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Solvability of Parametric Elliptic Systems with Variable Exponents

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ABSTRACT. In this paper, we study the solvability to the left of the positive infimum of all eigenvalues for some non-resonant quasilinear elliptic problems involving variable exponents. We first prove the existence of at least a weak solution for some non-variational systems by using a surjectivity result for pseudomonotone operators. Furthermore, under additional conditions, we show that the solution is unique and provide examples. Second, we deal with non-resonant gradient-type systems and obtain existence by using a variational approach.

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1. Introduction

In recent years, the $p(x)$ -Laplacian has gained increasing attention in various fields of science and engineering. This operator, defined as $\Delta_{p(x)}u = \operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right)$, is a generalization of the standard Laplacian operator and is commonly used in mathematics, physics, and engineering to describe physical phenomena such as heat flow, fluid dynamics, and electrostatics. One

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of the main advantages of the $p(x)$ -Laplacian is its ability to provide a more flexible description of nonlinear diffusion processes, which are often encountered in real-world applications. In addition, the $p(x)$ -Laplacian can also model non-Newtonian fluids, which exhibit nonlinear behavior due to the dependence of viscosity on the applied stress.

Furthermore, it's worth noting that there are other applications of the $p(x)$ -Laplacian, such as its use in electrorheological fluids (studied by Acerbi and Mingione [1] and Ružička [21]), image restoration (employed by Chen et al. [7]), and thermorheological fluids (explored by Antontsev et al. [3, 4]).

Given a bounded smooth domain $\Omega \subseteq \mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, this paper is concerned with the existence of a solution $(u_1, u_2) \in W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ for the following elliptic system

$$\begin{cases} -\Delta_{p_1(x)} u_1 - \mu_1 \Delta_{q_1(x)} u_1 = f_1(x, u_1, u_2, \nabla u_1, \nabla u_2) + h_1(x) & \text{in } \Omega, \\ -\Delta_{p_2(x)} u_2 - \mu_2 \Delta_{q_2(x)} u_2 = f_2(x, u_1, u_2, \nabla u_1, \nabla u_2) + h_2(x) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{PS})$$

where for $i \in \{1, 2\}$, $1 < q_i(x) \leq p_i(x) < N$ are variable exponents of class $C^1(\bar{\Omega})$, $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions, corresponding essentially to some perturbation below the positive infimum of all eigenvalues of a related eigenvalue problem defined in (1.3) below and satisfying some structure conditions, (see hypotheses (H_4) and (H_5) in Section 2), μ_i are positive constants and $h_i \in W^{-1,p'_i(x)}(\Omega)$ with $\frac{1}{p'_i(x)} + \frac{1}{p_i(x)} = 1$, for all $x \in \Omega$.

Many works have been dedicated to existence results for such quasilinear elliptic systems with constant or variable exponents. We quote for example Alves and Moussaoui [2], Boccardo and De Figueiredo [5], El Hachimi and De Thélin [10], Felmer et al. [15], Faria et al. [14], Hsu [16], Motreanu et al. [18], Nabab and Vélin [20] and the references therein. Recall that there are few works dealing with eigenvalue problems for elliptic systems. One can see De Thélin [8], El Khalil et al. [11], Kandilakis et al. [17], Moussaoui and Vélin [19], Zographopoulos [22] and the references therein. We refer especially to the paper of Moussaoui and Vélin [24], where the authors are concerned with the positive infimum $\lambda_c(p)$, with $p = (p_1, p_2)$, of all eigenvalues for a nonlinear gradient-type elliptic system with weight and involving variable exponents such as

$$\begin{cases} -\Delta_{p_1(x)} u = \lambda c(x) \alpha_1(x) |u_1|^{\alpha_1(x)-2} u_1 |u_2|^{\alpha_2(x)} & \text{in } \Omega, \\ -\Delta_{p_2(x)} v = \lambda c(x) \alpha_2(x) |u_1|^{\alpha_1(x)} u_2 |u_2|^{\alpha_2(x)-2} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \quad (\text{VP}_c)$$

Let be associated to (PS) the eigenvalue problem

$$\begin{cases} -\Delta_{p_1(x)} v_1 = \lambda \alpha_1(x) |v_1|^{\alpha_1(x)-2} v_1 |v_2|^{\alpha_2(x)} & \text{in } \Omega, \\ -\Delta_{p_2(x)} v_2 = \lambda \alpha_2(x) |v_2|^{\alpha_2(x)-2} v_2 |v_1|^{\alpha_1(x)} & \text{in } \Omega, \\ v_1 = v_2 = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{VP}_1)$$

where for $i \in \{1, 2\}$, α_i verify

$$\alpha_1(x), \alpha_2(x) > 0 \text{ and } \frac{\alpha_1(x)}{p_1(x)} + \frac{\alpha_2(x)}{p_2(x)} = 1, \text{ for all } x \in \Omega.$$

