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Article

A Triality Pattern in Entanglement Theory

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Abstract: In this work, we present new connections between three types of quantum states: positive under partial transpose states, symmetric with positive coefficients states and invariant under realignment states. First, we obtain a common upper bound for their spectral radii. Then we prove the existence of a lower bound for their ranks and the fact that whenever this bound is attained the states are separable. These connections add new evidence to the pattern that for every proven result for one of these types, there are counterparts for the other two, which is a potential source of information for entanglement theory.

Keywords: filter normal form; separability; complete reducibility property; spectral radius

1. Introduction

The separability problem in quantum theory [1] asks for a criterion to distinguish the separable states from the entangled states. It is known that separable states in $\mathcal{M}_k \otimes \mathcal{M}_m$ form a subset of the positive under partial transpose states (PPT states) and in low dimensions, $km \leq 6$, these two sets coincide [2,3] solving the problem. However, in larger dimensions, $km > 6$, there are entangled PPT states. In addition, for k, m arbitrary, this problem is known to be a hard problem [4], thus any reduction of this problem to a subset of PPT states of $\mathcal{M}_k \otimes \mathcal{M}_m$ is certainly important.

In [5], a procedure to reduce the separability problem to a proper subset of PPT states was presented. The idea behind this reduction can be summarized as follows. Let $A = \sum_{i=1}^n A_i \otimes B_i \in \mathcal{M}_k \otimes \mathcal{M}_m$ be a quantum state and consider $G_A : \mathcal{M}_k \to \mathcal{M}_m$ and $F_A : \mathcal{M}_m \to \mathcal{M}_k$ as $G_A(X) = \sum_{i=1}^n tr(A_i X) B_i$ and $F_A(X) = \sum_{i=1}^n tr(B_i X) A_i$, where $tr(X)$ stands for the trace of X.

Now, if A is PPT and X is a positive semidefinite Hermitian eigenvector of $F_A \circ G_A : \mathcal{M}_k \to \mathcal{M}_k$, then A decomposes as a sum of states with orthogonal local supports [5, Lemma 8], i.e.,

$$A = (V \otimes W)A(V \otimes W) + (V^\perp \otimes W^\perp)A(V^\perp \otimes W^\perp),$$

where $V, W, V^\perp, W^\perp$ are orthogonal projections onto $\mathrm{Im}(X)$, $\mathrm{Im}(G_A(X))$, $\ker(X)$ and $\ker(G_A(X))$, respectively.

Then the algorithm proceeds to decompose $(V \otimes W)A(V \otimes W)$ and $(V^\perp \otimes W^\perp)A(V^\perp \otimes W^\perp)$, since they are also PPT, whenever such X is found. Eventually this process stops with

$$A = \textstyle\sum_{i=1}^n (V_i \otimes W_i)A(V_i \otimes W_i),$$

where $(V_i \otimes W_i)A(V_i \otimes W_i)$ cannot be further decomposed for $1 \leq i \leq n$. These states are named weakly irreducible. Finally, A is separable if and only if each $(V_i \otimes W_i)A(V_i \otimes W_i)$ is separable, therefore this algorithm reduces the separability problem to the weakly irreducible PPT case.

Positive under partial transpose states are not the only type of states on which this procedure works because the key feature of this procedure, which is $A \in \mathcal{M}_k \otimes \mathcal{M}_m$ breaks whenever a certain positive semidefinite Hermitian eigenvector is found, is also true for two other types of quantum states. From now on we say that $A \in \mathcal{M}_k \otimes \mathcal{M}_m$ has the complete reducibility property, if for every positive semidefinite Hermitian eigenvector X of $F_A \circ G_A : \mathcal{M}_k \to \mathcal{M}_k$, we have

$$A = (V \otimes W)A(V \otimes W) + (V^\perp \otimes W^\perp)A(V^\perp \otimes W^\perp),$$

where $V, W, V^\perp, W^\perp$ are orthogonal projections onto $\mathrm{Im}(X)$, $\mathrm{Im}(G_A(X))$, $\ker(X)$ and $\ker(G_A(X))$, respectively. In [5], a search for other types of quantum states satisfying this property was conducted finding only three types of

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quantum states: positive under partial transpose states (PPT states), symmetric with positive coefficients states (SPC states) and invariant under realignment states. So far this property has been only verified for this triad of quantum states.

It is not hard to compute a single positive semidefinite Hermitian eigenvector for $F_A \circ G_A : \mathcal{M}_k \rightarrow \mathcal{M}_k$, when A is positive semidefinite. In the next section, we shall see that that under this hypothesis $G_A : \mathcal{M}_k \rightarrow \mathcal{M}_m$ and $F_A : \mathcal{M}_m \rightarrow \mathcal{M}_k$ are adjoint positive maps. So if λ is the largest positive eigenvalue of $F_A \circ G_A$, then $P = \lim_{n\rightarrow\infty}(\frac{F_A \circ G_A}{\lambda})^n$ is the orthogonal projection onto the eigenspace of $F_A \circ G_A$ associated to λ. As a limit of positive maps, P is also a positive map. Therefore $P(Id)$ is a positive semidefinite Hermitian matrix belonging to the eigenspace of $F_A \circ G_A$ associated to λ.

Now, the number of times that A breaks into weakly irreducible pieces is maximized when the non-null eigenvalues of $F_A \circ G_A : \mathcal{M}_k \rightarrow \mathcal{M}_k$ are equal, because in this case we are able to produce many positive semidefinite Hermitian eigenvectors. In this situation, we have the following theorem:

> If the non-null eigenvalues of $F_A \circ G_A$ are equal, then A is separable, when A is PPT or SPC or invariant under realignment [5, Proposition 15]. Notice that the eigenvalues of $F_A \circ G_A$ are the square of the Schmidt coefficients of A.

In [5, Lemma 42], it was also noticed that every SPC state and every invariant under realignment state is PPT in $\mathcal{M}_2 \otimes \mathcal{M}_2$. Before that in [6], it was proved that a state supported on the symmetric subspace of $\mathbb{C}^k \otimes \mathbb{C}^k$ is PPT if and only if it is SPC. This is plenty of evidence of how linked these three types of quantum states are.

In this work we prove new results for this triad of quantum states and a new consequence for their complete reducibility property. One of these results concerns the separability of these states. The aforementioned connections along with our new results lead us to notice a certain triality pattern:

> For each proven theorem for one of these three types of states, there are corresponding counterparts for the other two.

We believe that a solution to the separability problem for SPC states or invariant under realignment states would shed light on bound entanglement. This is our reason for studying these types of states.

Next, we would like to point out the origin of these connections which is also the source of the main tools used in this work. For this, we must consider the group of linear contractions [7] and some of its properties. For each permutation $\sigma \in S_4$, the linear transformation $L_\sigma : \mathcal{M}_k \otimes \mathcal{M}_k \rightarrow \mathcal{M}_k \otimes \mathcal{M}_k$ satisfying

$$L_\sigma(v_1 v_2^t \otimes v_3 v_4^t) = v_{\sigma(1)} v_{\sigma(2)}^t \otimes v_{\sigma(3)} v_{\sigma(4)}^t,$$

where $v_i \in \mathbb{C}^k$ is called a linear contraction.

The term contraction comes from the fact that $\|L_\sigma(\gamma)\|_1 \leq \|\gamma\|_1$, for every $\sigma \in S_4$, whenever $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ is a separable state and $\|\cdot\|_1$ is the trace norm of a matrix (i.e., the sum of its singular values). Hence, if $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ is a state and $\|L_\sigma(\gamma)\|_1 > \|\gamma\|_1$ for some $\sigma \in S_4$, then $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ is entangled. This observation provides two useful criteria for entanglement detection. In cycle notation, they are:

- the PPT criterion [2,3], when $\sigma = (34)$, and
- the CCNR criterion [8,9], when $\sigma = (23)$.

Despite the name – contraction – these linear maps are in fact isometries, as they map an orthonormal basis $\{e_i e_j^t \otimes e_k e_l^t,\ 1 \leq i,j,k,l \leq k\}$ of $\mathcal{M}_k \otimes \mathcal{M}_k$ onto itself. Hence, they preserve the Frobenius norm of $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$, denoted here by $\|\gamma\|_2$. This norm is an invariant exploited several times in this work.

In addition, the set of linear contractions is a group under composition generated by the partial transposes ($L_{(34)}(\gamma) = \gamma^\Gamma$ and $L_{(12)}$) and the realignment map ($L_{(23)}(\gamma) = \mathcal{R}(\gamma)$). The relations among the elements of this group are extremely useful as we shall see in the proofs of our novel results.

Another very useful fact about the realignment map is that this map is a homomorphism with respect to the usual matrix product and a new product which generalizes the Hadamard product (see item 8) of Lemma 1).

Finally, these maps allow connecting this triad of quantum states from their origins, that is, their definitions:

- PPT states are the states that remain positive under partial transpose ($\gamma \geq 0, \gamma^\Gamma \geq 0$) [2,3],
- SPC states are the states that remain positive under partial transpose composed with realignment ($\gamma \geq 0,\ \mathcal{R}(\gamma^\Gamma) \geq 0$) ([5, corollary 25] and [5, definition 17]),

- invariant under realignment states are the states that remain the same under realignment ($\gamma \geq 0$, $\mathcal{R}(\gamma) = \gamma$).

These observations on the group of linear contractions are used throughout this paper; they are the keys to obtain our novel results. Now, let us describe these results.

Our first result is an upper bound on the spectral radius for this triad of states. We show that if $\gamma = \sum_{i=1}^{m} C_i \otimes D_i \in \mathcal{M}_k \otimes \mathcal{M}_k$ is PPT or SPC or invariant under realignment, then

$$\|\gamma\|_\infty \leq \min\{\|\gamma_A\|_\infty, \|\gamma_B\|_\infty, \|\mathcal{R}(\gamma)\|_\infty\},$$

where $\|.\|_\infty$ is the operator norm, and $\gamma_A = \sum_{i=1}^{m} C_i tr(D_i)$ and $\gamma_B = \sum_{i=1}^{m} D_i tr(C_i)$ are the reduced states. Let us say that the ranks of γ_A and γ_B are the reduced ranks of γ.

Our second result regards the filter normal form of SPC states, invariant under realignment states and certain types of PPT states. This normal form has been used in entanglement theory, for example, to provide a different solution for the separability problem in $\mathcal{M}_2 \otimes \mathcal{M}_2$ [10–12] or to prove the equivalence of some criteria for entanglement detection [13]. Here we show that states which are SPC or invariant under realignment can be put in the filter normal form and their normal forms can still be chosen to be SPC and invariant under realignment, respectively. In other words, if $A, B \in \mathcal{M}_k \otimes \mathcal{M}_k$ are states such that

(1) A is SPC then there is an invertible matrix $R \in \mathcal{M}_k$ such that $(R \otimes R)A(R \otimes R)^* = \sum_{i=1}^{n} \lambda_i \delta_i \otimes \delta_i$, where $\lambda_1 = \frac{1}{k}$, $\delta_1 = \frac{Id}{\sqrt{k}}$, $\lambda_i > 0$ and δ_i is Hermitian for every i, and $tr(\delta_i \delta_j) = 0$ for $i \neq j$.

(2) B is invariant under realignment then there is an invertible matrix $R \in \mathcal{M}_k$ such that $(R \otimes \overline{R})B(R \otimes \overline{R})^* = \sum_{i=1}^{n} \lambda_i \delta_i \otimes \overline{\delta_i}$, where $\lambda_1 = \frac{1}{k}$, $\delta_1 = \frac{Id}{\sqrt{k}}$, $\lambda_i > 0$ and δ_i is Hermitian for every i, and $tr(\delta_i \delta_j) = 0$ for $i \neq j$.

Our PPT counterpart to these two results, Theorem 3, says that every PPT state whose rank is equal to its two reduced ranks can be put in the filter normal form, which we believe is a novel contribution to the filter normal form literature.

