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On a New Parabolic Sobolev Embedding Map

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ABSTRACT. The purpose of the present article is to provide a new parabolic Sobolev embedding map between a parabolic weighted Sobolev space and the space of square-integrable functions on a cylinder. Furthermore, the embedding constant is furnished explicitly.

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1. Introduction and preliminaries

We furnish a new embedding map between a parabolic weighted Sobolev space and the space of measurable square-integrable functions on $C_T = \Omega \times (0, T)$ for $T > 0$ such that Ω is a domain in $\mathbb{R}^n$, and the weight is a locally integrable function (Theorem 1.1). The essential tool that we use is the parabolic capacity of a compact set of C_T (Lemma 2.1). Let us define a few useful definitions.

Let $(L_2(C_T), \|\cdot\|_{L_2(C_T)})$ be the set of the classical square-integrable measurable functions on C_T and endowed with the classical norm $\|\cdot\|_{L_2(C_T)}$.

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The weighted parabolic Sobolev space is defined by $\mathcal{W}_V(C_T)$ as the closure of $C_0^\infty(C_T)$ with respect to the norm $\|\cdot\|_{\mathcal{W}_V(C_T)}$ such that for $u \in \mathcal{W}_V(C_T)$, we have

$$\begin{aligned} \|u\|_{\mathcal{W}_V(C_T)}^2 &= \|\nabla_x u\|_{L_2(C_T)}^2 + \|u\|_{L_2(C_T, V(x,t)dxdt)}^2 \\ &= \int_{C_T} |\nabla_x u(x,t)|^2 dxdt + \int_{C_T} |u(x,t)|^2 V(x,t) dxdt, \end{aligned} \quad (1.1)$$

such that V is a locally integrable positive function on C_T and $\nabla_x u(x,t)$ represents the gradient of $u(x,t)$ with respect to the space variable $x \in \Omega$.

Remark 1.1. The density of smooth functions with compact support in a weighted Sobolev space where the weight function is a positive locally integrable function is treated in [7, Chapter 7], so the potential V should be locally integrable.

Now, we are in the position to define the parabolic capacity of a compact set in C_T .

Definition 1.1. Let K be a compact set of C_T , the 2-variational parabolic capacity of K is defined by

$$cap_{vp}(K, C_T) = \inf_{u \in C_0^\infty(C_T)} \left\{ \|\nabla_x u\|_{L_2(C_T)}^2 : u = 1 \text{ on } K \right\},$$

and we note $cap_{vp}(K) = cap_{vp}(K, \mathbb{R}^n \times (0, \infty))$.

Remark 1.2. When $K = B_\delta \times I_{\delta^2}$, the Cartesian product of B_δ , the a ball of radius δ , with I_{δ^2} , the closed real interval of length δ^2 , Ziemer showed that $cap_{vp}(B_\delta \times I_{\delta^2})$ is equivalent to δ^n [12]. Apropos of properties of variational capacity, e.g., see [2, 11, 13].

By employing the definition of an elliptic negligibility set [7, §14.1.1], let us define the parabolic negligibility set in C_T .

Definition 1.2. Let Q_δ be a closed cube of side δ , $\gamma \in (0, 1)$. The negligibility set in C_T is defined by the set $\mathcal{N}_\gamma = \mathcal{N}_\gamma(Q_\delta \times I_{\delta^2}, C_T)$ comprised by compacts set $F \subset Q_\delta \times I_{\delta^2}$ such that

$$[Q_\delta \times I_{\delta^2}] \setminus C_T \subset F \subset Q_\delta \times I_{\delta^2} \text{ and } cap_{vp}(F) \leq \gamma cap_{vp}(Q_\delta \times I_{\delta^2}).$$

Each element of $\mathcal{N}_\gamma$ is called γ -negligible set in $Q_\delta \times I_{\delta^2}$.

Remark 1.3. If $\mathcal{N}_\gamma$ is not an empty set, then $Q_\delta \times I_{\delta^2}$ is almost completely inside in C_T such that a few parts of $Q_\delta \times I_{\delta^2}$ are not in C_T and have small parabolic capacities.