On one hand, we were motivated by the paper of Motreanu et al. [18] where the authors studied quasilinear parametric elliptic systems, and proved existence and uniqueness of solutions to the following elliptic system with constant exponents p_i, q_i

$$\begin{cases} -\Delta_{p_1} u_1 - \mu_1 \Delta_{q_1} u_1 = f_1(x, u_1, u_2, \nabla u_1, \nabla u_2) & \text{in } \Omega, \\ -\Delta_{p_2} u_2 - \mu_2 \Delta_{q_2} u_2 = f_2(x, u_1, u_2, \nabla u_1, \nabla u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{CPS})$$

Their results are obtained for functions

$$F^* : (u_1, u_2) \rightarrow f_1(x, u_1, u_2, \nabla u_1, \nabla u_2) u_1 + f_2(x, u_1, u_2, \nabla u_1, \nabla u_2) u_2,$$

that are, in some sense, below the first eigenvalues of the related p_i -Laplacian operators. Their principal condition reads

$$f_1(x, s_1, s_2, \xi_1, \xi_2) s_1 + f_2(x, s_1, s_2, \xi_1, \xi_2) s_2 \leq \omega(x) + c(|s_1|^{p_1} + |s_2|^{p_2}) + d(|\xi_1|^{p_1} + |\xi_2|^{p_2}), \quad (1.1)$$

for a.e $x \in \Omega$, all $(s_1, s_2) \in \mathbb{R}^2$ and all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{2N}$, with $\omega \in L^1(\Omega)$ and positive constants c and d such that

$$\min(\lambda_{1,p_1}, \lambda_{1,p_2}) < \frac{1-c}{d},$$

where λ_{1,p_i} denotes the first eigenvalue of the p_i -Laplacian operator for $i \in \{1, 2\}$. This condition ensures the coerciveness of the operator associated with the problem as defined in Motreanu et al. [18]. On the other hand, the motivation of the present work was to extend to the case of a non-gradient-type system with variable exponents, the earlier results by El Hachimi and De Th  lin [10] for the existence of solutions to the problem

$$\begin{cases} -\Delta_{p_i} u_i = \frac{\partial F}{\partial u_i}(x, u) + h_i(x) & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{GS})$$

obtained when the function F satisfies a non-resonance condition near the first eigenvalue of an associated eigenvalue system. Unlike the variational case used in Section 5, where solutions are obtained as critical points of C^1 -functionals, we shall use in Section 3, a topological method based on a fixed point theorem for bounded, pseudomonotone and coercive operators.

Our aim is to establish the existence of a solution for (PS) based on the findings of Moussaoui and V  lin [19]. Broadly speaking, if the condition

$$\begin{aligned} \forall \epsilon > 0, \exists \delta_\epsilon > 0, \exists M_\epsilon \in L^1(\Omega), \forall s = (s_1, s_2) \in \mathbb{R}^2 \text{ with } |s|_\infty > \delta_\epsilon, \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^{2N}, \forall x \in \Omega, \\ f_1(x, s_1, s_2, \xi_1, \xi_2) s_1 + f_2(x, s_1, s_2, \xi_1, \xi_2) s_2 \leq (\lambda_1(p) + \epsilon) \left(|s_1|^{\alpha_1(x)} |s_2|^{\alpha_2(x)} \right) \\ + c \left(|\xi_1|^{q_1(x)} + |\xi_2|^{q_2(x)} \right) + M_\epsilon(x), \end{aligned} \quad (1.2)$$

is satisfied, where c is a positive constant and $\lambda_1(p)$ is the positive infimum of all eigenvalues of (VP₁), then we prove the existence of at least one weak solution for problem (PS). Note that

(1.2) is verified if one has for example

$$f_{\infty}(x) \equiv \limsup_{|s|_{\infty} \rightarrow +\infty} \frac{f_1(x, s_1, s_2, \xi_1, \xi_2) s_1 + f_2(x, s_1, s_2, \xi_1, \xi_2) s_2}{|s_1|^{\alpha_1(x)} |s_2|^{\alpha_2(x)}} < \lambda_1(p), \quad (1.3)$$

uniformly with respect to $\xi = (\xi_1, \xi_2)$. Moreover, under additional conditions on the functions f_1 and f_2 and the variable exponents, we show that the solution is unique.