We are not aware of any general algorithm capable of determining whether a given state can be put in the filter normal form or not. From this point of view, these results are not only novel but are highly relevant to the literature on the filter normal form.

Our final result is a lower bound for the ranks of these three types of states. We show that the rank of PPT states, SPC states and invariant under realignment states cannot be inferior to their reduced ranks (when they are equal) and whenever this minimum is attained the states are separable.

In [14], it was proved that a state γ such that $\text{rank}(\gamma) \leq \max\{\text{rank}(\gamma_A), \text{rank}(\gamma_B)\}$ is separable by just being positive under partial transpose. In [15], it was shown that the rank of a separable state is greater or equal to its reduced ranks. So if γ is PPT and $\text{rank}(\gamma) \leq \max\{\text{rank}(\gamma_A), \text{rank}(\gamma_B)\}$, then it is separable and $\text{rank}(\gamma) = \max\{\text{rank}(\gamma_A), \text{rank}(\gamma_B)\}$.

Thus, our last result is known for PPT states, but it is original for SPC states and invariant under realignment states. The proofs presented here for these results are completely original. We use the facts that every PPT state, SPC state and invariant under realignment state with minimal rank can be put in the filter normal form together with the complete reducibility property to finish the proofs. These results show a nice interplay between the filter normal form and the complete reducibility property.

It is quite surprising that all three of these types of states guarantee separability of their states under such condition. In the entanglement literature, the opposite is usually the case.

For example, it is known that any given bipartite state γ satisfies $\text{rank}(\gamma) \geq \frac{\max\{\text{rank}(\gamma_A), \text{rank}(\gamma_B)\}}{\min\{\text{rank}(\gamma_A), \text{rank}(\gamma_B)\}}$ (see [16, Theorem 3.3]) and, whenever the equality holds, this γ becomes entangled (see [17, Theorem 1]). From this point of view, states that reach the minimum possible rank value are usually entangled.

The results described above show how fundamental the complete reducibility property is to entanglement theory, it acts as a unifying approach for many results. Another recent one on entanglement breaking Markovian dynamics was described in [18]. In fact, even outside entanglement theory, we can find consequences of that property, for example, a new proof of Weiner's theorem [19] on mutually unbiased bases found in [5].

The triality pattern described above together with the complete reducibility property form a potential source of information for entanglement theory.

This paper is organized as follows. In Section 2, we present some preliminary results, which are mainly facts about the group of linear contractions. In Section 3, we obtain an upper bound for the spectral radius of our special triad of quantum states. In Section 4, we show that SPC states and invariant under realignment states can be put in the filter normal form and their normal forms retain their shape. In addition, we show that PPT states with minimal rank can also be put in the filter normal form. In Section 5, we prove that the ranks of our triad of quantum states cannot be inferior to their reduced ranks and whenever this minimum is attained the states are separable.

2. Preliminary Results

In this section we present some preliminary results. We begin by noticing that $G_A : \mathcal{M}_k \rightarrow \mathcal{M}_m$ and $F_A : \mathcal{M}_m \rightarrow \mathcal{M}_k$ defined in the introduction are adjoint maps with respect to the trace inner product, when A is Hermitian. The reason behind this is quite simple: If A is Hermitian, then $F_A(Y)^* = F_{A^*}(Y^*) = F_A(Y^*)$, for every $Y \in \mathcal{M}_m$, hence

$$tr(G_A(X)Y^*) = tr(A(X \otimes Y^*)) = tr(XF_A(Y^*)) = tr(XF_A(Y)^*).$$

Notice that for positive semidefinite Hermitian matrices $X \in \mathcal{M}_k, Y \in \mathcal{M}_m$ and $A \in \mathcal{M}_k \otimes \mathcal{M}_m$, we have $tr(A(X \otimes Y^*)) \geq 0$. Thus, the equality above also shows that G_A and F_A are positive maps (Definition 2) when A is positive semidefinite. So in this case $F_A \circ G_A : \mathcal{M}_k \rightarrow \mathcal{M}_k$ is a self-adjoint positive map.

These maps $G_A : \mathcal{M}_k \rightarrow \mathcal{M}_m$ and $F_A : \mathcal{M}_m \rightarrow \mathcal{M}_k$ are connected to the following generalization of the Hadamard product extensively used in [20] and required here a few times.

Definition 1 (Generalization of the Hadamard product). *Let $\gamma = \sum_{i=1}^{n} A_i \otimes B_i \in \mathcal{M}_m \otimes M_k$, $\delta = \sum_{j=1}^{l} C_j \otimes D_j \in \mathcal{M}_k \otimes M_s$. Define the product $\gamma * \delta \in \mathcal{M}_m \otimes \mathcal{M}_s$ as*

$$\gamma * \delta = \sum_{i=1}^{n} \sum_{j=1}^{l} A_i \otimes D_j tr(B_i C_j^t).$$

Let us recall some facts regarding this product.

Remark 1. *Let $u = \sum_{i=1}^{k} e_i \otimes e_i \in \mathbb{C}^k \otimes \mathbb{C}^k$, where $e_1, \ldots, e_k$ is the canonical basis of $\mathbb{C}^k$.*

a) Notice that $u^t(B_i \otimes C_j)u = tr((B_i \otimes C_j)uu^t) = tr(B_i C_j^t)$, where $B_i, C_j \in \mathcal{M}_k$. Therefore,

$$\gamma * \delta = (Id_{m\times m} \otimes u^t \otimes Id_{s\times s})(\gamma \otimes \delta)(Id_{m\times m} \otimes u \otimes Id_{s\times s}),$$

which implies that $\gamma * \delta$ is positive semidefinite, whenever γ, δ are positive semidefinite. In addition, $tr(\gamma * \delta) = tr(\gamma \otimes \delta\, (Id \otimes uu^t \otimes Id)) = tr(\gamma_B \otimes \delta_A\, uu^t) = tr(\gamma_B \delta_A^t)$.

b) By [20, Proposition 8], $\gamma * \delta = (F_\gamma((\cdot)^t) \otimes Id)(\delta) = (Id \otimes G_\delta((\cdot)^t))(\gamma)$.

c) Let $F = (uu^t)^\Gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ and notice that if $\gamma = \sum_{i=1}^{n} A_i \otimes B_i \in \mathcal{M}_k \otimes \mathcal{M}_k$, then $F\gamma F = \sum_{i=1}^{n} B_i \otimes A_i$. This real symmetric matrix F is a unitary matrix and it is usually called the flip operator.

Now, notice that $\gamma * (F\gamma^t F) = \sum_j \sum_i A_i \otimes A_j^t tr(B_i B_j)$. Hence

$$tr(\gamma * (F\gamma^t F)) = tr((\textstyle\sum_i tr(A_i)B_i)(\sum_j tr(A_j)B_j)) = tr(\gamma_B^2) \text{ and}$$

$$G_{\gamma*(F\gamma^t F)}(X) = \textstyle\sum_j \sum_i tr(A_i X) tr(B_i B_j) A_j^t = \sum_j tr((\sum_i tr(A_i X) B_i) B_j) A_j^t = F_\gamma(G_\gamma(X))^t.$$

Next, we discuss some facts about the group of linear contractions. Actually, we need to focus only on three of these maps. The linear contractions important to us are

$$L_{(34)}(\gamma) = \gamma^\Gamma,\ L_{(24)}(\gamma) = \gamma F \text{ and } L_{(23)}(\gamma) = \mathcal{R}(\gamma).$$

Below we discuss several properties of these linear contractions such as relations among these elements and how they behave with respect to the product defined in Definition 1.

Lemma 1. *Let $\gamma, \delta \in \mathcal{M}_k \otimes \mathcal{M}_k$ and $v = \sum_i a_i \otimes b_i,\ w = \sum_j c_j \otimes d_j \in \mathbb{C}^k \otimes \mathbb{C}^k$.*

(1) $\mathcal{R}(vw^t) = V \otimes W$, where $V = \sum_i a_i b_i^t,\ W = \sum_j c_j d_j^t$.
(2) $\mathcal{R}(\mathcal{R}(\gamma)) = \gamma$
(3) $\mathcal{R}((V \otimes W)\gamma(M \otimes N)) = (V \otimes M^t)\mathcal{R}(\gamma)(W^t \otimes N)$

(4) $\mathcal{R}(\gamma F)F = \gamma^{\Gamma}$
(5) $\mathcal{R}(\gamma^{\Gamma}) = \mathcal{R}(\gamma)F$
(6) $\mathcal{R}(\gamma F) = \mathcal{R}(\gamma)^{\Gamma}$
(7) $\mathcal{R}(\gamma^{\Gamma})^{\Gamma} = \gamma F$
(8) $\mathcal{R}(\gamma * \delta) = \mathcal{R}(\gamma)\mathcal{R}(\delta)$ (i.e., $\mathcal{R}$ is a homomorphism).
(9) $\mathcal{R}(F\overline{\gamma}F) = \mathcal{R}(\gamma)^*$

Proof. Items (1–6) were proved in items (2–7) of [5, Lemma 23]. For the other three items, we just need to prove them when $\gamma = ab^t \otimes cd^t$ and $\delta = ef^t \otimes gh^t$, where $a, b, c, d, e, f, g, h \in \mathbb{C}^k$.

Item (7): $\mathcal{R}(ab^t \otimes cd^t)^{\Gamma})^{\Gamma} = \mathcal{R}(ab^t \otimes dc^t)^{\Gamma} = (ad^t \otimes bc^t)^{\Gamma} = ad^t \otimes cb^t$.
Now, $(ab^t \otimes cd^t)F = ad^t \otimes cb^t$. So $\mathcal{R}(\gamma^{\Gamma})^{\Gamma} = \gamma F$.

Item (8): $\mathcal{R}(ab^t \otimes cd^t * ef^t \otimes gh^t) = \mathcal{R}(ab^t \otimes gh^t)(d^t f)(c^t e) = (ag^t \otimes bh^t)(d^t f)(c^t e)$.
Now, $\mathcal{R}(ab^t \otimes cd^t)\mathcal{R}(ef^t \otimes gh^t) = (ac^t \otimes bd^t)(eg^t \otimes fh^t) = (ag^t \otimes bh^t)(c^t e)(d^t f)$.
So $\mathcal{R}(\gamma * \delta) = \mathcal{R}(\gamma)\mathcal{R}(\delta)$

Item (9): $\mathcal{R}(F\overline{a}\overline{b}^t \otimes \overline{c}\overline{d}^t F) = \mathcal{R}(\overline{c}\overline{d}^t \otimes \overline{a}\overline{b}^t) = \overline{ca}^t \otimes \overline{db}^t$
Now, $\mathcal{R}(ab^t \otimes cd^t)^* = (ac^t \otimes bd^t)^* = \overline{ca}^t \otimes \overline{db}^t$.
So $\mathcal{R}(F\overline{\gamma}F) = \mathcal{R}(\gamma)^*$. $\square$

The next lemma is important for Theorem 3, it says something very interesting about PPT states which remain PPT, under realignment: They must be invariant under realignment.

Lemma 2. *Let $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ be a positive semidefinite Hermitian matrix. If γ and $\mathcal{R}(\gamma)$ are PPT, then $\gamma = \mathcal{R}(\gamma)$.*

Proof. By item (4) of Lemma 1, $\gamma^{\Gamma} = \mathcal{R}(\gamma F)F$.
Now, since $F^2 = Id$, $\gamma^{\Gamma}F = \mathcal{R}(\gamma F)$.
Next, by item (6) of Lemma 1, $\mathcal{R}(\gamma F) = \mathcal{R}(\gamma)^{\Gamma}$. So $\gamma^{\Gamma}F = \mathcal{R}(\gamma)^{\Gamma}$ is a positive semidefinite Hermitian matrix by hypothesis.
Since F, γ^{Γ} and $\gamma^{\Gamma}F$ are Hermitian matrices, $\gamma^{\Gamma}F = F\gamma^{\Gamma}$. So there is an orthonormal basis of $\mathbb{C}^k \otimes \mathbb{C}^k$ formed by symmetric and anti-symmetric eigenvectors of γ^{Γ}.
Remind that γ^{Γ} and $\gamma^{\Gamma}F$ are positive semidefinite, hence $\gamma^{\Gamma} = \gamma^{\Gamma}F$.
Finally, we have noticed that $\gamma^{\Gamma}F = \mathcal{R}(\gamma)^{\Gamma}$. So $\gamma^{\Gamma} = \mathcal{R}(\gamma)^{\Gamma}$, which implies $\gamma = \mathcal{R}(\gamma)$. $\square$

The next lemma is used in this work a few times. Although simple, we state it here in order to better organize our arguments.

