore these three postulates in the transition probability framework, which is more specific than the GPTs since the existence of the transition probabilities for the quantum logical atoms is presupposed either directly or indirectly via a certain geometric property of the state space. This property for compact convex sets was introduced by the author in a recent paper. We show that bit symmetry implicates the symmetry of the transition probabilities between the atoms. Using a result by Barnum and Hilgert, we can then conclude that the third rather strong symmetry postulate rules out all models but the classical cases and the simple Euclidean Jordan algebras.

Keywords: quantum transition probability; convex sets; state spaces; quantum information; self-dual cones; Euclidean Jordan algebras; quantum logicss

1. Introduction

Generalized probabilistic theories (GPTs [1–5]) are commonly used as generic models to reconstruct quantum theory from a few basic principles and to gain a better understanding of the probabilistic and information theoretic foundations of quantum physics and quantum computing. A more specific model is the author's transition probability framework [6–9]. A distinguishing feature is the direct or indirect postulate that the transition probability must exist for the quantum logical atoms or, equivalently, that there is one unique state for each atom in which the atom carries the probability 1. This equivalent postulate is sometimes called *sharpness* [5].

A single geometric property of a compact convex set that gives rise to such a model was discovered in Ref. [9]. It is presupposed here as a given fact and we refer to Ref. [9] for background information and motivating considerations.

We use this model and the geometric property to study several well-known symmetry postulates of the GPTs and particularly Müller and Ududec's *bit symmetry*, which they motivate by quantum computational needs [4]. We show that bit symmetry implies the symmetry of the transitions probabilities between the atoms. Using a result by Barnum and Hilgert [10], we can then conclude that a stronger form of symmetry rules out all models but the classical cases and the simple Euclidean Jordan algebras.

The transition probability framework and some results from previous papers (particularly [9]) that will be needed here are briefly recapitulated in the next section. The symmetry postulates under consideration are defined and discussed in Section 3. Section 4 is dedicated to the first and most weak one. In Section 5 bit symmetry is studied and our main result (Theorem 1) is presented. The strong form of symmetry is considered in Section 6.

2. A Brief Synopsis of the Transition Probability Framework

Let Ω be any compact convex set in some finite-dimensional real vector space and let A_Ω denote the order unit space that consists of the *affine* real-valued functions on Ω; its order unit is the constant function $\mathbb{I} \equiv 1$. The state space of A_Ω consists of the positive linear functionals μ on A_Ω with $\mu(\mathbb{I}) = 1$ and is isomorphic to the convex set Ω

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with the mapping $\omega \to \delta_\omega$, $\omega \in \Omega$, where $\delta_\omega(a) := a(\omega)$ for $a \in A_\Omega$. We consider the following compact convex set in A_Ω

$$[0, \mathbb{I}] := \{a \in A_\Omega | 0 \leq a \leq \mathbb{I}\} = \{a \in A_\Omega | 0 \leq a, \|a\| \leq 1\}$$

and the set of its extreme points $ext([0, \mathbb{I}])$. For each $\omega \in \Omega$ we define the following function e_ω on Ω:

$$e_\omega(\zeta) := inf\{a(\zeta) : a \in A_\Omega, 0 \leq a \text{ and } a(\omega) = 1\}$$

for $\zeta \in \Omega$. Since $\mathbb{I}(\omega) = 1$, we have $e_\omega(\zeta) \leq 1$ for all $\zeta \in \Omega$. Generally, e_ω is not affine and does not belong to A_Ω. The following novel property of a compact convex set Ω was introduced in Ref. [9]:

($**$) *For each extreme point $\omega \in \Omega$, the function e_ω is affine (this means $e_\omega \in A_\Omega$) and $e_\omega(\zeta) \neq 1$ for all $\zeta \in \Omega$ with $\zeta \neq \omega$.*

This means that there is a smallest non-negative affine function with the value 1 at the extreme point and at no other point. If the condition ($**$) holds, then A_Ω has the following properties [9]:

(a) The set of extreme points $ext([0, \mathbb{I}])$ is an atomic orthomodular lattice with the orthocomplementation $p \to p' := \mathbb{I} - p$ for $p \in ext([0, \mathbb{I}])$. This set thus becomes a *quantum logic*; its *atoms* are the minimal non-zero elements.