If $\gamma \rightarrow 1^-$, then $\mathcal{N}_1$ is the set of all compacts set of the bounded closed cylinder $Q_\delta \times I_{\delta^2}$.

The interior parabolic capacity diameter (IPCD) associated to C_T is given in terms of an auxiliary averaged potential, the case of IPCD corresponding to Ω , see [7, §16.2], [9], [10, page 207]. Precisely, we have the following definition.

Definition 1.3. The parabolic capacity interior diameter of C_T is the number

$$\delta_{V, \gamma, C_T} = \sup_{Q_\delta \times I_{\delta^2} \in \mathcal{Q}_{C_T}} \left\{ \delta : \delta^n \geq \inf_{F \in \mathcal{N}_\gamma} \int_{[Q_\delta \times I_{\delta^2}] \setminus F} V(x,t) dxdt \right\}.$$

such that $\mathcal{Q}_{C_T} = \{Q_\delta \times I_{\delta^2} : cap_{vp}([Q_\delta \times I_{\delta^2}] \setminus C_T) \leq \gamma cap_{vp}(Q_\delta \times I_{\delta^2})\}$.

Remark 1.4. When δ is very small, we have $\delta_{V,\gamma,C_T} > 0$ and when C_T contains a large cylinder, i.e., C_T coincides with $\mathbb{R}^n \times \mathbb{R}_+$ and V is identically equal to zero then $\delta_{V,\gamma,C_T} = \infty$, therefore $\delta_{V,\gamma,C_T} \in (0, \infty]$. We recall that $\delta_{V,\gamma,\Omega}$ is useful for showing the discreteness spectrum of a Schrödinger operator [8].

Our following main theorem states that $\mathcal{W}(C_T)$ is embedded in $L_2(C_T)$. Furthermore, the embedding constant is furnished explicitly. Precisely, we have.

Theorem 1.1. Let $u \in \mathcal{W}(C_T)$ and V be a positive potential locally integrable in C_T such that $\delta^n \leq \int_{[Q_\delta \times I_{\delta^2}] \setminus F} V(x, t) dx dt$ for all negligible compact set F of $Q_\delta \times I_{\delta^2}$. Then there is an almost explicit positive constant $\mathcal{K}_{n,\gamma,\delta_{V,\gamma,C_T}}$ relying on $n, \gamma, \delta_{V,\gamma,C_T}$, such that

$$\|u\|_{L_2(C_T)}^2 \leq \mathcal{K}_{n,\gamma,\delta_{V,\gamma,C_T}} \|u\|_{\mathcal{W}_V(C_T)}^2, \quad (1.2)$$

with $\mathcal{K}_{n,\gamma,\delta_{V,\gamma,C_T}} := 2^{2(n+1)} \delta_{V,\gamma,C_T}^2 \left(4 + \frac{\mathcal{C}(n)}{\gamma}\right)$ such that $\mathcal{C}(n)$ is a positive constant depending on n .

A version of Theorem 1.1 in a closed bounded cylinder is given as follows.

Theorem 1.2. Let $u \in C^\infty(Q_\delta \times I_{\delta^2})$ and V be a positive integrable potential in $Q_\delta \times I_{\delta^2}$ such that $\delta^n \leq \int_{[Q_\delta \times I_{\delta^2}] \setminus F} V(x, t) dx dt$ for all negligible compact set F of $Q_\delta \times I_{\delta^2}$. Then there is an explicit positive constant $D_{n,\gamma,\delta}$ relying on n, γ, δ such that

$$\|u\|_{L_2(Q_\delta \times I_{\delta^2})}^2 \leq D_{n,\gamma,\delta} \|u\|_{\mathcal{W}_V(Q_\delta \times I_{\delta^2})}^2. \quad (1.3)$$

A version of Theorem 1.1 in a closed bounded cylinder is given as follows.