Lemma 3. *Let $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$. The singular values of the linear map $G_{\gamma} : \mathcal{M}_k \to \mathcal{M}_k$ and the matrix $\mathcal{R}(\gamma) \in \mathcal{M}_k \otimes \mathcal{M}_k$ are equal and their largest singular values can be computed as*

$$\|G_{\gamma}\|_{\infty} = \|\mathcal{R}(\gamma)\|_{\infty} = \max\{|tr(\gamma(X \otimes Y))|,\ \|X\|_2 = \|Y\|_2 = 1\}.$$

In addition, if γ is Hermitian, then there are Hermitian matrices $\gamma_1, \delta_1 \in \mathcal{M}_k$ such that $\|\mathcal{R}(\gamma)\|_{\infty} = tr(\gamma(\gamma_1 \otimes \delta_1))$, where $\|\gamma_1\|_2 = \|\delta_1\|_2 = 1$.

Proof. It was proved in item 1) of [5, Lemma 23] that

$$V \circ F_{\gamma^*} \circ V^* = \mathcal{R}((\gamma^*)^{\Gamma}),$$

where $V : \mathcal{M}_k \to \mathbb{C}^k \otimes \mathbb{C}^k$ is the vectorization map $V(\sum_i a_i b_i^t) = \sum_i a_i \otimes b_i$. Since the vectorization map is an isometry, the singular values of $F_{\gamma^*} : \mathcal{M}_k \to \mathcal{M}_k$ and $\mathcal{R}((\gamma^*)^{\Gamma}) \in \mathcal{M}_k \otimes \mathcal{M}_k$ are equal.

Now, by item $5)$ of Lemma 1, $\mathcal{R}((\gamma^*)^{\Gamma}) = \mathcal{R}(\gamma^*)F$, where F is the flip operator, which is unitary. Hence the singular values of $\mathcal{R}((\gamma^*)^{\Gamma})$ and $\mathcal{R}(\gamma^*)$ are the same.

Next, items 2) and 9) of Lemma 1 imply that $\mathcal{R}(\gamma^*) = F\overline{\mathcal{R}(\gamma)}F$. Again, since F is unitary $\mathcal{R}(\gamma^*)$ and $\mathcal{R}(\gamma)$ share their singular values. Hence the singular values of $F_{\gamma^*} : \mathcal{M}_k \to \mathcal{M}_k$ and $\mathcal{R}(\gamma) \in \mathcal{M}_k \otimes \mathcal{M}_k$ are equal.

In addition, $G_\gamma : \mathcal{M}_k \to \mathcal{M}_k$ is the adjoint of $F_{\gamma^*} : \mathcal{M}_k \to \mathcal{M}_k$, thus G_γ and F_{γ^*} have the same singular values. Therefore, the singular values of the linear map $G_\gamma : \mathcal{M}_k \to \mathcal{M}_k$ and the matrix $\mathcal{R}(\gamma) \in \mathcal{M}_k \otimes \mathcal{M}_k$ are equal. Thus, $\|G_\gamma\|_\infty = \|\mathcal{R}(\gamma)\|_\infty$.

In order to show that their largest singular values can be computed as stated above, recall that the largest singular value of $G_\gamma : \mathcal{M}_k \to \mathcal{M}_k$ is equal to

$$\begin{aligned}\max\{\|(G_\gamma(W)\|_2,\ \|W\|_2 = 1\} &= \max\{|tr(G_\gamma(W)V)|,\ W, V \in \mathcal{M}_k \text{ and } \|W\|_2 = \|V\|_2 = 1\},\\ &= \max\{|tr(\gamma(W \otimes V))|,\ W, V \in \mathcal{M}_k \text{ and } \|W\|_2 = \|V\|_2 = 1\}.\end{aligned}$$

This proves the first part of the lemma. Now for the second part, if γ is Hermitian, then the set of Hermitian matrices is left invariant by $G_\gamma : \mathcal{M}_k \to \mathcal{M}_k$. Therefore, there is an Hermitian matrix $\gamma_1 \in \mathcal{M}_k$ such that $\|\gamma_1\|_2 = 1$ and $\|G_\gamma(\gamma_1)\|_2 =$ the largest singular value of G_γ.

Notice that $\|G_\gamma(\gamma_1)\|_2 = tr(G_\gamma(\gamma_1)\delta_1) = tr(\gamma(\gamma_1 \otimes \delta_1))$, where $\delta_1 = G_\gamma(\gamma_1)/\|G_\gamma(\gamma_1)\|_2$.

So $\|\mathcal{R}(\gamma)\|_\infty = tr(\gamma(\gamma_1 \otimes \delta_1))$, where $\|\gamma_1\|_2 = \|\delta_1\|_2 = 1$ and γ_1, δ_1 are Hermitian matrices. □

Now, we have all the preliminary results required to discuss our new results.

3. An Upper Bound for the Spectral Radius of the Special Triad

In this section we obtain an upper bound for the spectral radius of PPT states, SPC states and invariant under realignment states (Theorem 1). In order to prove this theorem, two lemmas are required.

Our first lemma says that the largest singular value of the partial transpose of any state cannot exceed the largest singular values of its reduced states or of its own realignment.

Lemma 4. *Let $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ be any positive semidefinite Hermitian matrix. Then*

$$\|\gamma^\Gamma\|_\infty \leq \min\{\|\gamma_A\|_\infty, \|\gamma_B\|_\infty, \|\mathcal{R}(\gamma)\|_\infty\}.$$

Proof. Let $v \in \mathbb{C}^k \otimes \mathbb{C}^k$ be a unit vector such that $|tr(\gamma^\Gamma vv^*)| = \|\gamma^\Gamma\|_\infty$. Let n be its rank. Denote by $\{g_1, \dots, g_k\}$ and $\{e_1, \dots, e_n\}$ the canonical bases of $\mathbb{C}^k$ and $\mathbb{C}^n$, respectively.

Next, there are matrices $D \in \mathcal{M}_{k\times k}$, $E \in \mathcal{M}_{k\times k}$, $R \in \mathcal{M}_{k\times n}$ and $S \in \mathcal{M}_{k\times n}$ such that

(1) $v = (D \otimes Id)w$, where $tr(DD^*) = 1$, $w = \sum_{i=1}^k g_i \otimes g_i \in \mathbb{C}^k \otimes \mathbb{C}^k$,
(2) $v = (Id \otimes E)w$, where $tr(\overline{E}E^t) = 1$, $w = \sum_{i=1}^k g_i \otimes g_i \in \mathbb{C}^k \otimes \mathbb{C}^k$,
(3) $v = (R \otimes S)u$, where $u = \sum_{i=1}^n e_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^n$ and $tr((RR^*)^2) = tr((\overline{S}S^t)^2) = 1$.

Now,

(1) $\|\gamma^\Gamma\|_\infty = |tr(\gamma^\Gamma vv^*)| = |tr((D^* \otimes Id)\gamma(D \otimes Id)(ww^*)^\Gamma)| \leq tr((D^* \otimes Id)\gamma(D \otimes Id))$,
since $Id \otimes Id \pm (ww^*)^\Gamma$ and γ are positive semidefinite. Hence

$$\|\gamma^\Gamma\|_\infty \leq tr(\gamma(DD^* \otimes Id)) = tr(\gamma_A DD^*) \leq \|\gamma_A\|_\infty tr(DD^*) = \|\gamma_A\|_\infty.$$

(2) $\|\gamma^\Gamma\|_\infty = |tr(\gamma^\Gamma vv^*)| = |tr((Id \otimes E^t)\gamma(Id \otimes \overline{E})(ww^*)^\Gamma)| \leq tr((Id \otimes E^t)\gamma(Id \otimes \overline{E}))$,
since $Id \otimes Id \pm (ww^*)^\Gamma$ and γ are positive semidefinite. Hence

$$\|\gamma^\Gamma\|_\infty \leq tr(\gamma(Id \otimes \overline{E}E^t)) = tr(\gamma_B \overline{E}E^t) \leq \|\gamma_B\|_\infty tr(\overline{E}E^t) = \|\gamma_B\|_\infty.$$

(3) $\|\gamma^\Gamma\|_\infty = |tr(\gamma^\Gamma vv^*)| = |tr((R^* \otimes S^t)\gamma(R \otimes \overline{S})(uu^*)^\Gamma)| \leq tr((R^* \otimes S^t)\gamma(R \otimes \overline{S}))$,
since $Id \otimes Id \pm (uu^*)^\Gamma$ and γ are positive semidefinite. Hence

$$\|\gamma^\Gamma\|_\infty \leq tr(\gamma(RR^* \otimes \overline{S}S^t)) \leq \|\mathcal{R}(\gamma)\|_\infty, \text{ since } \|RR^*\|_2 = \|\overline{S}S^t\|_2 = 1, \text{ by Lemma 3.}$$

□

Our next lemma shows that the largest singular value of the realignment of any state cannot exceed the geometric mean of the largest singular values of its reduced states.

Lemma 5. *Let $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ be a positive semidefinite Hermitian matrix. Then*

$$\|\mathcal{R}(\gamma)\|_\infty^2 \leq \|\gamma_A\|_\infty \|\gamma_B\|_\infty.$$

Proof. By Lemma 3, there are Hermitian matrices $\gamma_1 \in \mathcal{M}_k$ and $\delta_1 \in \mathcal{M}_k$ such that

(1) $tr(\gamma_1^2) = tr(\delta_1^2) = 1$

(2) $tr(\gamma(\gamma_1 \otimes \delta_1)) = \|\mathcal{R}(\gamma)\|_\infty$.

Consider the following positive semidefinite Hermitian matrix

$$\begin{pmatrix} \gamma^{\frac{1}{2}} & 0 \\ 0 & \gamma^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} Id \otimes \delta_1 \\ \gamma_1 \otimes Id \end{pmatrix} \begin{pmatrix} Id \otimes \delta_1 & \gamma_1 \otimes Id \end{pmatrix} \begin{pmatrix} \gamma^{\frac{1}{2}} & 0 \\ 0 & \gamma^{\frac{1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^{\frac{1}{2}}(Id \otimes \delta_1^2)\gamma^{\frac{1}{2}} & \gamma^{\frac{1}{2}}(\gamma_1 \otimes \delta_1)\gamma^{\frac{1}{2}} \\ \gamma^{\frac{1}{2}}(\gamma_1 \otimes \delta_1)\gamma^{\frac{1}{2}} & \gamma^{\frac{1}{2}}(\gamma_1^2 \otimes Id)\gamma^{\frac{1}{2}} \end{pmatrix}.$$

Its partial trace, $D = \begin{pmatrix} tr(\gamma(Id_k \otimes \delta_1^2)) & tr(\gamma(\gamma_1 \otimes \delta_1)) \\ tr(\gamma(\gamma_1 \otimes \delta_1)) & tr(\gamma(\gamma_1^2 \otimes Id_k)) \end{pmatrix}_{2\times 2}$ is also positive semidefinite.

Thus $0 \leq \det(D) = tr(\gamma(Id_k \otimes \delta_1^2))tr(\gamma(\gamma_1^2 \otimes Id_m)) - tr(\gamma(\gamma_1 \otimes \delta_1))^2$.