It should be noted that, even in the case of constant exponents and functions f_i that are not depending on the gradients of the unknown functions, little is known about resonance and nonresonance issues, when considering the interaction with the variational spectrum of the (p_1, p_2) -Laplacians eigenvalue system (VP₁) of the function

$$(s_1, s_2) \longrightarrow \frac{f_1(x, s_1, s_2) s_1 + f_2(x, s_1, s_2) s_2}{|s_1|^{\alpha_1} |s_2|^{\alpha_2}}.$$

The same remark is also valid in the case of gradient-type systems and functions f_i not depending on ∇u_i for the function

$$(s_1, s_2) \longrightarrow \frac{F(x, s_1, s_2)}{|s_1|^{\alpha_1} |s_2|^{\alpha_2}},$$

where $\frac{\partial F}{\partial s_i}(x, s_1, s_2) = f_i(x, s_1, s_2)$ for $i \in \{1, 2\}$. Let us recall that the positive infimum $\lambda_1(p)$ of all eigenvalues can be defined by the expression

$$\lambda_1(p) = \inf_{(u_1, u_2) \in \mathcal{V} \setminus \{0\}} \frac{\mathcal{A}_p(u_1, u_2)}{\mathcal{B}_p(u_1, u_2)}, \quad (1.4)$$

where

$$\mathcal{A}_p(u_1, u_2) = \int_{\Omega} \left(\frac{1}{p_1(x)} |\nabla u_1|^{p_1(x)} + \frac{1}{p_2(x)} |\nabla u_2|^{p_2(x)} \right) dx \quad \text{and} \quad \mathcal{B}_p(u_1, u_2) = \int_{\Omega} |u_1|^{\alpha_1(x)} |u_2|^{\alpha_2(x)} dx.$$

It is worth noting that condition (1.1) implies (1.2) which clearly implies

$$\begin{aligned} \forall \epsilon > 0, \exists \delta_{\epsilon} > 0, \exists M_{\epsilon} \in L^1(\Omega), \forall s = (s_1, s_2) \in \mathbb{R}^2, \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^{2N}, \forall x \in \Omega, \\ f_1(x, s_1, s_2, \xi_1, \xi_2) s_1 + f_2(x, s_1, s_2, \xi_1, \xi_2) s_2 \leq (\lambda_1(p) + \epsilon) \left(|s_1|^{\alpha_1(x)} |s_2|^{\alpha_2(x)} \right) \\ + c \left(|\xi_1|^{p_1(x)} + |\xi_2|^{p_2(x)} \right) + M_{\epsilon}(x). \end{aligned} \quad (1.5)$$

Now, according to a recent result by Bozorgnia et al. [6, Theorem 3.3] obtained in the case of constant exponents, the following situation may occur

$$\min(\lambda_{1,p_1}, \lambda_{1,p_2}) < \lambda_1(p).$$

Therefore, in the case of constant exponents, condition (1.3) is weaker than condition (1.1).

The remainder of this paper is organized as follows. In Section 2, we present preliminary results and state some assumptions. In Section 3, we present and prove our main existence result for the problem (PS) and, in Section 4, we provide conditions to obtain the uniqueness of the solution. We conclude this section by giving examples of applications. In Section 5, we treat the case of a gradient-type system and demonstrate an existence result using a variational

approach, similar to that employed in El Hachimi and de Th  lin [10]. Finally, in Section 6, we provide some concluding remarks.

2. Functional framework

2.1. Notations. For $s = (s_1, s_2)$ and $\beta = (\beta_1, \beta_2)$ in $\mathbb{R}^2$ with $\beta_i > 0$, for $i \in \{1, 2\}$, denote

$$|s|_\infty = \max\{|s_1|, |s_2|\} \quad \text{and} \quad |s|^\beta = |s_1|^{\beta_1} |s_2|^{\beta_2}.$$

For $q \in]1, +\infty[$, let

$$q' = \frac{q}{q-1} \quad \text{and} \quad q^* = \begin{cases} \frac{Nq}{N-q} & \text{if } N > q, \\ +\infty & \text{if } N \leq q. \end{cases}$$

Let p be a log-H  lder function in the class $C^1(\bar{\Omega})$, such that

$$1 < p^- = \inf_{x \in \Omega} p(x) \leq p(x) \leq p^+ = \sup_{x \in \Omega} p(x) < \infty.$$