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$$\|u\|_{L_2(Q_\delta \times I_{\delta^2})}^2 \leq D_{n,\gamma,\delta} \|u\|_{\mathcal{W}_V(Q_\delta \times I_{\delta^2})}^2. \quad (1.4)$$

The structure of the article. The second section focuses on a few meaningful lemmas allowing the proof Theorem 1.3. The third section concentrates on providing the proof of Theorem 1.1.

2. Miscellaneous lemmas and the proof of Theorem 1.3

The following lemma states a result on a parabolic capacity inequality.

Lemma 2.1. Let $u \in C^\infty(Q_\delta \times I_{\delta^2})$ such that u is equal to zero on a compact set $K \subset Q_\delta \times I_{\delta^2}$ and u satisfies

$$\frac{1}{\delta^{n+2}} \int_{Q_\delta \times I_{\delta^2}} |u(x, t)|^2 dx dt = 1. \quad (2.1)$$

Then, there is constant $\mathcal{C}(n)$ relying on n such that

$$\text{cap}_{vp}(K) \leq \mathcal{C}(n) \|\nabla_x u\|_{L_2(Q_\delta \times I_{\delta^2})}^2.$$

Proof. We use the scheme for proving elliptic capacity inequality in [5]. Without loss of generality, we consider the case when u is a smooth positive function. Let $\varphi = 1 - u$ and by assumption we have $u|_K = 0$, thus $\varphi|_K = 1$ and $\varphi_{av(Q_\delta \times I_{\delta^2})}$, the average of φ over $Q_\delta \times I_{\delta^2}$, is equal to one minus $u_{av(Q_\delta \times I_{\delta^2})}$. Then, by employing (2.1) and the Cauchy-Schwartz inequality, we have $\varphi_{av(Q_\delta \times I_{\delta^2})} \geq 0$. By applying norms properties, we have

$$\varphi_{av(Q_\delta \times I_{\delta^2})} \leq \delta^{-\frac{n+2}{2}} \|u - u_{av(Q_\delta \times I_{\delta^2})}\|_{L_2(Q_\delta \times I_{\delta^2})}. \quad (2.2)$$

Let us recall that there is a version of the parabolic Poincaré type inequality, see Theorem 2.2 in [1]. Whence, we apply it to Inequality (2.2), and we get

$$\varphi_{av(Q_\delta \times I_{\delta^2})} \leq A\delta^{-\frac{n+2}{2}+1} \|\nabla_x \varphi\|_{L_2(Q_\delta \times I_{\delta^2})}, \quad (2.3)$$

such that A is a real constant. Consequently, by using (2.3) and the parabolic Poincaré inequality, we have

$$\begin{aligned} \|\varphi\|_{L_2(Q_\delta \times I_{\delta^2})}^2 &\leq 2(\|\varphi - \varphi_{av(Q_\delta \times I_{\delta^2})}\|_{L_2(Q_\delta \times I_{\delta^2})}^2 + \|\varphi_{av(Q_\delta \times I_{\delta^2})}\|_{L_2(Q_\delta \times I_{\delta^2})}^2) \\ &\leq A\delta^2 \|\nabla_x \varphi\|_{L_2(Q_\delta \times I_{\delta^2})}^2. \end{aligned} \quad (2.4)$$

By definition of the parabolic capacity the infimum is taken over smooth functions with compact support in $\mathbb{R}^n \times (0, \infty)$, thus let us set up a suitable one by extending φ to a function φ_1 defined in $Q_{3\delta} \times I_{(3\delta)^2}$ as $\varphi_1(x, t) = \varphi(\frac{x}{3}, \frac{t}{3^2})$, and by applying a simple substitution we obtain

$$\int_{Q_{3\delta} \times I_{(3\delta)^2}} |\nabla_x \varphi_1(x, t)|^2 dx dt = 3^{n+2} \int_{Q_\delta \times I_{\delta^2}} |\nabla_x \varphi(x, t)|^2 dx dt, \quad (2.5)$$

$$\int_{Q_{3\delta} \times I_{(3\delta)^2}} |\varphi_1(x, t)|^2 dx dt = 3^{n+2} \int_{Q_\delta \times I_{\delta^2}} |\varphi(x, t)|^2 dx dt. \quad (2.6)$$