(b) For each atom e in $ext([0, \mathbb{I}])$ there is one unique state $\mathbb{P}_e$ with $\mathbb{P}_e(e) = 1$. This state is pure (an extreme point of the state space).

(c) For each pure state μ there is an atom e with $\mu = \mathbb{P}_e$.

(d) Two atoms e_1 and e_2 are *orthogonal* if one of the following three equivalent conditions holds: (1) $e_1 + e_2 \leq \mathbb{I}$, (2) $\mathbb{P}_{e_1}(e_2) = 0$, (3) $\mathbb{P}_{e_2}(e_1) = 0$. Note that the orthogonality of atoms implies their linear independence.

(e) A_Ω is *spectral*; this means that each $a \in A_\Omega$ can be represented as $a = \sum_{k=1}^{n} s_k e_k$ with $s_k \in \mathbb{R}$ and pairwise orthogonal atoms $e_k \in ext([0, \mathbb{I}])$, $k = 1, 2, ..., n$, $n \in \mathbb{N}$. Here we have $0 \leq a$ iff $0 \leq s_k$ for each $k = 1, ..., n$.

Property (b), which is sometimes called *sharpness* [5], means that the *transition probability* [7] exists for each atom. It is invariant under the automorphisms U of the order-unit space A_Ω: $\mathbb{P}_e(a) = \mathbb{P}_{Ue}(Ua)$ for any atom e and any element a in A_Ω [7]. The transition probability is called *symmetric* if $\mathbb{P}_{e_2}(e_1) = \mathbb{P}_{e_1}(e_2)$ holds for each pair of atoms e_1 and e_2.

Following the notation of Ref. [1], the maximum number m of pairwise orthogonal atoms is called the *information capacity*; m is finite, since A_Ω has a finite dimension. Then $\mathbb{I} = e_1 + ... + e_m$ with some pairwise orthogonal atoms $e_1, ..., e_m$ and the sum of any m pairwise orthogonal atoms equals $\mathbb{I}$. Furthermore we have $n \leq m$ for the number n in the spectral decomposition (e).

First examples of compact convex sets with ($**$) are the strictly convex and smooth compact convex sets [8]; they all have information capacity $m = 2$. Here the transition probability is symmetric only for the Euclidean unit balls.

Further examples are the state spaces of the finite-dimensional Euclidean (formally real) Jordan algebras [9,11,12]. Their transition probabilities are always symmetric and they include the finite-dimensional classical state spaces (simplexes) as well as the state spaces of finite-dimensional quantum theory.

A more exotic example is the *triangular pillow* [11]. Its information capacity is $m = 3$, its transition probabilities are not symmetric [8] and it satisfies ($**$) [9].

The mathematical property ($**$) appears to emerge without any motivation here. However, it does follow from four physically more plausible conditions [9]; these are the above conditions (b), (c) and (e) together with a further condition that is not discussed here and that more closely ties the order relation to the state space [9].

3. The Symmetry Postulates

We consider the automorphism groups $Aut(\Omega)$ and $Aut(A_\Omega)$ of Ω and A_Ω, repectively; $Aut(\Omega)$ consists of the bijective affine transformations of Ω and $Aut(A_\Omega)$ consists of the bijective positive linear transformations U of A_Ω with a positive inverse and $U(\mathbb{I}) = \mathbb{I}$. Each one is a compact group [13,14].