We define $L^{p(x)}(\Omega)$ as the generalized Lebesgue space that consists of all measurable real-valued functions u satisfying

$$\rho_{p(x)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx < +\infty,$$

and is endowed with the Luxemburg norm

$$\|u\|_{p(x)} = \inf \left\{ \tau > 0 : \rho_{p(x)} \left(\frac{u}{\tau} \right) \leq 1 \right\}.$$

Throughout this paper, and to simplify, we use the notation $\|u\|_{p(x)}$ instead of $\|u\|_{L^{p(x)}(\Omega)}$. The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with its associated norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

$W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm of $W^{1,p(x)}(\Omega)$ and its norm is denoted by $\|u\|_{W_0^{1,p(x)}(\Omega)}$, that is

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)}.$$

The variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$ becomes a Banach space because of its associated norm, and the following embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ is compact when $1 < r(x) < p^*(x)$, for all $x \in \bar{\Omega}$. In the following, we shall use the simplified notation $\|\cdot\|_{1,p(x)}$ instead of $\|\cdot\|_{W_0^{1,p(x)}(\Omega)}$.

Proposition 2.1. (see Diening et al. [9]) For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the following inequality holds

$$\left| \int_{\Omega} uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'} \right) \|u\|_{p(x)} \|v\|_{p'(x)} \leq 2 \|u\|_{p(x)} \|v\|_{p'(x)}.$$

Moreover, if $r_1, r_2, r_3 : \bar{\Omega} \rightarrow]1, \infty[$ are Lipschitz continuous functions such that

$$\frac{1}{r_1(x)} + \frac{1}{r_2(x)} + \frac{1}{r_3(x)} = 1, \quad \forall x \in \Omega,$$

then, for all $u_i \in L^{r_i(x)}(\Omega)$, for $i \in \{1, 2, 3\}$, we have

$$\left| \int_{\Omega} u_1(x) u_2(x) u_3(x) dx \right| \leq \left(\frac{1}{r_1^-} + \frac{1}{r_2^-} + \frac{1}{r_3^-} \right) \|u_1\|_{r_1(x)} \|u_2\|_{r_2(x)} \|u_3\|_{r_3(x)}.$$

Proposition 2.2. (see Fan and Zhao [13, Theorems 1.2 and 1.3]) For any $u \in L^{p(x)}(\Omega)$, we have

- (a) $\|u\|_{p(\cdot)} > 1 \implies \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}.$
- (b) $\|u\|_{p(\cdot)} < 1 \implies \|u\|_{p(\cdot)}^{p^-} \geq \rho_{p(\cdot)}(u) \geq \|u\|_{p(\cdot)}^{p^+}.$
- (c) $\|u\|_{p(\cdot)} = a > 0 \iff \rho_{p(\cdot)}\left(\frac{u}{a}\right) = 1.$
- (d) $\|u\|_{p(\cdot)} < 1 (= 1; > 1) \iff \rho_{p(\cdot)}(u) < 1 (= 1; > 1).$

Remark 2.1. From Proposition 2.2, we have

$$\rho_{p(\cdot)}(u) \leq \max \left\{ \|u\|_{p(\cdot)}^{p^+}, \|u\|_{p(\cdot)}^{p^-} \right\} \leq \|u\|_{p(\cdot)}^{p^+} + \|u\|_{p(\cdot)}^{p^-}.$$

2.2. General hypotheses.

- (H₁) Ω is a bounded open of $\mathbb{R}^N$, with boundary $\partial\Omega$ of class $C^{2,\delta}$, for some $0 < \delta < 1$.
- (H₂) μ_1 and μ_2 are two positive constants and p_i and q_i , for $i \in \{1, 2\}$, are two functions in $C^1(\bar{\Omega})$ satisfying $p_i(x) < p_i^*(x)$ and $q_i(x) < q_i^*(x)$, respectively, for all $x \in \bar{\Omega}$, with

$$1 < p_i^- = \inf_{x \in \Omega} p_i(x) \leq p_i(x) \leq p_i^+ = \sup_{x \in \Omega} p_i(x) < \infty,$$

$$1 < q_i^- = \inf_{x \in \Omega} q_i(x) \leq q_i(x) \leq q_i^+ = \sup_{x \in \Omega} q_i(x) < \infty.$$