Now, we consider ζ a smooth positive function bounded by one and with compact support in $Q_{2\delta} \times I_{(2\delta)^2}$ such that $\zeta(x, t)|_{Q_\delta \times I_{\delta^2}} = 1$, $\zeta(x, t)|_{\mathbb{R}^n \times (0, \infty) \setminus Q_{2\delta} \times I_{(2\delta)^2}} = 0$ and $|\nabla_x \zeta(x, t)| \leq \delta^{-1}$.

Consequently, the function $\varphi_1 \zeta$ is a smooth function with compact support in $Q_{2\delta} \times I_{(2\delta)^2}$ and $\varphi_1 \zeta|_K = 1$, whence $\varphi_1 \zeta$ is a well candidate. Therefore, by applying successively the definition of the parabolic capacity, Inequalities (2.1), (2.4), (2.5), and (2.6), we get

$$\begin{aligned} cap_{vp}(K) &\leq \int_{Q_{2\delta} \times I_{4\delta^2}} |\nabla_x(\varphi_1 \zeta)(x, t)|^2 dx dt \\ &\leq 2 \left(\int_{Q_{2\delta} \times I_{4\delta^2}} |\nabla_x(\varphi_1)(x, t)|^2 dx dt + \int_{Q_{2\delta} \times I_{\delta^2}} |\varphi_1(x, t)|^2 |\nabla_x \zeta(x, t)|^2 dx \right) \\ &\leq 2.3^{n+2} \left(\int_{Q_\delta \times I_{\delta^2}} |\nabla_x \varphi(x, t)|^2 dx dt + \delta^{-2} \int_{Q_\delta \times I_{\delta^2}} |\varphi(x, t)|^2 dx dt \right) \\ &\leq 2.3^{n+2} \left(\int_{Q_\delta \times I_{\delta^2}} |\nabla_x \varphi(x, t)|^2 dx dt + A \int_{Q_\delta \times I_{\delta^2}} |\nabla_x \varphi(x, t)|^2 dx dt \right) \end{aligned}$$

$$\begin{aligned}
&\leq 4.3^n(1+A) \left[\int_{Q_\delta \times I_{\delta^2}} |\nabla_x u(x,t)|^2 dx dt \right] \\
&= \mathcal{C}(n) \|\nabla_x u\|_{L_2(Q_\delta \times I_{\delta^2})}^2.
\end{aligned}$$

The case when the function $u \in C^\infty(Q_\delta \times I_{\delta^2})$ and does not satisfy Equality (2.1), we have the following corollary.

Corollary 2.1. *Let $u \in C^\infty(Q_\delta \times I_{\delta^2})$ and we assume that u is equal to zero on a compact set $K \subset Q_\delta \times I_{\delta^2}$. Then we have*

$$cap_{vp}(K) \leq \mathcal{C}(n) \delta^{n+2} \frac{\|\nabla_x u\|_{L_2(Q_\delta \times I_{\delta^2})}^2}{\|u\|_{L_2(Q_\delta \times I_{\delta^2})}^2}.$$

Let $\tau^2 := \frac{1}{\delta^{n+2}} \int_{Q_\delta \times I_{\delta^2}} |u(x,t)|^2 dx dt$ and $\mathcal{M}_\tau := \{(x,t) \in Q_\delta \times I_{\delta^2} : |u(x,t)| > \tau\}$. Since that $[Q_\delta \times I_{\delta^2}] \setminus [\Omega \times (0, T)] \subset [Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau$, so from the monotonicity of the parabolic capacity we have

$$cap_{vp}([Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau) \leq cap_{vp}(Q_\delta \times I_{\delta^2}).$$

Thus, we need to consider the following two cases:

when

$$i) cap_{vp}([Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau) \geq \gamma cap_{vp}(Q_\delta \times I_{\delta^2}),$$

and when

$$ii) cap_{vp}([Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau) \leq \gamma cap_{vp}(Q_\delta \times I_{\delta^2}).$$

Concerning the case i), we have the following result.