For each $T \in Aut(\Omega)$ we get $T^* \in Aut(A_\Omega)$ by defining $(T^*a)(\omega) := a(T\omega)$ for $a \in A_\Omega$ and $\omega \in \Omega$. Vice versa, for each $U \in Aut(A_\Omega)$ we get $U^* \in Aut(\Omega)$ by defining $U^*\omega := \theta$, where θ is the element in Ω with $\delta_\theta = \delta_\omega U$.

The first symmetry postulate becomes that $Aut(\Omega)$ acts transitively on the extreme points of Ω [14,15]: For any two extreme points ω_1 and ω_2 there shall be $T \in Aut(\Omega)$ with $T\omega_1 = T\omega_2$. Because of (b) and (c) in section 2 this is equivalent to the property that $Aut(A_\Omega)$ acts transitively on the atoms in $ext([0, \mathbb{I}])$. This property is often called *transitivity*, but here it shall be named *weak symmetry*, which better matches our terminology of symmetry conditions.

We say that the *exchange symmetry* holds if there is $T \in Aut(\Omega)$ with $T\omega_1 = T\omega_2$ and $T\omega_2 = T\omega_1$ for any two extreme points ω_1 and ω_2 in Ω. This is equivalent to the property that there is $U \in Aut(A_\Omega)$ with $Up = q$ and $Uq = p$ for any two atoms p and q in $ext([0,\mathbb{I}])$.

We call a pair of extreme points ω_1 and ω_2 in Ω *orthogonal* if the atoms belonging to the corresponding pure states are orthogonal. In the generalized probabilistic theories, the term "*perfectly distinguishable*" [1,4,10] is often used in this case. The set Ω is called *bit-symmetric* [4] if there is a transformation $T \in Aut(\Omega)$ with $T\omega_1 = T\theta_1$ and $T\omega_2 = T\theta_2$, whenever ω_1 and ω_2 form any orthogonal pair of extreme points and θ_1 and θ_2 form any further orthogonal pair of extreme points in Ω. This becomes equivalent to the condition that any two orthogonal pairs of atoms in $ext([0,\mathbb{I}])$ can be mapped to each other by a transformation in $Aut(A_\Omega)$. Bit symmetry is considered a quantum computational requirement, since it is thought that a quantum computer must be capable to reversibly transfer any logical bit to any other logical bit.

Strong symmetry [10] means that any family of pairwise orthogonal extreme points of Ω (equivalently, atoms in $ext([0,\mathbb{I}])$) can be mapped to any other such family with the same number of elements by a transformation in $Aut(\Omega)$ (or $Aut(A_\Omega)$). Such families are called *frames* in the GPTs [1,5,10].

Obviously, the strong symmetry implies the bit symmetry which again implies weak symmetry. Moreover, the exchange symmetry implies the weak symmetry, but its relation to the bit symmetry and the strong symmetry is not clear.

An immediate important consequence of the exchange symmetry is that the transition probability becomes symmetric. For any two atoms e_1 and e_2 we have an automorphism U with $Ue_1 = e_2$ and $Ue_2 = e_1$ and then $\mathbb{P}_{e_1}(e_2) = \mathbb{P}_{Ue_1}(Ue_2) = \mathbb{P}_{e_2}(e_1)$. As shown in Ref. [9] the symmetric transition probability then results in an inner product $\langle \, | \, \rangle$ on A_Ω with a self-dual cone and $\mathbb{P}_{e_1}(e_2) = \langle e_1 | e_2 \rangle$ for any atoms e_1 and e_2. The main result of this paper will be that bit symmetry has the same consequences, but this is somewhat more difficult to show.

The finite-dimensional Euclidean Jordan algebras do not generally satisfy the above symmetry postulates. Bit symmetry and strong symmetry hold only in the simple (or non-decomposable or irreducible) algebras and in the abelian algebras. Weak symmetry and exchange symmetry hold also in the decomposable algebras, if they are direct sums of *isomorphic* factors, but do not hold in direct sums of non-isomorphic factors.