- (H₃) For $i \in \{1, 2\}$, $\alpha_i : \bar{\Omega} \rightarrow]0, +\infty[$ are two continuous functions satisfying

$$0 < \alpha_i^- = \inf_{x \in \Omega} \alpha_i(x) \leq \alpha_i(x) \leq \alpha_i^+ = \sup_{x \in \Omega} \alpha_i(x) < \infty,$$

with

$$\frac{\alpha_1(x)}{p_1(x)} + \frac{\alpha_2(x)}{p_2(x)} = 1, \quad \text{for all } x \in \bar{\Omega}.$$

- (H₄) For $i \in \{1, 2\}$, $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions such that there exist three positive continuous functions α_i^* , q_i^* and δ_i , and a nonnegative function $e_i \in L^{\frac{\gamma_i(x)}{\gamma_i(x)-1}}(\Omega)$ with $1 < \gamma_i(x) < p_i^*(x)$, satisfying the following conditions

$$0 < \alpha_i^*(x) < p_i^*(x) - 1, \quad 0 < \delta_i(x) < \frac{q_i(x)}{(q_i^*(x))'}, \quad \frac{\alpha_1^*(x)}{\gamma_1(x)} + \frac{\alpha_2^*(x)}{\gamma_2(x)} = \frac{\gamma_1(x) - 1}{\gamma_1(x)},$$

and nonnegative constants a_i, b_i , such that for a.e. $x \in \Omega$ and all $s_1, s_2 \in \mathbb{R}$, we have

$$|f_1(x, s_1, s_2, \xi_1, \xi_2)| \leq a_1 |s_1|^{\alpha_1^*(x)} |s_2|^{\alpha_2^*(x)} + b_1 \left(|\xi_1|^{\delta_1(x)} + |\xi_2|^{\frac{\delta_1(x)q_2(x)}{q_1(x)}} \right) + e_1(x), \quad (2.1)$$

and

$$|f_2(x, s_1, s_2, \xi_1, \xi_2)| \leq a_2 |s_1|^{\alpha_1^*(x)} |s_2|^{\alpha_2^*(x)} + b_2 \left(|\xi_1|^{\frac{\delta_2(x)q_1(x)}{q_2(x)}} + |\xi_2|^{\delta_2(x)} \right) + e_2(x). \quad (2.2)$$

$$(H_5) \quad \forall \epsilon > 0, \exists \delta_\epsilon > 0, \exists M_\epsilon \in L^1(\Omega), \forall s = (s_1, s_2) \in \mathbb{R}^2 \text{ with } |s|_\infty > \delta_\epsilon, \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^{2N}, \forall x \in \Omega$$

$$\begin{aligned} f_1(x, s_1, s_2, \xi_1, \xi_2) s_1 + f_2(x, s_1, s_2, \xi_1, \xi_2) s_2 &\leq (\lambda_1(p) + \epsilon) |s|^{\alpha(x)} \\ &+ c \left(|\xi_1|^{q_1(x)} + |\xi_2|^{q_2(x)} \right) + M_\epsilon(x). \end{aligned} \quad (2.3)$$

2.3. Preliminary results. Let us denote $\mathcal{V} = W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$, $\mathcal{V}' = W^{-1,p'_1(x)}(\Omega) \times W^{-1,p'_2(x)}(\Omega)$ and for $u = (u_1, u_2) \in \mathcal{V}$ and $\varphi = (\varphi_1, \varphi_2) \in \mathcal{V}'$

$$\langle u, \varphi \rangle = \langle u_1, \varphi_1 \rangle_{W_0^{1,p_1(x)}(\Omega), W^{-1,p'_1(x)}(\Omega)} + \langle u_2, \varphi_2 \rangle_{W_0^{1,p_2(x)}(\Omega), W^{-1,p'_2(x)}(\Omega)}.$$

The norm considered on $\mathcal{V}$ is given by

$$\|u\|_{\mathcal{V}} = \max \left\{ \|u_1\|_{W_0^{1,p_1(x)}(\Omega)}, \|u_2\|_{W_0^{1,p_2(x)}(\Omega)} \right\}, \quad \forall u = (u_1, u_2) \in \mathcal{V}.$$

For $u = (u_1, u_2) \in \mathcal{V}$, define the functionals $\mathcal{A}_p$ and $\mathcal{B}_p$ on $\mathcal{V}$ by

$$\begin{cases} \mathcal{A}_p(u) = \int_{\Omega} \left(\frac{1}{p_1(x)} |\nabla u_1|^{p_1(x)} + \frac{1}{p_2(x)} |\nabla u_2|^{p_2(x)} \right) dx, \\ \mathcal{B}_p(u) = \int_{\Omega} |u_1|^{\alpha_1(x)} |u_2|^{\alpha_2(x)} dx. \end{cases}$$