Lemma 2.2. *Let $\gamma \in (0,1)$, $cap_{vp}([Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau) \geq \gamma cap_{vp}(Q_\delta \times I_{\delta^2})$, and $u \in C^\infty(Q_\delta \times I_{\delta^2})$. Then*

$$\|u\|_{L_2(Q_\delta \times I_{\delta^2})}^2 \leq \frac{\mathcal{C}(n)\delta^2}{\gamma} \|\nabla_x u\|_{L_2(Q_\delta \times I_{\delta^2})}^2. \quad (2.7)$$

Proof. By assumption, we have $cap_{vp}([Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau) \geq \gamma cap_{vp}(Q_\delta \times I_{\delta^2})$, thus we apply Corollary 2.1, to the function $(|u(x,t)| - \tau)_+$ which is equal by definition to $|u(x,t)| - \tau$ on $\mathcal{M}_\tau$ and equal to zero on the complement of $\mathcal{M}_\tau$ in $Q_\delta \times I_{\delta^2}$, and by taking into account that $\mathcal{M}_\tau \subset ([Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau) \cup \mathcal{M}_\tau$ we obtain

$$\begin{aligned}
\gamma cap_{vp}(Q_\delta \times I_{\delta^2}) &\leq cap_{vp}([Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau) \\
&\leq \mathcal{C}(n) \delta^{n+2} \frac{\| |u| - \tau \|_{L_2(Q_\delta \times I_{\delta^2})}^2}{\| |u| - \tau \|_{L_2(Q_\delta \times I_{\delta^2})}^2} \\
&\leq \mathcal{C}(n) \delta^{n+2} \frac{\int_0^{\delta^2} \int_{Q_\delta} |\nabla_x u(x,t)|^2 dx dt}{\|u\|_{L_2(Q_\delta \times I_{\delta^2})}^2}
\end{aligned}$$

$$\leq C(n)\delta^{n+2} \frac{\int_0^{\delta^2} \int_{Q_\delta} |\nabla_x u(x,t)|^2 dx dt}{\|u\|_{L_2(Q_\delta \times I_{\delta^2})}^2}. \quad (2.8)$$

Consequently, by using Inequality (2.8) and the fact that $\text{cap}_{vp}(Q_\delta \times I_{\delta^2})$ is equivalent to δ^n , see Remark 1.2, we have Inequality (2.7).

Apropos of the case $\text{cap}_{vp}([Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau) \leq \gamma \text{cap}_{vp}(Q_\delta \times I_{\delta^2})$, we have the following two lemma.

Lemma 2.3. *Let $\gamma \in (0,1)$, $\text{cap}_{vp}([Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau) \leq \gamma \text{cap}_{vp}(Q_\delta \times I_{\delta^2})$, $u \in C^\infty(Q_\delta \times I_{\delta^2})$, $V \in L_1(Q_\delta \times I_{\delta^2})$ such that $\delta^n \leq \int_{[Q_\delta \times I_{\delta^2}] \setminus F} V(x,t) dx dt$ for all $F \in \mathcal{N}_\gamma$. Then*

$$\|u\|_{L_2(Q_\delta \times I_{\delta^2})}^2 \leq 4\delta^2 \int_{Q_\delta \times I_{\delta^2}} |u(x,t)|^2 V(x,t) dx dt.$$

Proof. Since that $\mathcal{M}_\tau \subset Q_\delta \times I_{\delta^2}$ then $[Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau \supset [Q_\delta \times I_{\delta^2}] \setminus [\Omega \times (0,T)]$. Therefore, $\mathcal{N}_\gamma \neq \emptyset$. By assumption, we have $\delta^n \leq \int_{[Q_\delta \times I_{\delta^2}] \setminus F} V(x,t) dx dt$ for all negligible compact set F in $Q_\delta \times I_{\delta^2}$, thus we obtain

$$\delta^n \leq \inf_{F \in \mathcal{N}_\gamma} \int_{[Q_\delta \times I_{\delta^2}] \setminus F} V(x,t) dx dt.$$