Note that, with information capacity $m = 2$, weak symmetry, bit symmetry and strong symmetry become immediately equivalent. If $Up = Uq$ holds for two atoms p and q and $U \in Aut(A_\Omega)$, then $U(\mathbb{I} - p) = U(\mathbb{I} - q)$, but any orthogonal pair of atoms consists of an atom e and the atom $e' = \mathbb{I} - e$ in this case. Moreover frames with more than two elements do not exist.

Several further symmetry postulates for the compact convex sets and the GPTs can be found in the mathematical and physical literature, but do not play any role here.

4. Weak Symmetry

Lemma 1. *Let Ω be any finite-dimensional weakly symmetric compact convex set with the property (**) and information capacity m.*

(i) There exists an $Aut(A_\Omega)$-invariant state μ_{inv} on A_Ω and $\mu_{inv}(e) = 1/m$ for every atom e in $ext([0,\mathbb{I}])$.

(ii) There exists an $Aut(A_\Omega)$-invariant inner product $\langle \, | \, \rangle_o$ on A_Ω with $\langle e | e \rangle_o = 1$ for every atom e.

(iii) If $\mathbb{I} = q_1 + ... + q_n$ holds with pairwise orthogonal atoms $q_1, ..., q_n$, then $n = m$.

Proof. (i) and (iii): Using the normalized Haar measure on $Aut(A_\Omega)$ and arbitrarily selecting any state μ_o, we define

$$\mu_{inv}(a) := \int_{U \in Aut(A_\Omega)} \mu_o(Ua) dU$$

for $a \in A_\Omega$. This becomes an $Aut(A_\Omega)$-invariant state. Since $Aut(A_\Omega)$ acts transitively on the atoms, we get $\mu_{inv}(p) = \mu_{inv}(q)$ for any atoms p and q.

There are m pairwise orthogonal atoms $e_1, ..., e_m$ with $\mathbb{I} = e_1 + ... + e_m$. Then $1 = \mu_{inv}(\mathbb{I}) = \mu_{inv}(e_1) + ... + \mu_{inv}(e_m) = m\mu(p)$ and thus $\mu(p) = 1/m$ for every atom p. So far we have (i). Now suppose $\mathbb{I} = q_1 + ... + q_n$ with any further pairwise orthogonal atoms $q_1, ..., q_n$. Then $1 = \mu_{inv}(\mathbb{I}) = \mu_{inv}(q_1) + ... + \mu_{inv}(q_n) = n/m$. Therefore $n = m$ and we have (iii).

(ii): Let $(\,|\,)$ be any inner product on A_Ω. Using again the normalized Haar measure on $Aut(A_\Omega)$, we construct a further $Aut(A_\Omega)$-invariant inner product via

$$\langle a|b\rangle_o := \int_{U\in Aut(A_\Omega)} (Ua|Ub)dU$$

for any $a,b \in A_\Omega$. Since $Aut(A_\Omega)$ acts transitively on the atoms, we have $\langle p|p\rangle_o = \langle q|q\rangle_o$ for any two atoms p and q and we can normalize $\langle\,|\,\rangle_o$ in such a way that $\langle e|e\rangle_o = 1$ for every atom e. $\square$

Lemma 1 and its proof constitute a small piece of a proof in Müller and Ududec's paper [4]. The same thing has been shown by Wilce in another way [16].

5. Bit Symmetry

So far there is no connection between the orthogonality in the quantum logic $ext([0,\mathbb{I}])$ and the orthogonality in the Euclidean space A_Ω with the inner product $\langle\,|\,\rangle_o$ from Lemma 1 (ii). Here we will construct a further inner product with such a connection, preassuming bit symmetry and using the state μ_{inv} and the inner product $\langle\,|\,\rangle_o$ from Lemma 1.

Theorem 1. *Let Ω be any bit-symmetric finite-dimensional compact convex set with the property $(**)$ and information capacity m.*

Then A_Ω possesses an inner product $\langle\,|\,\rangle$ such that the positive cone becomes self-dual and $\mathbb{P}_p(q) = \langle p|q\rangle$ holds for any atoms p and q. Therefore the transition probability is symmetric: $\mathbb{P}_p(q) = \mathbb{P}_q(p)$.