Notice that

- $tr(\gamma(Id_k \otimes \delta_1^2)) = tr(\gamma_B \delta_1^2) \leq \|\gamma_B\|_\infty tr(\delta_1^2) = \|\gamma_B\|_\infty$,
- $tr(\gamma(\gamma_1^2 \otimes Id_m)) = tr(\gamma_A \gamma_1^2) \leq \|\gamma_A\|_\infty tr(\gamma_1^2) = \|\gamma_A\|_\infty$,
- $tr(\gamma(\gamma_1 \otimes \delta_1))^2 = \|\mathcal{R}(\gamma)\|_\infty^2$.

Hence $\|\mathcal{R}(\gamma)\|_\infty^2 \leq \|\gamma_A\|_\infty \|\gamma_B\|_\infty$. $\square$

These lemmas imply the first new connection for our special triad of quantum states. This theorem says that the spectral radius of any state that is PPT or SPC or invariant under realignment cannot exceed the largest singular values of its reduced states or of its own realignment.

Theorem 1. *Let $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ be a positive semidefinite Hermitian matrix. If γ is PPT or SPC or invariant under realignment, then*

$$\|\gamma\|_\infty \leq \min\{\|\gamma_A\|_\infty, \|\gamma_B\|_\infty, \|\mathcal{R}(\gamma)\|_\infty\}.$$

Proof. First, let γ be a PPT state. Hence γ^Γ is also a state.

Notice that $(\gamma^\Gamma)_A = \gamma_A$, $(\gamma^\Gamma)_B = \gamma_B^t$ and, by lemma 3, $\|\mathcal{R}(\gamma^\Gamma)\|_\infty = \|\mathcal{R}(\gamma)\|_\infty$.

By applying Lemma 4 on γ^Γ, we obtain

$$\|\gamma\|_\infty \leq \min\{\|(\gamma^\Gamma)_A\|_\infty, \|(\gamma^\Gamma)_B\|_\infty, \|\mathcal{R}(\gamma^\Gamma)\|_\infty\} = \min\{\|\gamma_A\|_\infty, \|\gamma_B\|_\infty, \|\mathcal{R}(\gamma)\|_\infty\}.$$

So the proof of the PPT case is complete.

Next, if γ is SPC or invariant under realignment, then $\gamma_A = \gamma_B$ or $\gamma_A = \gamma_B^t$ by [5, Corollary 25]. Hence, by Lemma 5, $\|\mathcal{R}(\gamma)\|_\infty \leq \|\gamma_A\|_\infty$.

It remains to prove that $\|\gamma\|_\infty \leq \|\mathcal{R}(\gamma)\|_\infty$, whenever γ is SPC or invariant under realignment. Notice that this inequality is trivial for matrices invariant under realignment. Thus, let γ be an SPC state.

As defined in the introduction, $\mathcal{R}(\gamma^\Gamma)$ is positive semidefinite. Applying Lemma 4 on $\mathcal{R}(\gamma^\Gamma)$, we obtain

$$\|\mathcal{R}(\gamma^\Gamma)^\Gamma\|_\infty \leq \|\mathcal{R}(\mathcal{R}(\gamma^\Gamma))\|_\infty.$$

Now, by items (7) and (2) of Lemma 1, $\mathcal{R}(\gamma^\Gamma)^\Gamma = \gamma F$ and $\mathcal{R}(\mathcal{R}(\gamma^\Gamma)) = \gamma^\Gamma$, where F is the flip operator. Therefore

$$\|\gamma F\|_\infty \leq \|\gamma^\Gamma\|_\infty.$$

Finally, $\|\gamma^\Gamma\|_\infty \leq \|\mathcal{R}(\gamma)\|_\infty$ by Lemma 4, and $\|\gamma\|_\infty = \|\gamma F\|_\infty$, since F is unitary. $\square$

4. Filter Normal Form for SPC States and Invariant under Realignment States

In this section we show that every SPC state and every invariant under realignment state can be put in the filter normal form. In addition their filter normal forms can still be chosen to be SPC and invariant under realignment, respectively (Theorem 2).

Then we show that every PPT state whose rank is equal to its two reduced ranks can be put in the filter normal form (see Theorem 3).

As described in the introduction, there are applications of this normal form in entanglement theory. Now, it has been noticed that this normal form is connected to an extension of Sinkhorn-Knopp theorem for positive maps [4,21]. This theorem concerns the existence of invertible matrices R, S such that $R^*T(SXS^*)R$ is doubly stochastic for a positive map $T(X)$ satisfying suitable conditions. So we start this section with some definitions and lemmas related to this theorem.

Let $V \in \mathcal{M}_k$ be an orthogonal projection and consider the sub-algebra of $\mathcal{M}_k : V\mathcal{M}_kV = \{VXV,\ X \in \mathcal{M}_k\}$. Let P_k denote the set of positive semidefinite Hermitian matrices of $\mathcal{M}_k$.

Definition 2. *Let us say that $T : V\mathcal{M}_kV \to V\mathcal{M}_kV$ is a positive map if $T(X) \in P_k \cap V\mathcal{M}_kV$ for every $X \in P_k \cap V\mathcal{M}_kV$. In addition, we say that a positive map $T : V\mathcal{M}_kV \to V\mathcal{M}_kV$ is doubly stochastic if the following equivalent conditions hold:*

(1) *the matrix $A_{m\times m}$, defined as $A_{ij} = tr(T(v_iv_i^*)w_jw_j^*)$, is doubly stochastic for every choice of orthonormal bases $v_1, \dots, v_m$ and $w_1, \dots, w_m$ of* $\mathrm{Im}(V)$,

(2) *$T(V) = T^*(V) = V$, where $T^* : V\mathcal{M}_kV \to V\mathcal{M}_kV$ is the adjoint of $T : V\mathcal{M}_kV \to V\mathcal{M}_kV$ with respect to the trace inner product.*

Definition 3. *A positive map $T : V\mathcal{M}_kV \to V\mathcal{M}_kV$ is said to be fully indecomposable if the following equivalent conditions hold:*

(1) *the matrix $A_{m\times m}$, defined as $A_{ij} = tr(T(v_iv_i^*)w_jw_j^*)$, is fully indecomposable [22] for every choice of orthonormal bases $v_1, \dots, v_m$ and $w_1, \dots, w_m$ of* $\mathrm{Im}(V)$,

(2) $\mathrm{rank}(X) + \mathrm{rank}(Y) < \mathrm{rank}(V)$, *whenever $X, Y \in (V\mathcal{M}_kV \cap P_k) \setminus \{0\}$and $tr(T(X)Y) = 0$,*

(3) $\mathrm{rank}(T(X)) > \mathrm{rank}(X)$, *$\forall X \in V\mathcal{M}_kV \cap P_k$ such that* $0 < \mathrm{rank}(X) < \mathrm{rank}(V)$.

Below we prove two lemmas concerning self-adjoint maps with respect to the trace inner product.

Lemma 6. *Let $T : V\mathcal{M}_kV \to V\mathcal{M}_kV$ be a fully indecomposable self-adjoint map. There is $R \in V\mathcal{M}_kV$ such that $R^*T(R(\cdot)R^*)R : V\mathcal{M}_kV \to V\mathcal{M}_kV$ is doubly stochastic.*

Proof. Since $T : V\mathcal{M}_kV \to V\mathcal{M}_kV$ is fully indecomposable, it has total support [21, Lemma 2.3] or it has a positive achievable capacity [4]. So there are matrices $A, B \in V\mathcal{M}_kV$ such that $\mathrm{rank}(A) = \mathrm{rank}(B) = \mathrm{rank}(V)$ and $T_1(X) = B^*T(AXA^*)B$ is doubly stochastic [21, Theorem 3.7]. Notice that T_1 still is fully indecomposable.

Now, $T_2(X) = T_1^*(X) = A^*T(B(\cdot)B^*)A$ is also doubly stochastic and

$$T_2(X) = C^*T_1(DXD^*)C, \tag{1}$$

where $C = B^+A$, $D = A^+B$ and Y^+ is the pseudo-inverse of Y.

Let $C = EFG^*$ and $D = HLJ^*$ be the SVD decompositions of C and D, where

(1) $E = (e_1, \dots, e_m), G = (g_1, \dots, g_m), H = (h_1, \dots, h_m), J = (j_1, \dots, j_m) \in \mathcal{M}_{k\times m}$ and the columns of each of these matrices form an orthonormal basis of $\mathrm{Im}(V)$.

(2) $F = diagonal(f_1, \dots, f_m), L = diagonal(l_1, \dots, l_m)$ and $f_i > 0, l_i > 0$ for every i.

Next, define $R, S \in \mathcal{M}_{m\times m}$ as

$$R_{ik} = tr(T_2(j_i j_i^*) g_k g_k^*) \text{ and } S_{ik} = tr(T_1(h_i h_i^*) e_k e_k^*).$$

By Equation (1), $R_{ik} = l_i^2 f_k^2 \, S_{ik}$, i.e., $R = L^2 S F^2$.

Thus, L^2, F^2 are positive diagonal matrices such that L^2SF^2 is doubly stochastic by Definition 2. Recall that S is a fully indecomposable matrix by Definition 3.

Since S is fully indecomposable, by a theorem proved in [23], the diagonal matrices L^2 and F^2 such that L^2SF^2 is doubly stochastic must be unique up to multiplication by positive numbers, but $Id.S.Id$ is also doubly stochastic. Thus, $L = a^{-2}Id$ and $F = a^2 Id$ for some $a > 0$.

Therefore, $B^+A = C = a^2U$, where $U = EG^*$. Notice that $UVU^* = V$.

In addition, $BB^+A = a^2BU$. Since $BB^+ = V$ and $VA = A$, we obtain $A = a^2BU$.

Thus,

$$B^*T(A(\cdot)A^*)B = B^*T((a^2B)U(\cdot)U^*(a^2B)^*)B = (aB)^*T((aB)U(\cdot)U^*(aB)^*)(aB).$$

Finally, $(aB)^*T((aB)(\cdot)(aB)^*)(aB)$ is doubly stochastic too, since $V = UVU^*$. $\square$

Lemma 7. *Let $T : V\mathcal{M}_kV \to V\mathcal{M}_kV$ be a self-adjoint positive map such that $v \notin \ker(T(vv^*))$ for every $v \in \text{Im}(V) \setminus \{\vec{0}\}$. Then there is $R \in V\mathcal{M}_kV$ such that $R^*T(R(\cdot)R^*)R : V\mathcal{M}_kV \to V\mathcal{M}_kV$ is doubly stochastic.*

Proof. This proof is an induction on the $\text{rank}(V)$.

If $\text{rank}(V) = 1$, then $V\mathcal{M}_kV = \{\lambda vv^*,\ \lambda \in \mathbb{C}\}$. Thus, $T(vv^*) = \mu vv^*$, where $\mu > 0$ by hypothesis.

Define $R = \frac{1}{\sqrt[4]{\mu}} vv^*$. So $R^*T(Rvv^*R^*)R = vv^*$. Thus, $R^*T(R(\cdot)R^*)R : V\mathcal{M}_kV \to V\mathcal{M}_kV$ is a self-adjoint doubly stochastic map.

Let $\text{rank}(V) = n > 1$ and assume the validity of this theorem whenever the rank of the orthogonal projection is less than n.

Consider all pairs of orthogonal projections (V_1, W_1) such that

$$V_1, W_1 \in V\mathcal{M}_kV \setminus \{0\} \text{ and } 0 = tr(T(V_1)W_1).$$

Since there is no $v \in \text{Im}(V) \setminus \{\vec{0}\}$ such that $tr(T(vv^*)vv^*) = 0$, $\text{Im}(V_1) \cap \text{Im}(W_1) = \{\vec{0}\}$. So

$$\text{rank}(V_1) + \text{rank}(W_1) \leq \text{rank}(V).$$

If for every aforementioned pair (V_1, W_1), we have $\text{rank}(V_1) + \text{rank}(W_1) < \text{rank}(V)$, then T is fully indecomposable by Definition 3. So the result follows by Lemma 6.

Let us assume that there is such a pair (V_1, W_1) satisfying $\text{rank}(V_1) + \text{rank}(W_1) = \text{rank}(V)$.