Now by using the fact that $\mathcal{M}_\tau = [Q_\delta \times I_{\delta^2}] \setminus ([Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau)$, and $[Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau \in \mathcal{N}_\gamma$, we have

$$\begin{aligned} \int_{Q_\delta \times I_{\delta^2}} |u(x,t)|^2 V(x,t) dx dt &\geq \tau^2 \int_{\mathcal{M}_\tau} V(x,t) dx dt \\ &= \left(\frac{1}{4\delta^{n+2}} \int_{Q_\delta \times I_{\delta^2}} |u(x,t)|^2 dx dt \right) \\ &\quad \times \int_{[Q_\delta \times I_{\delta^2}] \setminus ([Q_\delta \times I_{\delta^2}] \setminus \mathcal{M}_\tau)} V(x,t) dx dt \\ &\geq \left(\frac{1}{4\delta^{n+2}} \int_{Q_\delta \times I_{\delta^2}} |u(x,t)|^2 dx dt \right) \\ &\quad \times \inf_{F \in \mathcal{N}_\gamma} \int_{[Q_\delta \times I_{\delta^2}] \setminus F} V(x,t) dx dt \\ &\geq \left(\frac{1}{4\delta^2} \right) \int_{Q_\delta \times I_{\delta^2}} |u(x,t)|^2 dx dt. \end{aligned}$$

The proof of Theorem 1.3 is the synthesis of the Lemma 2.2 and the Lemma 2.3 and the $D_{n,\gamma,\delta} = \delta^2 \left(4 + \frac{C(n)}{\gamma} \right)$.

3. Proof of Theorem 1.1

Let us cover the compact support of $u \in C_0^\infty(C_T)$ by a finite family of bounded closed cylinders with same volume $\left((Q_\delta \times I_{\delta^2})^{(k)}\right)_{k \in J}$ such that J is a finite set of the enter number. As expected we consider bounded closed cylinders such that $\delta < \sqrt{T}$. Let us choose a suitable partition of unity subordinated to the covering $\left((Q_\delta \times I_{\delta^2})^{(k)}\right)_{k \in J}$ such that the multiplicity of this covering is at most equal to $2^{2(n+1)}$ [4] or [6, Lemma 2.5]. Therefore, by applying Theorem 1.3 for each bounded closed cylinder, we obtain

$$\begin{aligned}
 \|u\|_{L_2(C_T)}^2 &\leq \sum_{k=1}^{2^{2(n+1)}} \|u\|_{L_2((Q_\delta \times I_{\delta^2})^{(k)})}^2 \\
 &\leq \sum_{k=1}^{2^{2(n+1)}} D_{n,\gamma,\delta} \|u\|_{\mathcal{W}_V(Q_\delta \times I_{\delta^2})^{(k)}}^2 \\
 &\leq \sum_{k=1}^{2^{2(n+1)}} D_{n,\gamma,\delta} \|u\|_{\mathcal{W}_V(C_T)}^2 \\
 &= 2^{2(n+1)} D_{n,\gamma,\delta} \|u\|_{\mathcal{W}_V(C_T)}^2.
 \end{aligned} \tag{3.1}$$

and from Inequality (3.1) we extend δ to δ_{V,γ,C_T} and by density the proof of Theorem 1.1 is complete.

Remark 3.1. By defining the m -variational parabolic capacity of a compact set in $\mathbb{R}^n \times (0, \infty)$ (corresponding to the generalized parabolic operator $\partial_t + (-\Delta)^m$ such that $n > 2m$), we pretend that it is possible to provide a generalized version of Theorem 1.1, and principally we need to define the parabolic Sobolev space of order $m \in \mathbb{N}$ on C_T and a general version of Lemma 2.1 where we can use Lemma 2 in [3]. Also, we pretend that it is possible to provide a version of Theorem 1.1 in a two-dimension subspace.

The corresponding author states that there is no conflict of interest.

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