Furthermore, $\langle \mathbb{I}|x\rangle = \langle x|\mathbb{I}\rangle = m\mu_{inv}(x)$ for $x \in A_\Omega$. The atoms are the extreme points of the set $\{a \in A_\Omega | 0 \le a, \mu_{inv}(a) = 1/m\}$ and this set is affinely isomorphic to the state space of A_Ω as well as to Ω itself.

Proof. We use the state μ_{inv} and the $Aut(A_\Omega)$-invariant inner product $\langle\,|\,\rangle_o$ from Lemma 1. The information capacity is again denoted by m. From the bit symmetry we get a real number ϵ such that $\epsilon = \langle p|q\rangle_o$ for any atoms p,q with $p+q \le \mathbb{I}$. Together with Lemma 1 (ii) the Cauchy-Schwarz inequality implicates $|\epsilon| \le 1$. Moreover, the case $|\epsilon| = 1$ is impossible, since atoms p,q with $p+q \le \mathbb{I}$ are linearly independent, and we thus have $|\epsilon| < 1$. We now define

$$\langle a|b\rangle := \frac{1}{1-\epsilon}\left[\langle a|b\rangle_o - m^2\epsilon\mu_{inv}(a)\mu_{inv}(b)\right]$$

for $a,b \in A_\Omega$. Then $\langle p|p\rangle = 1$ for each atom p and $\langle p|q\rangle = 0$ for atoms p and q with $p+q \le \mathbb{I}$.

Each $a \in A_\Omega$ has a spectral decomposition $a = s_1e_1 + ... + s_ne_n$ with pairwise orthogonal atoms e_k and $s_k \in \mathbb{R}$. Thus $\langle a|a\rangle = s_1^2 + ... + s_n^2 \ge 0$ and $\langle a|a\rangle = 0$ iff $a = 0$. If $0 \le \langle a|b\rangle$ for all $b \in A_\Omega$ with $0 \le b$, select $b = e_k$ and get $0 \le s_k$ for each k; thus $0 \le a$.

Now let p_o be any atom. By the Riesz representation theorem there is an element $a_o \in A_\Omega$ with $\mathbb{P}_{p_o}(x) = \langle a_o|x\rangle$ for all $x \in A_\Omega$. Let $a_o = s_1e_1 + ... + s_ne_n$ be its spectral decomposition with pairwise orthogonal atoms $e_1,...,e_n$ and real numbers $s_1,...,s_n$. From $\mathbb{P}_{p_o}(e_j) = \left\langle a_o|e_j\right\rangle = \sum_k s_k\left\langle e_k|e_j\right\rangle = s_j$ we get $0 \le s_j \le 1$ for $j = 1,...,n$. Moreover $1 = \mathbb{P}_{p_o}(\mathbb{I}) = \langle a_o|e_1 + ... + e_n\rangle = s_1 + ... + s_n$ and $1 = \mathbb{P}_{p_o}(p_o) = \langle a_o|p_o\rangle = \sum_k s_k\langle e_k|p_o\rangle$. The Cauchy-Schwarz inequality implicates $|\langle e_k|p_o\rangle| \le 1$ for each k.

For those k with $s_k \ne 0$ we then get $\langle e_k|p_o\rangle = 1$. Thus e_k and p_o become linearly dependent. From $\|e_k\| = 1 = \|p_o\|$ we get $e_k = \pm p_o$. Since $0 \le e_k$ and $0 \le p_o$ we have $e_k = p_o$. However $e_k \ne e_j$ for $k \ne j$ and e_k can coincide with p_o for only one single k_o. Therefore $s_k = 0$ for $k \ne k_o$, $s_{k_o} = 1$ and $a_o = e_{k_o} = p_o$.