Since $\text{Im}(V_1) \cap \text{Im}(W_1) = \{\vec{0}\}$, there is $S \in V\mathcal{M}_kV$ such that $SV_1S^* = V_1$ and $S(V - V_1)S^* = W_1$. Define $T'(X) = S^*T(SXS^*)S$. Notice that $tr(T'(V_1)(V - V_1)) = 0$.

Next, since T is self-adjoint so is T', hence $tr(T'(V - V_1)V_1) = 0$.

These last two equalities imply that

$$T'(V_1\mathcal{M}_kV_1) \subset V_1\mathcal{M}_kV_1 \text{ and } T'((V - V_1)\mathcal{M}_k(V - V_1)) \subset (V - V_1)\mathcal{M}_k(V - V_1). \quad (2)$$

Of course the restrictions $T'|_{V_1\mathcal{M}_kV_1}$ and $T'|_{(V-V_1)\mathcal{M}_k(V-V_1)}$ are self-adjoint and there is no $v \in \text{Im}(V_1) \setminus \{\vec{0}\}$ or $v \in \text{Im}(V - V_1) \setminus \{\vec{0}\}$ such that $tr(T'(vv^*)vv^*) = 0$.

By induction hypothesis, there are $R_1 \in V_1\mathcal{M}_kV_1$ and $R_2 \in (V - V_1)\mathcal{M}_k(V - V_1)$ such that

- $R_1^*T'(R_1(\cdot)R_1^*)R_1 : V_1\mathcal{M}_kV_1 \to V_1\mathcal{M}_kV_1$ is doubly stochastic, i.e.,

$$R_1^*T'(R_1(V_1)R_1^*)R_1 = V_1 \quad (3)$$

- $R_2^*T'(R_2(\cdot)R_2^*)R_2 : (V - V_1)\mathcal{M}_k(V - V_1) \to (V - V_1)\mathcal{M}_k(V - V_1)$ is doubly stochastic, i.e.,

$$R_2^*T'(R_2(V - V_1)R_2^*)R_2 = V - V_1 \quad (4)$$

Set $R = R_1 + R_2 \in V\mathcal{M}_kV$. Notice that $T''(X) = R^*T'(RXR^*)R$ is self-adjoint and

$$\begin{aligned} T''(V) &= T''(V_1 + V - V_1) = T''(V_1) + T''(V - V_1) \\ &= R^*T'(RV_1R^*)R + R^*T'(R(V - V_1)R^*)R \\ &= R^*T'(R_1V_1R_1^*)R + R^*T'(R_2(V - V_1)R_2^*)R \\ &= R_1^*T'(R_1V_1R_1^*)R_1 + R_2^*T'(R_2(V - V_1)R_2^*)R_2, \text{ by equation 2,} \\ &= V_1 + (V - V_1) = V, \text{ by equations 3 and 4.} \end{aligned}$$

Hence, $T'' : V\mathcal{M}_kV \to V\mathcal{M}_kV$ is doubly stochastic. □

Corollary 1. *Let $A \in \mathcal{M}_k \otimes \mathcal{M}_k$ be an Hermitian matrix such that $G_A : \mathcal{M}_k \to \mathcal{M}_k$ is a self-adjoint positive map and $tr(A(vv^* \otimes vv^*)) > 0$ for every $v \in \mathbb{C}^k \setminus \{0\}$. There is an invertible matrix $R \in \mathcal{M}_k$ such that $(R^* \otimes R^*)A(R \otimes R) = \sum_{i=1}^n \lambda_i\gamma_i \otimes \gamma_i$, where*

(1) $\lambda_1 = 1$ and $\gamma_1 = \frac{Id}{\sqrt{k}}$
(2) $\lambda_i \in \mathbb{R}$ and $\gamma_i = \gamma_i^*$ for every i,
(3) $1 \geq |\lambda_i|$ for every i,
(4) $tr(\gamma_i\gamma_j) = 0$ for every $i \neq j$ and $tr(\gamma_i^2) = 1$ for every i.

Proof. By the definition of $G_A : \mathcal{M}_k \to \mathcal{M}_k$ (given in the introduction), notice that

$$0 < tr(A(vv^* \otimes vv^*)) = tr(G_A(vv^*)vv^*).$$

Hence $v \notin \ker G_A(vv^*)$ for every $v \in \mathbb{C}^k \setminus \{0\}$.

By Lemma 7, there is an invertible matrix $R \in \mathcal{M}_k$ such that $R^*G_A(RXR^*)R$ is doubly stochastic.

Define $B = (R^* \otimes R^*)A(R \otimes R)$ and notice that $G_B(X) = R^*G_A(RXR^*)R$. Therefore, G_B is a self-adjoint doubly stochastic map, i.e., $G_B(\frac{Id}{\sqrt{k}}) = \frac{Id}{\sqrt{k}}$.

Let $\gamma_1 = \frac{Id}{\sqrt{k}}, \gamma_2, \ldots, \gamma_{k^2}$ be an orthonormal basis of $\mathcal{M}_k$ formed by Hermitian eigenvectors of the self-adjoint positive map $G_B : \mathcal{M}_k \to \mathcal{M}_k$ such that

- $G_B(\gamma_i) = \lambda_i\gamma_i$, where $|\lambda_i| > 0$ for $1 \leq i \leq n$
- $G_B(\gamma_i) = 0$, for $i > n$.

Since G_B is a positive map satisfying $G_B(Id) = Id$, its spectral radius is 1 [24, Theorem 2.3.7]. So $|\lambda_i| \leq 1$ for every i.

Finally, by the definition of G_B,

$$B = \frac{Id}{\sqrt{k}} \otimes G_B\left(\frac{Id}{\sqrt{k}}\right) + \gamma_2 \otimes G_B(\gamma_2) + \ldots + \gamma_{k^2} \otimes G_B(\gamma_{k^2}) = \sum_{i=1}^n \lambda_i\gamma_i \otimes \gamma_i.$$

□

The next theorem shows that SPC states and invariant under realignment states can be put in the filter normal form retaining their SPC structure and invariance under realignment, respectively.

Theorem 2. *Let $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ be a positive semidefinite Hermitian matrix such that* $\text{rank}(\gamma_A) = k$. *There is an invertible matrix $R \in \mathcal{M}_k$ such that*

(1) $(R^* \otimes R^*)\gamma(R \otimes R) = \sum_{i=1}^n \lambda_i\gamma_i \otimes \gamma_i$, if $\mathcal{R}(\gamma^\Gamma)$ is positive semidefinite;
(2) $(R^* \otimes R^t)\gamma(R \otimes \overline{R}) = \sum_{i=1}^n \lambda_i\gamma_i \otimes \overline{\gamma_i}$, if $\mathcal{R}(\gamma)$ is positive semidefinite,

where

a) $\lambda_1 = \frac{1}{k}$ and $\gamma_1 = \frac{Id}{\sqrt{k}}$
b) $\frac{1}{k} \geq \lambda_i > 0$ and $\gamma_i = \gamma_i^*$ for every i,
c) $tr(\gamma_i\gamma_j) = 0$ for every $i \neq j$ and $tr(\gamma_i^2) = 1$ for every i.

Proof. (1) If γ is a state such that $\mathcal{R}(\gamma^{\Gamma})$ is positive semidefinite, then, by [5, corollary 25], γ can be written as $\gamma = \sum_{i=1}^{n} a_i B_i \otimes B_i$, where $a_i > 0$, $B_i = B_i^*$ and $tr(B_i^2) = 1$ for every i, and $tr(B_i B_j) = 0$ for $i \neq j$.

Hence $G_\gamma(X) = \sum_{i=1}^{n} a_i B_i tr(B_i X)$ is a self-adjoint map with positive eigenvalues $a_1, \dots, a_n$ and possibly some null eigenvalues. In addition, since γ is positive semidefinite, $G_\gamma(X)$ is a positive map.

Now, let $v \in \mathbb{C}^k$ be such that

$$0 = tr(\gamma(vv^* \otimes vv^*)) = \sum_{i=1}^{n} a_i tr(B_i vv^*)^2.$$

Since $a_i > 0$ and $tr(B_i vv^*) \in \mathbb{R}$ for every i, $tr(B_i vv^*) = 0$ for every i. Therefore,

$$tr(\gamma_A vv^*) = \sum_{i=1}^{n} a_i tr(B_i) tr(B_i vv^*) = 0.$$

By hypothesis γ_A is positive definite, hence $v = 0$.

So, by Corollary 1, there is a invertible matrix R such that

$$(R^* \otimes R^*)\gamma(R \otimes R) = \sum_{i=1}^{n} \lambda_i \gamma_i \otimes \gamma_i \tag{5}$$

satisfies the four conditions of that corollary. It remains to show that $\lambda_i > 0$ and then we multiply Equation (5) by $\frac{1}{k}$ to obtain our desired result.

Finally, since G_γ has only non-negative eigenvalues and $\lambda_1, \dots, \lambda_n$ are non-null eigenvalues of $R^* G_\gamma(RXR^*)R$ (as seen in the proof of Corollary 1), $\lambda_1, \dots, \lambda_n$ are positive.

(2) If γ is a state such that $\mathcal{R}(\gamma)$ is positive semidefinite, then, by [5, Corollary 25], γ can be written as $\gamma = \sum_{i=1}^{n} a_i B_i \otimes \overline{B_i}$, where $a_i > 0$, $B_i = B_i^*$ and $tr(B_i^2) = 1$ for every i, and $tr(B_i B_j) = 0$ for $i \neq j$.

Consider $\gamma^{\Gamma} = \sum_{i=1}^{n} a_i B_i \otimes B_i$ and notice that $G_{\gamma^{\Gamma}}(X) = G_\gamma(X)^t$ is also a positive map. Now repeat the proof of item (1) for γ^{Γ}. Hence there is a invertible matrix R such that

$$(R^* \otimes R^*)\gamma^{\Gamma}(R \otimes R) = \sum_{i=1}^{n} \lambda_i \gamma_i \otimes \gamma_i, \tag{6}$$

where γ_i and λ_i satisfy all the required conditions. Finally,

$$(R^* \otimes R^t)\gamma(R \otimes \overline{R}) = \sum_{i=1}^{n} \lambda_i \gamma_i \otimes \overline{\gamma_i}.$$

□

The next corollary says that after a local operation any state possess a Schmidt decomposition such that one of its matrices is a multiple of the identity.

Corollary 2. *Let $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ be a state such that* rank$(\gamma_A) = k$. *There is an invertible matrix $R \in \mathcal{M}_k$ such that $(R^* \otimes Id)\gamma(R \otimes Id) = \sum_{i=1}^{n} a_i \gamma_i \otimes \delta_i$, where*

a) $a_1 \geq a_i > 0$, for every $1 \leq i \leq n$, and $\gamma_1 = \frac{Id}{\sqrt{k}}$,

b) $\gamma_i = \gamma_i^*$, $\delta_i = \delta_i^*$ for every i,

c) $tr(\gamma_i \gamma_j) = tr(\delta_i \delta_j) = 0$ for every $i \neq j$ and $tr(\gamma_i^2) = tr(\delta_i^2) = 1$.

Proof. First, since γ is a state, so is $F\overline{\gamma}F$. Therefore, $\gamma * F\overline{\gamma}F$ is positive semidefinite by item $a)$ of Remark 1.

Now, by items (8) and (9) of Lemma 1,

$$\mathcal{R}(\gamma * F\overline{\gamma}F) = \mathcal{R}(\gamma)\mathcal{R}(\gamma)^*.$$

It is not difficult to check that rank$((\gamma * F\overline{\gamma}F)_A) = k$, we leave it for last.

By item (2) of Theorem 2, there is an invertible matrix $R \in \mathcal{M}_k$ such that

$$(R^* \otimes R^t)(\gamma * F\overline{\gamma}F)(R \otimes \overline{R}) = \sum_{i=1}^{n} \lambda_i \gamma_i \otimes \gamma_i,$$

where

$a)$ $\lambda_1 = \frac{1}{k}$ and $\gamma_1 = \frac{Id}{\sqrt{k}}$,

$b)$ $\frac{1}{k} \geq \lambda_i > 0$ and $\gamma_i = \gamma_i^*$ for every i,

$c)$ $tr(\gamma_i\gamma_j) = 0$ for every $i \neq j$ and $tr(\gamma_i^2) = 1$ for every i.