For fixed $R > 0$, set $\Omega_R = \{u \in \mathcal{V}, \mathcal{B}_p(u) = R\}$. It is clear that Ω_R is not empty, see Moussaoui and Vélin [19]. Define the Rayleigh quotients as follows

$$\begin{cases} \lambda_R(p) = \inf_{u \in \Omega_R} \frac{\mathcal{A}_p(u)}{\mathcal{B}_p(u)}, \\ \lambda_1(p) = \inf_{u \in \mathcal{V} \setminus \{0\}} \frac{\mathcal{A}_p(u)}{\mathcal{B}_p(u)}. \end{cases}$$

Now, we recall the following result by Moussaoui and Vélin.

Theorem 2.1. (see Moussaoui and Vélin [19, Theorem 2.2]) Assume that $(H_1) - (H_3)$ hold. Then, system (VP₁) has a one-parameter family of nontrivial solutions $((\hat{u}_{1,R}, \hat{u}_{2,R}), \lambda_R^*)$ for all $R \in]0, +\infty[$. Moreover, if one of the following conditions holds

- (P₁) There are two vectors $l_1, l_2 \in \mathbb{R}^N \setminus \{0\}$ such that for all $x \in \Omega$, $\theta_1(t_1) = p_1(x + t_1 l_1)$ and $\theta_2(t_2) = p_2(x + t_2 l_2)$ are monotone for $t_i \in I_{i,x} = \{t_i, x + t_i l_i \in \Omega\}$, for $i \in \{1, 2\}$.
- (P₂) There are x_1 and $x_2 \notin \Omega$ such that for all $w_1, w_2 \in \mathbb{R}^N \setminus \{0\}$ with $\|w_1\|, \|w_2\| = 1$, the functions $\theta_1(t_1) = p_1(x_0 + t_1 w_1)$ and $\theta_2(t_2) = p_2(x_2 + t_2 w_2)$ are monotone for $t_i \in I_{x_i, w_i} = \{t_i \in \mathbb{R}, x_i + t_i w_i \in \Omega\}$, for $i \in \{1, 2\}$.

Subsequently, $\lambda_1(p) = \inf_{R>0} \lambda_R(p) > 0$ is the positive infimum eigenvalue of problem (VP₁).

Definition 2.1. A couple $(u_1, u_2) \in \mathcal{V}$ is called a weak solution of problem (PS) if

$$\int_{\Omega} \left(\mathcal{F}_{p_1(x)}(\nabla u_1) + \mu_1 \mathcal{F}_{q_1(x)}(\nabla u_1) \right) \cdot \nabla \varphi_1 + \left(\mathcal{F}_{p_2(x)}(\nabla u_1) + \mu_2 \mathcal{F}_{q_2(x)}(\nabla u_1) \right) \cdot \nabla \varphi_2 dx = \int_{\Omega} \mathbf{f}(x, u, \nabla u_1, \nabla u_2) \cdot \varphi dx + \langle h, \varphi \rangle,$$

for any couple $\varphi = (\varphi_1, \varphi_2) \in \mathcal{V}$, where $\mathcal{F}_{r(x)}(t) = |t|^{r(x)-2}t$, for $t \in \mathbb{R}^N$ and $\langle h, \varphi \rangle = \langle h_1, \varphi_1 \rangle + \langle h_2, \varphi_2 \rangle$ is the duality pairing between $\mathcal{V}$ and its dual space, where we denote

$$\mathbf{f}(x, s, \xi_1, \xi_2) \cdot s = f_1(x, s_1, s_2, \xi_1, \xi_2) s_1 + f_2(x, s_1, s_2, \xi_1, \xi_2) s_2.$$

Definition 2.2. Let X be a reflexive Banach space, X^* its dual space and denote by $\langle \cdot, \cdot \rangle$ its duality pairing. Let $\mathcal{I} : X \rightarrow X^*$, then $\mathcal{I}$ is called