We now have that $\mathbb{P}_p(x) = \langle p|x\rangle$ holds for each atom p and for all $x \in A_\Omega$. For any two atoms we get $\mathbb{P}_p(q) = \langle p|q\rangle = \langle q|p\rangle = \mathbb{P}_q(p)$. It remains to show that $0 \le \langle a|b\rangle$ holds for $0 \le a$ and $0 \le b$ ($a,b \in A_\Omega$). Due to the spectral decomposition it is sufficient to prove this inequality for atoms p an q and here we have already $\langle p|q\rangle = \mathbb{P}_p(q) \ge 0$.

For $x \in A_\Omega$ with the spectral decomposition $x = s_1e_1 + ... + s_ne_n$ ($n \le m$) with pairwise orthogonal atoms e_k and $s_k \in \mathbb{R}$ we get $\langle x|\mathbb{I}\rangle = \sum s_k\langle e_k|\mathbb{I}\rangle = \sum s_k\mathbb{P}_{e_k}(\mathbb{I}) = \sum s_k = m\mu_{inv}(x)$. For $0 \le x$ we have $0 \le s_k$ for each k and $\|x\| = max\{s_1,...,s_n\} \le \sum s_k = m\mu_{inv}(x)$. Therefore $\{a \in A_\Omega | 0 \le a, \mu_{inv}(a) = 1/m\}$ $\subseteq [0,\mathbb{I}]$.

Each atom is extreme in $[0,\mathbb{I}]$ and thus in $\{a \in A_\Omega | 0 \le a, \mu_{inv}(a) = 1/m\}$. Vice versa, if x is an extreme point in $\{a \in A_\Omega | 0 \le a, \mu_{inv}(a) = 1/m\}$, its spectral decomposition becomes a non-trivial convex combination of pairwise orthogonal atoms unless x itself is an atom.

For $a \in A_\Omega$ with $0 \leq a$ and $\mu_{inv}(a) = 1/m$ the map $A_\Omega \ni x \to \langle a|x \rangle$ defines a state μ_a and the map $a \to \mu_a$ yields an affine isomorphism onto the state space. The Riesz representation theorem and the self-duality make sure that each state has this form. □

With Theorem 1 we come rather close to the familiar situation of Hilbert space quantum mechanics, where $\mu_{inv}(x) = trace(x)/m$, $\langle x|y \rangle = trace(xy)$ for the self-adjoint operators x and y (the observables) and where the positive operators with normalized trace represent the (mixed) states.

The cases with information capacity $m = 2$ are the generalized qubit models (or binary models) considered in Ref. [8]. Their state spaces are the smooth and strictly convex compact convex sets. If the automorphism group acts transitively on the pure states (weak symmetry), we get the bit symmetry from the remarks at the end of Section 3, the transition probability then becomes symmetric by Theorem 1 and the state space is affinely isomorphic to the unit ball in an Euclidean space, as already shown in Ref. [8]. We can thus conclude that the ellipsoids are the only smooth and strictly convex compact convex sets where the automorphism group acts transitively on the extreme points. In these cases, A_Ω becomes a so-called spin factor which is a special type of Euclidean (formally real) Jordan algebra [8]. Since the exchange symmetry holds in the spin factors, we now know that, in the case of information capacity $m = 2$ or the smooth and strictly convex compact convex sets, the exchange symmetry is equivalent to the other three symmetry postulates considered here.

The above proof is partly adopted from Müller and Ududec [4], who derive a weaker version of Theorem 1 for a more general situation. They show that any bit-symmetric compact convex set is self-dual. They do not have the property $(**)$ with its implications like a reasonable atomic quantum logic and spectrality and they do not consider the transition probability between the atoms. Moreover their proof becomes more tricky and must use other methods. A simple example of a bit-symmetric compact convex set that does not have the property $(**)$ is the pentagon [4].