Define $\delta = (R^* \otimes Id)\gamma(R \otimes Id)$ and notice that

$$\delta * F\delta^t F = \delta * F\overline{\delta}F = (R^* \otimes R^t)(\gamma * F\overline{\gamma}F)(R \otimes \overline{R}).$$

Thus, $G_{\delta * F\delta^t F}(\frac{Id}{k}) = \lambda_1 \frac{Id}{k}$.

By item $c)$ of Remark 1, $F_\delta(G_\delta(\frac{Id}{\sqrt{k}})) = G_{\delta * F\delta^t F}(\frac{Id}{\sqrt{k}})^t = \lambda_1 \frac{Id}{\sqrt{k}}$. So $F_\delta(G_\delta(Id)) = \lambda_1 Id$.

By [24, Theorem 2.3.7], λ_1 is the largest eigenvalue of the positive map $F_\delta \circ G_\delta$. So $\sqrt{\lambda_1}$ is the largest singular value of G_δ and F_δ, since they are adjoints.

Next, let $\delta_1 = \frac{Id}{\sqrt{k}}, \delta_2, \ldots, \delta_{k^2}$ be an orthonormal basis of $\mathcal{M}_k$ formed by Hermitian eigenvectors of $F_\delta \circ G_\delta : \mathcal{M}_k \to \mathcal{M}_k$.

Notice that $(R^* \otimes Id)\gamma(R \otimes Id) = \delta = \sum_{i=1}^{k^2} \delta_i \otimes G_\delta(\delta_i)$.

If $G_\delta(\delta_i) \neq 0_{k\times k}$, for $1 \leq i \leq n$, then define $a_i = \|G_\delta(\delta_i)\|_2 > 0$. Thus, $\delta = \sum_{i=1}^n a_i\delta_i \otimes \frac{1}{a_i}G_\delta(\delta_i)$.

Notice that $G_\delta(\delta_i)^* = G_{\delta^*}(\delta_i^*) = G_\delta(\delta_i)$, since δ and δ_i are Hermitian matrices. Moreover, by the definition of a_i,

$$tr(\tfrac{1}{a_i}G_\delta(\delta_i)\tfrac{1}{a_i}G_\delta(\delta_i)) = 1, \quad 1 \leq i \leq n.$$

In addition, since $F_\delta(G_\delta(\delta_j))$ is a multiple of δ_j, δ_i is orthogonal to $F_\delta(G_\delta(\delta_j))$ for $i \neq j$,

$$tr(\tfrac{1}{a_i}G_\delta(\delta_i)\tfrac{1}{a_j}G_\delta(\delta_j)) = \tfrac{1}{a_i a_j} tr(\delta_i F_\delta(G_\delta(\delta_j))) = 0.$$

Finally, $a_1^2 = tr(G_\delta(\delta_1)^2) = tr(\delta_1 F_\delta(G_\delta(\delta_1))) = \lambda_1 tr(\delta_1^2) = \lambda_1$, so $a_1 = \sqrt{\lambda_1}$. Notice also that, for every i, $a_i \leq$ the largest singular value of G_δ, which is $\sqrt{\lambda_1} = a_1$.

It remains to check that $\text{rank}((\gamma * F\overline{\gamma}F)_A) = k$. Indeed, since F_γ and G_γ are adjoints, $\text{Im}(F_\gamma \circ G_\gamma) = \text{Im}(F_\gamma)$. So there are Hermitian matrices $Y, Z \in \mathcal{M}_k$ such that $F_\gamma(G_\gamma(Y + iZ)) = F_\gamma(Id) = \gamma_A$, which is positive definite. Since F_γ and G_γ are also positive maps, they leave the set of Hermitian matrices invariant, hence $F_\gamma(G_\gamma(Y)) = \gamma_A$.

There is $\lambda > 0$ such that $Id - \lambda Y$ is positive semidefinite, so $F_\gamma(G_\gamma(Id - \lambda Y))$ is positive semidefinite too. Thus, $F_\gamma(G_\gamma(Id)) = F_\gamma(G_\gamma(Id - \lambda Y)) + \lambda\gamma_A$, which is positive definite.

Finally, since γ and $\gamma * F\overline{\gamma}F$ are Hermitian matrices

$$F_{\gamma * F\overline{\gamma}F}(\cdot) = G^*_{\gamma * F\overline{\gamma}F}(\cdot) = ((F_\gamma \circ G_\gamma)^t)^* = G_\gamma^* \circ F_\gamma^*((\cdot)^t) = F_\gamma \circ G_\gamma((\cdot)^t),$$

where the second equality comes from item $c)$ of Remark 1.

Therefore $(\gamma * F\overline{\gamma}F)_A = F_{\gamma * F\overline{\gamma}F}(Id) = F_\gamma(G_\gamma(Id))$, which is positive definite. $\square$

For the last result of this section, we finally prove that every PPT state whose rank coincides with its two reduced ranks can be put in the filter normal form. This is the key result to prove Theorem 6. See how the relations between linear contractions (Proposition 1) are important in proving this result.

Theorem 3. *If $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ is a PPT state such that* $\text{rank}(\gamma) = \text{rank}(\gamma_A) = \text{rank}(\gamma_B) = k$*, then there are invertible matrices $R, S \in \mathcal{M}_k$ such that $\delta_A = \delta_B = \frac{1}{k}Id$, where $\delta = (R \otimes S)\gamma(R^* \otimes S^*)$.*

Proof. First, define $\gamma_1 = (Id \otimes S)\gamma(Id \otimes S^*)$ such that $(\gamma_1)_B = \frac{1}{k}Id$, where S is invertible.

Since γ_1 is also PPT, $\|\gamma_1\|_\infty \leq \|(\gamma_1)_B\|_\infty = \frac{1}{k}$, by Theorem 1.

Hence $1 = tr((\gamma_1)_B) = tr(\gamma_1) \leq \|\gamma_1\|_\infty \text{rank}(\gamma_1) = \frac{1}{k}.k = 1$. So γ_1 has k eigenvalues equal to $\frac{1}{k}$ and the others 0.

Notice that $(F\overline{\gamma_1}F)^\Gamma$ is positive semidefinite and so is $(\gamma_1 * F\overline{\gamma_1}F)^\Gamma = \gamma_1 * (F\overline{\gamma_1}F)^\Gamma$ as a $*-$product of two positive semidefinite Hermitian matrices by item $a)$ of Remark 1. So the positive semidefinite Hermitian matrix $\gamma_1 * F\overline{\gamma_1}F$ is PPT.

Now, by items (8) and (9) of Lemma 1, $\mathcal{R}(\gamma_1 * F\overline{\gamma_1}F) = \mathcal{R}(\gamma_1)\mathcal{R}(\gamma_1)^*$, which is positive semidefinite.

Next, on the one hand $tr(\mathcal{R}(\gamma_1 * F\overline{\gamma_1}F)) = tr(\mathcal{R}(\gamma_1)\mathcal{R}(\gamma_1)^*) = tr(\gamma_1\gamma_1^*) = \frac{1}{k}$, since $\mathcal{R}$ is an isometry.

On the other hand $\|\mathcal{R}(\gamma_1 * F\overline{\gamma_1}F)^\Gamma\|_1 = \|\mathcal{R}(\gamma_1 * (F\overline{\gamma_1}F)F)\|_1$ (by item (6) of Lemma 1)

$$= \|\mathcal{R}(\gamma_1 * (F\overline{\gamma_1}F)F)F\|_1 \text{ (since } F \text{ is unitary)}$$

$$= \|(\gamma_1 * (F\overline{\gamma_1}F))^{\Gamma}\|_1 \text{ (by item (4) of Lemma 1)}$$

$$= tr(\gamma_1 * (F\overline{\gamma_1}F)) \text{ (since } \gamma_1 * (F\overline{\gamma_1}F) \text{ is PPT)}$$

$$= tr((\gamma_1)_B(\gamma_1)_B^*) = \tfrac{1}{k} \text{ (by item } c) \text{ of Remark 1).}$$

Therefore, $tr(\mathcal{R}(\gamma_1 * F\overline{\gamma_1}F)) = \|\mathcal{R}(\gamma_1 * F\overline{\gamma_1}F)^{\Gamma}\|_1$.

Since $\mathcal{R}(\gamma_1 * F\overline{\gamma_1}F)$ is positive semidefinite, this last equality means that $\mathcal{R}(\gamma_1 * F\overline{\gamma_1}F)^{\Gamma}$ is positive semidefinite, i.e., $\mathcal{R}(\gamma_1 * F\overline{\gamma_1}F)$ is PPT.

We have just discovered that $\mathcal{R}(\gamma_1 * F\overline{\gamma_1}F)$ and $\gamma_1 * F\overline{\gamma_1}F$ are PPT, but in this situation Lemma 2 says that $\gamma_1 * F\overline{\gamma_1}F = \mathcal{R}(\gamma_1 * F\overline{\gamma_1}F)$.

Now, by Corollary 2, there is an invertible matrix $R \in \mathcal{M}_k$ such that

$$\gamma_2 = (R^* \otimes Id)\gamma_1(R \otimes Id) = \sum_{i=1}^{n} \lambda_i A_i \otimes B_i,$$

where

$a)$ $\lambda_1 \geq \lambda_i > 0$ for every i and $A_1 = \frac{Id}{\sqrt{k}}$,

$b)$ $A_i = A_i^*$, $B_i = B_i^*$ for every i,

$c)$ $tr(A_iA_j) = tr(B_iB_j) = 0$ for every $i \neq j$ and $tr(A_i^2) = tr(B_i^2) = 1$.

In addition, we can normalize its trace, so assume that $tr(\gamma_2) = 1$.

Hence, $\gamma_2 * (F\overline{\gamma_2}F) = \sum_{i=1}^{n} \lambda_i^2 A_i \otimes \overline{A_i} = (R^* \otimes R^t)(\gamma_1 * (F\overline{\gamma_1}F))(R \otimes \overline{R})$.

As $\gamma_1 * (F\overline{\gamma_1}F)$, the positive semidefinite Hermitian matrix $\gamma_2 * (F\overline{\gamma_2}F)$ is also invariant under realignment because

$$\begin{aligned}\mathcal{R}(\gamma_2 * (F\overline{\gamma_2}F)) &= \mathcal{R}((R^* \otimes R^t)(\gamma_1 * (F\overline{\gamma_1}F))(R \otimes \overline{R}))\\ &= (R^* \otimes R^t)\mathcal{R}(\gamma_1 * (F\overline{\gamma_1}F))(R \otimes \overline{R}), \text{ by item (3) of lemma 1},\\ &= (R^* \otimes R^t)(\gamma_1 * (F\overline{\gamma_1}F))(R \otimes \overline{R}), \text{ since } \gamma_1 * F\overline{\gamma_1}F = \mathcal{R}(\gamma_1 * F\overline{\gamma_1}F).\\ &= \gamma_2 * (F\overline{\gamma_2}F).\end{aligned}$$

Now, notice that
$$\begin{aligned}k\lambda_1^2 &= tr\Big(\gamma_2 * (F\overline{\gamma_2}F)\ (A_1 \otimes \overline{A_1})\Big)k\\ &= tr\Big(\gamma_2 * (F\overline{\gamma_2}F)\ (\tfrac{Id}{\sqrt{k}} \otimes \tfrac{Id}{\sqrt{k}})\Big)k\\ &= tr(\gamma_2 * (F\overline{\gamma_2}F))\\ &= tr(\mathcal{R}(\gamma_2 * (F\overline{\gamma_2}F))), \text{ since } \gamma_2 * F\overline{\gamma_2}F = \mathcal{R}(\gamma_2 * F\overline{\gamma_2}F)\\ &= tr(\mathcal{R}(\gamma_2)\mathcal{R}(\gamma_2)^*), \text{ by items (8–9) of Lemma 1}\\ &= tr(\gamma_2\gamma_2^*), \text{ since } \mathcal{R} \text{ is an isometry.}\end{aligned}$$

Next, $\|\gamma_2\|_\infty^2 \leq \|\mathcal{R}(\gamma_2)\|_\infty^2$, since γ_2 is PPT and Theorem 1.

