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On some generalization of order cancellation law for subsets of topological vector space

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ABSTRACT. In this paper we generalize a result of R. Urbański from paper [7] which states that for subsets A, B, C of topological vector space X the following implication holds

$$A + B \subset B + C \implies A \subset C$$

provided that B is bounded and C is closed and convex. The generalization is given in Theorem 5 where we prove this result for k -convex subsets of a topological vector space. Also we introduce a notion of some abstract like-closure operation for subsets of linear space and we study its connections to order cancellation law.

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1. Introduction

Cancellation law plays an important role in many areas of mathematics and was studied by many authors (see for example [1], [2], [3], [5], [6], [7]). Namely if we had a commutative structure with one binary operation say $(S, +)$ and not every element in S is invertible then the cancellation law guarantees that the equation $x + b = b + c$ has exactly one solution $x = c$. This observation is crucial and allows us to introduce an equivalence relation in the cartesian

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product $S \times S$ in the following way

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c.$$

The quotient set $S \times S / \sim$ with a suitable defined operation has a structure of group and S can be embedded into it. This kind of constructions of embedding of a structure into a richer structure was used by many authors in many areas of mathematics even the well known construction of integers is made in this way.

2. Basic notations and Definitions

Let X be a vector space and let $A, B \subset X, \lambda \in \mathbb{R}$ then we define

$$A + B = \{a + b : a \in A, b \in B\},$$

$$\lambda A = \{\lambda a : a \in A\}.$$

It is clear that generally $2A \neq A + A$, thus the symbol $\sum_{j=1}^k A$ denotes the following sum

$$A + A + \dots + A$$

where the number of summands is equal k and it is generally not equal to the set kA .

We say that set A is *convex* if $\lambda A + (1 - \lambda)A = A$ for all $0 < \lambda < 1$.

If X is a topological vector space and let $A \subset X$, then the symbol $\overline{A}$ denotes the topological closure of this set and the symbol $\text{Int}(A)$ denotes its topological interior.

3. Main result

We start with the definition of the operation "like-closure" of a set.

Definition 1. Let X be a vector space and let $A, B \subset X$. A *like-closure* of the set A with respect to set B and point $b \in B$ is defined by

$$\overline{A}_b^B = \bigcap_{n \in \mathbb{N}} \left(\frac{1}{n}(B - b) + A \right)$$

Generally for fixed B the set $\overline{A}_b^B$ depends of the choice of the point b as shows the following example. Take $X = \mathbb{R}$ and $A = (0, 1), B = [0, 1]$ then $0 \in \overline{A}_b^B$ for $0 < b \leq 1$ but $0 \notin \overline{A}_0^B$.

But is easy to observe that in the case of normed space and B bounded set we have

$$\overline{A}_{b_1}^B = \overline{A}_{b_2}^B = \overline{A}$$

for all $b_1, b_2 \in \text{Int}(B)$.

Observe that for a convex set C the inclusion $A + B \subset B + C$ implies

$$A \subset \bigcap_{b \in B} \overline{C}_b^B.$$

To see this notice that if C is convex and $A + B \subset B + C$ then for all $n \in \mathbb{N}$ we have

$$nA + B \subset B + nC$$

hence for any $b \in B$ and $n \in \mathbb{N}$ we get

$$A \subset \frac{1}{n}(B - b) + C$$

but this implies that

$$A \subset \bigcap_{n \in \mathbb{N}} \left(\frac{1}{n}(B - b) + C \right) = \overline{C}_b^B$$

for any $b \in B$, hence

$$A \subset \bigcap_{b \in B} \overline{C}_b^B.$$

The above consideration shows that if $\bigcap_{b \in B} \overline{C}_b^B \subset C$ then we can cancel the set B from the inclusion $A + B \subset B + C$ in other words for such sets B, C the cancellation law holds.

Definition 2 Let X be a vector space and let $A \subset X$. We say that a set A is k -convex ($k > 0$) if for every $n \in \mathbb{N}$ the inclusion $\sum_{j=1}^n A \subset knA$ holds true.

Lemma 3 Let X be a vector and let $A, B, C \subset X$. If the set C is k -convex then the inclusion $A + B \subset B + C$ implies that

$$A + \frac{1}{n}(B - z) \subset \frac{1}{n}(B - z) + kC,$$

for any $z \in X$ and $n \in \mathbb{N}$.

Proof. Since $A + (B - z) \subset (B - z) + C$ we get

$$2A + (B - z) \subset A + A + (B - z) \subset A + (B - z) + C \subset (B - z) + C + C \subset (B - z) + 2kC,$$

and analogously

$$nA + (B - z) \subset (B - z) + \sum_{j=1}^n A \subset (B - z) + knC,$$

dividing the last inclusion by n we get thesis of this lemma.

Lemma 4. Let X be a topological vector and let $A, B, C \subset X$. If the set B is bounded then

$$\overline{C}_x^B \subset \overline{C}$$

for all $x \in X$.

Proof. Take any neighbourhood U of 0 then since B is bounded therefore for some n_0 we have $\frac{1}{n_0}(B - x) \subset U$ and hence

$$\overline{C}_x^B \subset \frac{1}{n_0}(B - x) + C \subset C + U$$

thus

$$\overline{C}_x^B \subset \bigcap_U (C + U) = \overline{C}.$$

Theorem 5 (Cancellation law for k -convex sets) Let X be a topological vector space and let $A, B, C \subset X$. Assume that B is bounded and C is k -convex then the implication

$$A + B \subset B + C \Rightarrow A \subset k\overline{C}$$

holds true.

Proof. By lemma 3 we get that

$$A + \frac{1}{n}(B - z) \subset \frac{1}{n}(B - z) + kC,$$

for any $z \in X$ and $n \in \mathbb{N}$. If we take any $b \in B$ then

$$A \subset A + \frac{1}{n}(B - b) \subset \frac{1}{n}(B - b) + kC,$$

for $n \in \mathbb{N}$. Hence

$$A \subset \bigcap_{n \in \mathbb{N}} \left(\frac{1}{n}(B - b) + kC \right) = \overline{kC}_b^B$$

but since B is bounded by lemma 4 we have $\overline{kC}_b^B \subset \overline{kC} = k\overline{C}$.

Since every convex set is 1-convex we obtain a cancellation law in version which was first proved by Urbański in [7] as following corollary

Corollary 6 Let X be a topological vector space and let $A, B, C \subset X$. Assume that B is bounded and C is closed and convex then the implication

$$A + B \subset B + C \Rightarrow A \subset C$$

holds true..

Remark 7. The class of k -convex sets is wider than the class of convex sets for example if $A \subset \mathbb{R}$ is bounded and $0 \in \text{Int}(A)$ then A is k -convex for some $k \in \mathbb{N}$.

Below in the Example 8 we give a construction of sets in arbitrary topological vector space, which are not convex nor p -convex ($0 < p < 1$), but they are k -convex for some $k \in \mathbb{N}$.

Example 8. Let X be a topological vector space and $A, B \subset X$ be a bounded convex sets such that $\overline{A} \cap \overline{B} = \emptyset$, $0 \in B$ and $A \subset kB$ for some $k \in \mathbb{N}$. Then the set $D = A \cup B$ is of course not p -convex nor convex. But

$$\text{conv}(A \cup B) \subset kB$$

hence for any $n \in \mathbb{N}$ we have

$$\frac{1}{n} \sum_{j=1}^n D \subset \text{conv}(D) = \text{conv}(A \cup B) \subset kB \subset kD$$

hence

$$\sum_{j=1}^n D \subset knD$$

therefore the set D is k -convex.

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