- (a) to satisfy the (S_+) -property if $u_n \rightharpoonup u$ in X and $\limsup_{n \rightarrow \infty} \langle \mathcal{I}u_n, u_n - u \rangle \leq 0$ imply $u_n \rightarrow u$ in X ,
- (b) pseudomonotone if $u_n \rightharpoonup u$ in X and $\limsup_{n \rightarrow \infty} \langle \mathcal{I}u_n, u_n - u \rangle \leq 0$ imply $\mathcal{I}u_n \rightarrow u$ and $\langle \mathcal{I}u_n, u_n \rangle \rightarrow \langle \mathcal{I}u, u \rangle$,
- (c) coercive if

$$\lim_{\|u\|_X \rightarrow \infty} \frac{\langle \mathcal{I}(u), u \rangle}{\|u\|_X} = \infty.$$

Theorem 2.2. (see Motreanu et al. [18]) Let X be a real, reflexive Banach space, $\mathcal{I} : X \rightarrow X^*$ be a bounded, pseudomonotone, and coercive operator, and $h \in X^*$. Then, a solution to the equation $\mathcal{I}u = h$ exists.

For all $u = (u_1, u_2) \in \mathcal{V}$, define

$$\begin{aligned} \tilde{\mathcal{A}}(u) &= \left(-\Delta_{p_1(x)}u_1 - \mu_1 \Delta_{q_1(x)}u_1, -\Delta_{p_2(x)}u_1 - \mu_2 \Delta_{q_2(x)}u_2 \right) \\ &= \tilde{\mathcal{A}}_p(u) + \tilde{\mathcal{A}}_q(u), \end{aligned} \tag{2.4}$$

where

$$\tilde{\mathcal{A}}_p(u) = \left(-\Delta_{p_1(x)}u_1, -\Delta_{p_2(x)}u_2 \right) \quad \text{and} \quad \tilde{\mathcal{A}}_q(u) = \left(-\mu_1 \Delta_{q_1(x)}u_1, -\mu_2 \Delta_{q_2(x)}u_2 \right).$$

The following result summarizes the properties of the operator $\tilde{\mathcal{A}}$.

Lemma 2.1. The operator $\tilde{\mathcal{A}} : \mathcal{V} \rightarrow \mathcal{V}'$, defined in (2.4), is bounded, continuous, monotone (hence maximal monotone), and of type (S_+) .

Proof. The proof is similar to that obtained by Fan and Zhang [12, Theorem 3.1] for a single equation and $\mu_i = 0$ for $i \in \{1, 2\}$, and is omitted here.

3. Existence result

In this section, we provide and prove our first main result.

Theorem 3.1. Suppose that the hypotheses (P_1) or (P_2) and $(H_1) - (H_5)$ are verified. Moreover, suppose that $c < \frac{1}{\mu_0 \theta_q}$, where $\theta_q := \max \left\{ \frac{1}{q_1}, \frac{1}{q_2} \right\} < 1$ and $\mu_0 = \min \{\mu_1, \mu_2\}$. Then, for all $h \in \mathcal{V}'$, problem (PS) admits at least one solution, $u = (u_1, u_2) \in \mathcal{V}$.

Proof. In order to apply Theorem 2.2, we need to show that the operator $\mathcal{I}$ associated with (PS), which is defined on $\mathcal{V}$ by

$$\mathcal{I}(u) := \tilde{\mathcal{A}}(u) - N(u) - h,$$

where $N(u) = (N_1(u), N_2(u))$, and for $i \in \{1, 2\}$, $N_i : \mathcal{V} \rightarrow \mathcal{V}'$ denotes the Nemytskii operator associated with f_i , that is, $N_i(u) = f_i(x, u_1, u_2, \nabla u_1, \nabla u_2)$.

• **$\mathcal{I}$ is bounded:**

From (H_4) , it is easy to see that the Nemytskii operators N_1 and N_2 are well-defined, continuous and bounded. Because $\tilde{\mathcal{A}} : \mathcal{V} \rightarrow \mathcal{V}'$ is bounded, we conclude that $\mathcal{I}$ is bounded.

• **$\mathcal{I}$ is pseudomonotone:**

Let $\{u_n = (u_{n,1}, u_{n,2})\}_{n \in \mathbb{N}} \subset \mathcal{V}$ be a sequence such that

$$u_n \rightharpoonup u \text{ in } \mathcal{V} \quad \text{and} \quad \limsup_{n \rightarrow \infty} \langle \mathcal{I}(u_n), (u_n - u) \rangle \leq 0. \quad (3.1)$$