Notice that the largest singular value of G_{γ_2} is λ_1 by item $a)$ above and the definition of G_{γ_2}. Hence, by Lemma 3, $\|\mathcal{R}(\gamma_2)\|_\infty = \lambda_1$. Moreover, remind that rank$(\gamma_2) =$ rank$(\gamma_1) =$ rank$(\gamma) = k$. Therefore,

$$k\lambda_1^2 = tr(\gamma_2^2) \leq \|\gamma_2\|_\infty^2 \operatorname{rank}(\gamma_2) \leq \lambda_1^2.k.$$

The inequalities above are, in fact, equalities, which only happens when all the k non-null eigenvalues of γ_2 are equal to λ_1. Therefore, $1 = tr(\gamma_2) = k\lambda_1$. So $\lambda_1 = \frac{1}{k}$. In addition,

$$1 = tr(\gamma_2) = \lambda_1 tr(A_1)tr(B_1) = \frac{1}{k}\sqrt{k}\,tr(B_1).$$

So $tr(B_1) = \sqrt{k}$ and $tr(B_1^2) = 1$. Recall that $G_{\gamma_2}(\frac{1}{\lambda_1}A_1) = B_1$ is a positive semidefinite Hermitian matrix, since G_{γ_2} is a positive map and $\frac{1}{\lambda_1}A_1 = \sqrt{k}Id$. Under these conditions the only possibility for B_1 is $B_1 = \frac{Id}{\sqrt{k}}$. Therefore

$$(\gamma_2)_B = G_{\gamma_2}(Id) = G_{\gamma_2}(\sqrt{k}A_1) = \tfrac{Id}{k}, \quad (\gamma_2)_A = F_{\gamma_2}(Id) = F_{\gamma_2}(\sqrt{k}B_1) = \tfrac{Id}{k}.$$

Finally, notice that $\gamma_2 = (R^* \otimes Id)\gamma_1(R \otimes Id) = (R^* \otimes S)\gamma(R \otimes S^*)$. The proof is complete. □

5. Lower Bound for the Rank of the Special Triad

In this section, we prove that $\text{rank}(\gamma) \geq k$, whenever a state γ is PPT or SPC or invariant under realignment and $\text{rank}(\gamma_A) = \text{rank}(\gamma_B) = k$. Then we show that if $\text{rank}(\gamma) = k$, then γ is separable in each of these cases.

Explicit examples of states satisfying the hypotheses of these theorems are the separable states

$$\gamma_1 = \textstyle\sum_{i=1}^k a_i a_i^* \otimes a_i a_i^*, \quad \gamma_2 = \sum_{i=1}^k a_i a_i^* \otimes \overline{a_i} a_i^t,$$

where $a_1, \dots, a_k$ is any basis of $\mathbb{C}^k$. It is clear that their reduced ranks and their ranks are equal to k. In addition, it is clear that γ_1 is positive under partial transpose and $\mathcal{R}(\gamma_1^\Gamma) = \mathcal{R}(\gamma_2) = \gamma_2$. Thus γ_1 is also SPC and γ_2 is invariant under realignment.

We start this section by proving the SPC and the invariant under realignment cases. In their proofs we use the result that guarantees their separability whenever their non-null Schmidt coefficients are equal, which follows from the complete reducibility property [5, Proposition 15].

5.1. First Cases: SPC States and Invariant under Realignment States

Before proving the inequality, notice that by the symmetry of their Schmidt decompositions [5, Corollary 25], $\gamma_B = \gamma_A$ or $\gamma_B = \overline{\gamma_A}$, when γ is SPC or invariant under realignment, respectively. Hence $\text{rank}(\gamma_A) = \text{rank}(\gamma_B)$. In addition, we can assume without loss of generality that $\text{rank}(\gamma_A) = k$, otherwise we would be able to embed γ in $\mathcal{M}_s \otimes \mathcal{M}_s$, where $s = \text{rank}(\gamma_A)$, and obtain the same result.

Theorem 4. *If $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ is an SPC state such that $\text{rank}(\gamma_A) = k$, then $\text{rank}(\gamma) \geq k$. In addition, if the equality holds, then γ is separable.*

Proof. By Theorem 2, there is a invertible matrix R such that

$$\delta = (R^* \otimes R^*)\gamma(R \otimes R) = \textstyle\sum_{i=1}^n \lambda_i \gamma_i \otimes \gamma_i,$$

where $\lambda_1 = \dfrac{1}{k}$, $\gamma_1 = \dfrac{Id}{\sqrt{k}}$, $tr(\gamma_i \gamma_1) = \dfrac{tr(\gamma_i)}{\sqrt{k}} = 0$ for $i > 1$. So $\delta_A = \displaystyle\sum_{i=1}^n \lambda_i \gamma_i tr(\gamma_i) = \frac{Id}{k}$.

Now, notice that δ still is SPC by [5, Corollary 25]. Hence $\|\delta\|_\infty \leq \|\delta_A\|_\infty = \frac{1}{k}$, by Theorem 1.

So $\dfrac{1}{k} \geq \|\delta\|_\infty \geq \dfrac{tr(\delta)}{\text{rank}(\delta)} = \dfrac{1}{\text{rank}(\delta)}$. Hence $\text{rank}(\delta) \geq k$.

Since R is invertible, $\text{rank}(\gamma) = \text{rank}(\delta) \geq k$.

For the next part, assume $\text{rank}(\gamma) = k$. Then $\text{rank}(\delta) = k$. Therefore,

$$1 = tr(\delta) \leq \|\delta\|_\infty \,\text{rank}(\delta) = \frac{1}{k}.k = 1.$$

Since the equality $tr(\delta) = \|\delta\|_\infty \,\text{rank}(\delta)$ holds, the non-null eigenvalues of δ are equal to $\|\delta\|_\infty$. So $tr(\delta) = k\|\delta\|_\infty = 1$. Hence $\|\delta\|_\infty = \frac{1}{k}$ and $tr(\delta^2) = \frac{1}{k}$.

Next, since the linear contraction – $R((\cdot)^\Gamma)$ – preserves the Frobenius norm of δ,

$$\frac{1}{k} = tr(\delta^2) = tr(\mathcal{R}(\delta^\Gamma)\mathcal{R}(\delta^\Gamma)^*) \leq \|\mathcal{R}(\delta^\Gamma)\|_\infty \|\mathcal{R}(\delta^\Gamma)\|_1. \tag{7}$$

By item (5) of Lemma 1, $\mathcal{R}(\delta^\Gamma) = \mathcal{R}(\delta)F$. Since F is unitary, $\|\mathcal{R}(\delta)F\|_\infty = \|\mathcal{R}(\delta)\|_\infty$.

Therefore,

$$\|\mathcal{R}(\delta^\Gamma)\|_\infty = \|\mathcal{R}(\delta)F\|_\infty = \|\mathcal{R}(\delta)\|_\infty = \lambda_1 = \frac{1}{k}.$$

Now, since δ is SPC, by its definition, $\mathcal{R}(\delta^\Gamma)$ is positive semidefinite. Hence

$$\|\mathcal{R}(\delta^\Gamma)\|_1 \leq \|\mathcal{R}(\delta^\Gamma)^\Gamma\|_1.$$

By item (7) of Lemma 1, $\mathcal{R}(\delta^{\Gamma})^{\Gamma} = \delta F$. Thus, $\|\mathcal{R}(\delta^{\Gamma})\|_1 \leq \|\mathcal{R}(\delta^{\Gamma})^{\Gamma}\|_1 = \|\delta F\|_1 = \|\delta\|_1 = 1$.

Using these pieces of information in Equation (7) we obtain

$$\frac{1}{k} = tr(\mathcal{R}(\delta^{\Gamma})\mathcal{R}(\delta^{\Gamma})^*) \leq \|\mathcal{R}(\delta^{\Gamma})\|_\infty \|\mathcal{R}(\delta^{\Gamma})\|_1 \leq \frac{1}{k}.1. \tag{8}$$

Again, $tr(\mathcal{R}(\delta^{\Gamma})^2) = \|\mathcal{R}(\delta^{\Gamma})\|_\infty \|\mathcal{R}(\delta^{\Gamma})\|_1$ only holds if the non-null eigenvalues of the positive semidefinite Hermitian matrix $\mathcal{R}(\delta^{\Gamma})$ are equal to $\|\mathcal{R}(\delta^{\Gamma})\|_\infty = \lambda_1 = \dfrac{1}{k}$.

Finally, the non-null eigenvalues of $\mathcal{R}(\delta^{\Gamma})$ are the singular values of G_γ, which are the non-null Schmidt coefficients of δ, as explained in the proof of Lemma 3. Since δ is SPC and its non-null Schmidt coefficients are equal, δ is separable by [5, Proposition 15]. Since R is invertible, γ is separable too. □

The invariant under realignment counterpart is proved next in a similar way with minor modifications.

Theorem 5. *If $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_k$ is an invariant under realignment state such that* $\text{rank}(\gamma_A) = k$, *then* $\text{rank}(\gamma) \geq k$. *In addition, if the equality holds, then γ is separable.*

Proof. By Theorem 2, there is a invertible matrix R such that

$$\delta = (R^* \otimes R^t)\gamma(R \otimes \overline{R}) = \textstyle\sum_{i=1}^n \lambda_i \gamma_i \otimes \overline{\gamma_i},$$

where $\lambda_1 = \dfrac{1}{k}$, $\gamma_1 = \dfrac{Id}{\sqrt{k}}$, $tr(\gamma_i\gamma_1) = \dfrac{tr(\gamma_i)}{\sqrt{k}} = 0$ for $i > 1$. So $\delta_A = \displaystyle\sum_{i=1}^n \lambda_i \gamma_i tr(\gamma_i) = \dfrac{Id}{k}$.

Now, by item (3) of Lemma 1,

$$\mathcal{R}(\delta) = \mathcal{R}((R^* \otimes R^t)\gamma(R \otimes \overline{R})) = (R^* \otimes R^t)\mathcal{R}(\gamma)(R \otimes \overline{R}) = (R^* \otimes R^t)\gamma(R \otimes \overline{R}) = \delta.$$

Thus, δ is invariant under realignment. Hence $\|\delta\|_\infty \leq \|\delta_A\|_\infty = \frac{1}{k}$, by Theorem 1.

So $\dfrac{1}{k} \geq \|\delta\|_\infty \geq \dfrac{tr(\delta)}{\text{rank}(\delta)} = \dfrac{1}{\text{rank}(\delta)}$. Hence $\text{rank}(\delta) \geq k$.

Since R is invertible, $\text{rank}(\gamma) = \text{rank}(\delta) \geq k$.

For the next part, assume $\text{rank}(\gamma) = k$. Then $\text{rank}(\delta) = k$. Therefore,

$$1 = tr(\delta) \leq \|\delta\|_\infty \,\text{rank}(\delta) = \frac{1}{k}.k = 1.$$

Since the equality $tr(\delta) = \|\delta\|_\infty \text{rank}(\delta)$ holds, the non-null eigenvalues of δ are equal to $\|\delta\|_\infty$. Moreover, $tr(\delta) = k\|\delta\|_\infty = 1$. Hence $\|\delta\|_\infty = \frac{1}{k}$.

Since $\mathcal{R}(\delta) = \delta$, the non-null singular values of $\mathcal{R}(\delta)$ are the non-null eigenvalues of δ, which are equal in this case. Hence the non-null Schmidt coefficients of the Schmidt decomposition of δ are equal by Lemma 3. We know that every invariant under realignment state with equal non-null Schmidt coefficients is separable by [5, Proposition 15]. So δ is separable and so is γ, since R is invertible. □

5.2. The PPT Counterpart

For our final results, we need some tools developed in Sections 2 and 3 together with the complete reducibility property. First, we show that the rank of a PPT state is greater or equal to its reduced ranks in the next lemma.