6. Strong Symmetry

Corollary 1. *Let Ω be any strongly symmetric finite-dimensional compact convex set with the property $(**)$. Then Ω is either a simplex or the state space of a simple Euclidean Jordan algebra.*

Proof. Since the strong symmetry implies the bit symmetry, we can apply Theorem 1 and get the inner product $\langle \,|\, \rangle$. Now let μ be any state. By the Riesz representation theorem there is $a \in A_\Omega$ with $\mu(x) = \langle a|x \rangle$ for all $x \in A_\Omega$. The self-duality gives $0 \leq a$. Let $a = s_1 e_1 + ... + s_n e_n$ be the spectral decomposition of a, where $0 \leq s_k$ holds for each k and the e_k are pairwise orthogonal atoms (section 2 (e)). Then

$$\mu(x) = \langle a|x \rangle = \sum s_k \langle e_k|x \rangle = \sum s_k \mathbb{P}_{e_k}(x)$$

for all $x \in A_\Omega$ and $1 = \mu(\mathbb{I}) = \sum s_k$. From $0 \leq s_k$ we get that μ becomes the convex combination of the orthogonal (perfectly distinguishable) pure states $\mathbb{P}_{e_k}$, $k = 1, ..., n$. This is another type of spectrality, which differs from (e) in section 2 and which is used by Barnum and Hilgert [10]. We are now able to apply their theorem (*Every strongly symmetric compact convex set with their type of spectrality is a simplex or the state space of a simple Euclidean Jordan algebra*) and get the desired result. □

The simplexes represent the state spaces of classical probability theory with finite dimension. An n-simplex is the set

$$\{(s_1, ..., s_n) \in \mathbb{R}^n | 0 \leq s_1, ..., s_n, \ s_1 + ... + s_n = 1\}$$

or any other set that is affinely isomorphic to this one.

In Ref. [9] it was concluded at the end of section 6 that the property $(**)$, *strong symmetry and the symmetry of the transition probability* are possible only with the simplexes and the state spaces of the simple Euclidean Jordan algebras. Corollary 1 now reveals that the assumption that the transition probability is symmetric was redundant there, since this follows from the strong symmetry.

Thus we have almost reconstructed finite dimensional quantum theory from the strong symmetry and the property $(**)$. Still included is the exceptional Jordan algebra that consists of the Hermitian 3×3-matrices over the octonions and that does not possess a representation as Hilbert space operators.

7. Conclusions

The usual transition probability in Hilbert space quantum theory is symmetric. Is there a deeper physical, probabilistic or information theoretical reason why nature chose this or are extensions of the theory with non-symmetric transition probability conceivable? Here we have revealed a connection to another kind of symmetry and we now know that we must drop the bit symmetry when we abandon the symmetry of the transition probability.

In quantum computation reversible transformations from any logical bit to any other logical bit are usually regarded as necessary and this is considered an information theoretic reason for the bit symmetry [4]. A deeper look at some quantum informational procedures, however, shows that this rationale is not as clear as it seems. Grover's search algorithm and teleportation require assumptions that implicate the symmetry of the transition probability, but not the bit symmetry [17]. The no-cloning theorem and the quantum key distribution protocols need neither the bit symmetry nor the general validity of the symmetry of the transition probability [7,18].

From the physical point of view, a continuous reversible time evolution from one atom p to any other atom q (or from one pure state to any other one) might be a crucial requirement. This means that there are automorphisms U_t, $t \in [0, 1]$, such that U_0 is the identity, $U_1(p) = q$ and the function $t \rightarrow U_t$ is continuous. This requirement, which has not been considered here, implicates only the weak symmetry among the symmetry features we have looked at.

So we must finally conclude that, neither for the bit symmetry (and even less for the strong symmetry) nor for the symmetry of the transition probability, any truly convincing physical or information theoretical reason is seen. The symmetry of the transition probability appears to be needed in some cases of the quantum informational procedures mentioned above, whereas it may surprise that the bit symmetry plays no role there.

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