Lemma 8. *Let $\gamma \in \mathcal{M}_k \otimes \mathcal{M}_m$ be a PPT state. Then* $\text{rank}(\gamma) \geq \max\{\text{rank}(\gamma_A), \text{rank}(\gamma_B)\}$.

Proof. Let us assume without loss of generality that $\max\{\text{rank}(\gamma_A), \text{rank}(\gamma_B)\} = \text{rank}(\gamma_A) = k$. So there is an invertible matrix $R \in \mathcal{M}_k$ such that $R\gamma_A R^* = \frac{1}{k}Id$. Define $\delta = (R \otimes Id)\gamma(R^* \otimes Id)$ and notice that $\delta_A = \frac{1}{k}Id$.

Since δ is PPT, by Theorem 1, $\|\delta\|_\infty \leq \|\delta_A\|_\infty = \frac{1}{k}$. Hence

$$1 = tr(\delta_A) = tr(\delta) \leq \|\delta\|_\infty \,\text{rank}(\delta) \leq \frac{1}{k}\text{rank}(\delta).$$

Thus, $\text{rank}(\gamma) = \text{rank}(\delta) \geq \max\{\text{rank}(\gamma_A), \text{rank}(\gamma_B)\}$. □

We complete this paper by proving that a PPT state with minimal rank must be separable which is the PPT counterpart of the Theorems 4 and 5.

Theorem 6. *If $\delta \in \mathcal{M}_k \otimes \mathcal{M}_k$ is a PPT state such that* $\text{rank}(\delta) = \text{rank}(\delta_A) = \text{rank}(\delta_B) = k$*, then δ is separable.*

Proof. This result is trivial in $\mathcal{M}_2 \otimes \mathcal{M}_2$, since every PPT state there is separable. Assume the result is true in $\mathcal{M}_i \otimes \mathcal{M}_i$ for $i < k$. Let us prove the result in $\mathcal{M}_k \otimes \mathcal{M}_k$.

First of all, by Theorem 3, there are invertible matrices $M, N \in \mathcal{M}_k$ such that $\gamma = (M \otimes N)\delta(M^* \otimes N^*)$ satisfies $\gamma_A = \gamma_B = \frac{1}{k} Id$. Notice that γ is also PPT and its rank is equal to k.

Next, by Theorem 1, since $\|\gamma\|_\infty \leq \|\gamma_A\| = \frac{1}{k}$ and

$$1 = tr(\gamma_A) = tr(\gamma) \leq \|\gamma\|_\infty \,\text{rank}(\gamma) \leq \frac{1}{k} k = 1,$$

we obtain all k non-null eigenvalues of γ equal to $\frac{1}{k}$.

Now, since γ is a positive semidefinite Hermitian matrix, the linear transformations F_γ and G_γ are positive maps and adjoints with respect to the trace inner product.

In addition, notice that $\frac{Id}{k} = \gamma_A = F_\gamma(Id)$ and $\frac{Id}{k} = \gamma_B = G_\gamma(Id)$. Hence $F_\gamma(G_\gamma(Id)) = \frac{1}{k^2} Id$.

By [24, Theorem 2.3.7], the spectral radius of the positive operator $F_\gamma \circ G_\gamma$ is $\frac{1}{k^2}$. Hence the largest singular value of G_γ and F_γ is $\frac{1}{k}$, since they are adjoints. Thus,

$$\|G_\gamma(X)\|_2 \leq \tfrac{1}{k} \text{ and } \|F_\gamma(X)\|_2 \leq \tfrac{1}{k}, \text{ whenever } \|X\|_2 = 1.$$

Next, since γ has k linearly independent eigenvectors associated to $\frac{1}{k}$, by combining 2 of them we can find an eigenvector $v \in \mathbb{C}^k \otimes \mathbb{C}^k$ associated to $\frac{1}{k}$ such that $\|v\|_2 = 1$ and $\text{rank}(v) = m < k$.

Notice that there are $R, S \in \mathcal{M}_k$ with rank m such that

$$v = (R \otimes S)u \ (u \text{ as defined in Remark 1}) \text{ and } \|RR^*\|_2 = \|SS^*\|_2 = 1.$$

$$\begin{aligned}
\text{Therefore } \tfrac{1}{k} &= tr(\gamma v v^*)\\
&= tr((R^* \otimes S^*)\gamma(R \otimes S)uu^t)\\
&= tr((R^* \otimes S^t)\gamma^\Gamma(R \otimes \overline{S})F) \ (F \text{ as defined in item } c) \text{ of Remark 1})\\
&\leq tr((R^* \otimes S^t)\gamma^\Gamma(R \otimes \overline{S})) \ (\text{since } \gamma^\Gamma \text{ and } Id - F \text{ are positive semidefinite})\\
&= tr(\gamma(RR^* \otimes SS^*))\\
&= tr(G_\gamma(RR^*)SS^*)\\
&\leq \|G_\gamma(RR^*)\|_2 \|SS^*\|_2 \leq \tfrac{1}{k}.1 \ (\text{since the largest singular value of } G_\gamma \text{ is } \tfrac{1}{k})
\end{aligned}$$

Therefore, all the inequalities above are equalities, which imply

(1) $G_\gamma(RR^*) = \lambda SS^*$ for some $\lambda > 0$, since G_γ is a positive map, and

(2) $\frac{1}{k} = \|G_\gamma(RR^*)\|_2 = \lambda \|SS^*\|_2 = \lambda$.

Hence $G_\gamma(RR^*) = \frac{1}{k} SS^*$. Analogously, we get $F_\gamma(SS^*) = \frac{1}{k} RR^*$, since $tr(\gamma(RR^* \otimes SS^*)) = tr(RR^* F_\gamma(SS^*))$.

Therefore, $F_\gamma(G_\gamma(RR^*)) = \frac{1}{k^2} RR^*$ and $\text{rank}(RR^*) = m < k$.

Since γ is PPT, by the complete reducibility property,

$$\gamma = (V \otimes W)\gamma(V \otimes W) + (V^\perp \otimes W^\perp)\gamma(V^\perp \otimes W^\perp), \tag{9}$$

where $V, W, V^\perp, W^\perp$ are orthogonal projections onto $\text{Im}(RR^*)$, $\text{Im}(SS^*)$, $\ker(RR^*)$ and $\ker(SS^*)$, respectively.

The proof is almost done, we just need to check that $(V \otimes W)\gamma(V \otimes W)$, $(V^\perp \otimes W^\perp)\gamma(V^\perp \otimes W^\perp)$ are multiples of states satisfying the hypotheses and since $\text{rank}(V) = \text{rank}(W) < k$ and $\text{rank}(V^\perp) = \text{rank}(W^\perp) < k$, the result follows by induction.

Notice that by Equation (9) and the definition of G_γ,

$$\text{Im}(G_\gamma(V)) \subset \text{Im}(W), \text{Im}(G_\gamma(V^\perp)) \subset \text{Im}(W^\perp) \text{ and}$$

$$\text{Im}(F_\gamma(W)) \subset \text{Im}(V), \text{Im}(F_\gamma(W^\perp)) \subset \text{Im}(V^\perp).$$

Next, recall that $V + V^{\perp} = W + W^{\perp} = Id$, $VV^{\perp} = WW^{\perp} = 0$ and

$$G_{\gamma}(V) + G_{\gamma}(V^{\perp}) = G_{\gamma}(Id) = \tfrac{1}{k} Id = \tfrac{1}{k} W + \tfrac{1}{k} W^{\perp},$$

$$F_{\gamma}(W) + F_{\gamma}(W^{\perp}) = F_{\gamma}(Id) = \tfrac{1}{k} Id = \tfrac{1}{k} V + \tfrac{1}{k} V^{\perp}.$$

Therefore

$$G_{\gamma}(V) = \tfrac{1}{k} W,\ F_{\gamma}(W) = \tfrac{1}{k} V \text{ and } G_{\gamma}(V^{\perp}) = \tfrac{1}{k} W^{\perp},\ F_{\gamma}(W^{\perp}) = \tfrac{1}{k} V^{\perp}.$$

Now, define

$$\gamma_1 = \tfrac{k}{m}(V \otimes W)\gamma(V \otimes W) \text{ and } \gamma_2 = \tfrac{k}{k-m}(V^{\perp} \otimes W^{\perp})\gamma(V^{\perp} \otimes W^{\perp}).$$

Notice that

$$(\gamma_1)_A = F_{\gamma_1}(Id) = \tfrac{k}{m} F_{\gamma}(W) = \tfrac{1}{m} V \ \text{ and } \ (\gamma_1)_B = G_{\gamma_1}(Id) = \tfrac{k}{m} G_{\gamma}(V) = \tfrac{1}{m} W.$$

Thus, $\max\{\text{rank}((\gamma_1)_A), \text{rank}((\gamma_1)_B)\} = \max\{\text{rank}(V), \text{rank}(W)\} = \max\{\text{rank}(R), \text{rank}(S)\} = m$.
Moreover, notice that

$$(\gamma_2)_A = F_{\gamma_2}(Id) = \tfrac{k}{k-m} F_{\gamma}(W^{\perp}) = \tfrac{1}{k-m} V^{\perp} \ \text{ and } \ (\gamma_2)_B = G_{\gamma_2}(Id) = \tfrac{k}{k-m} G_{\gamma}(V^{\perp}) = \tfrac{1}{k-m} W^{\perp}.$$

Therefore $\max\{\text{rank}((\gamma_2)_A), \text{rank}((\gamma_2)_B)\} = \max\{\text{rank}(V^{\perp}), \text{rank}(W^{\perp})\} = k - m$.
By their definitions, γ_1 and γ_2 are PPT. So, by Lemma 8, $\text{rank}(\gamma_1) \geq m$ and $\text{rank}(\gamma_2) \geq k - m$.
Recall that $k = \text{rank}(\gamma) = \text{rank}(\gamma_1) + \text{rank}(\gamma_2) \geq m + (k - m)$. Thus $\text{rank}(\gamma_1) = m$ and $\text{rank}(\gamma_2) = k - m$.
Since $\gamma = \frac{m}{k}\gamma_1 + \frac{k-m}{k}\gamma_2$, γ has k eigenvalues equal to $\frac{1}{k}$ and $\gamma_1\gamma_2 = 0$,

- γ_1 has m eigenvalues equal to $\frac{1}{m}$ and the others 0,
- γ_2 has $k - m$ eigenvalues equal to $\frac{1}{k-m}$ and the others 0.

Hence,

- γ_1 has m eigenvalues equal to $\frac{1}{m}$, $(\gamma_1)_A = \frac{1}{m}V$, $(\gamma_1)_B = \frac{1}{m}W$ and $\text{rank}(V) = \text{rank}(W) = m$.
- γ_2 has $k - m$ eigenvalues equal to $\frac{1}{k-m}$, $(\gamma_2)_A = \frac{1}{k-m}V^{\perp}$, $(\gamma_1)_B = \frac{1}{k-m}W^{\perp}$ and $\text{rank}(V^{\perp}) = \text{rank}(W^{\perp}) = k - m$.

By induction hypothesis, γ_1 and γ_2 are separable and so is γ and δ. $\square$

6. Summary and Conclusions

In this paper we proved new results for a triad of quantum state types which includes the positive under partial transpose type. We obtained the same upper bound for the spectral radius of these types of quantum states. We then showed that the first two types can be put in the filter normal form and the third type only under some restriction. Finally, we proved that there is a lower bound for their ranks and whenever this lower bound is attained these states are separable. This last result is another consequence of their complete reducibility property. This is sufficient evidence that these states are deeply connected. Moreover, their complete reducibility property is a unifying force connecting and providing many results in entanglement theory.

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Conflicts of Interest
No potential conflict of interest was reported by the author.

Data Availability Statement
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