Taking the compact embedding $W_0^{1,p_i(x)}(\Omega) \hookrightarrow L^{\alpha_i^*(x)}(\Omega)$ into consideration gives

$$u_{n,1} \rightarrow u_1 \text{ in } L^{\alpha_1^*(x)}(\Omega) \quad \text{and} \quad u_{n,2} \rightarrow u_2 \text{ in } L^{\alpha_2^*(x)}(\Omega), \quad (3.2)$$

since $\alpha_1^*(x) < p_1^*(x)$ and $\alpha_2^*(x) < p_2^*(x)$, respectively. We need to show that

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_1(x, u_{n,1}, u_{n,2}, \nabla u_{n,1}, \nabla u_{n,2}) (u_{n,1} - u_1) dx = 0 \quad (3.3)$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_2(x, u_{n,1}, u_{n,2}, \nabla u_{n,1}, \nabla u_{n,2}) (u_{n,2} - u_2) dx = 0. \quad (3.4)$$

We proceed as described in Motreanu et al. [18]. In order to establish (3.3), we use hypothesis (H_4) , Hölder's inequality and compactness embedding relation (3.2). We have

$$\left| \int_{\Omega} f_1(x, u_{n,1}, u_{n,2}, \nabla u_{n,1}, \nabla u_{n,2}) (u_{n,1} - u_1) dx \right| \leq K_{n,1} + K_{n,2} + K_{n,3} + K_{n,4},$$

where, by using the relation $\frac{\alpha_1^*(x)}{\gamma_1(x)} + \frac{\alpha_2^*(x)}{\gamma_2(x)} + \frac{1}{\gamma_1(x)} = 1$ with $\rho = \frac{(\alpha_1^*)^-}{\gamma_1^+} + \frac{(\alpha_2^*)^-}{\gamma_2^+} + \frac{1}{\gamma_1^+}$ and applying Remark 2.1, we have

$$\begin{aligned}
K_{n,1} &= \int_{\Omega} a_1 |u_{n,1}|^{\alpha_1^*(x)} |u_{n,2}|^{\alpha_2^*(x)} |u_{n,1} - u_1| dx \\
&\leq \rho a_1 \left(\|u_{n,1}\|_{L^{\gamma_1(\cdot)}(\Omega)}^{(\alpha_1^*)^+} + \|u_{n,1}\|_{L^{\gamma_1(\cdot)}(\Omega)}^{(\alpha_1^*)^-} \right) \left(\|u_{n,1}\|_{L^{\gamma_2(\cdot)}(\Omega)}^{(\alpha_2^*)^+} + \|u_{n,1}\|_{L^{\gamma_2(\cdot)}(\Omega)}^{(\alpha_2^*)^-} \right) \|u_{n,1} - u_1\|_{L^{\gamma_1(\cdot)}(\Omega)} \\
&\rightarrow 0, \\
K_{n,2} &= \int_{\Omega} b_1 |\nabla u_{n,2}|^{\delta_1(x)} |u_{n,1} - u_1| dx \leq 2 \left\| |\nabla u_{n,2}|^{q_1(x)} \right\|_{L^{\frac{q_1(x)}{\delta_1(x)}}(\Omega)} \|u_{n,1} - u_1\|_{L^{\frac{q_1(x)}{q_1(x)-\delta_1(x)}}(\Omega)} \rightarrow 0, \\
K_{n,3} &= \int_{\Omega} b_1 |\nabla u_{n,2}|^{\frac{\delta_1(x)q_2(x)}{q_1(x)}} |u_{n,1} - u_1| dx \leq 2 \left\| |\nabla u_{n,2}|^{q_2(x)} \right\|_{L^{\frac{q_1(x)}{\delta_1(x)}}(\Omega)} \|u_{n,1} - u_1\|_{L^{\frac{q_1(x)}{q_1(x)-\delta_1(x)}}(\Omega)} \rightarrow 0, \\
K_{n,4} &= \int_{\Omega} e_1(x) |u_{n,1} - u_1| dx \leq 2 \|e_1\|_{L^{\gamma_1'(x)}(\Omega)} \|u_{n,1} - u_1\|_{L^{\gamma_1(x)}(\Omega)} \rightarrow 0.
\end{aligned}$$

Therefore, (3.3) is fulfilled. Applying similar calculations yields (3.4). Taking the weak formulation in Definition 2.1, replacing u_1 by $u_{n,1}$, u_2 by $u_{n,2}$, φ_1 by $u_{n,1} - u_1$ and φ_2 by $u_{n,2} - u_2$ and using (3.1) as well as (3.3) and (3.4) lead to

$$\limsup_{n \rightarrow \infty} \langle \tilde{\mathcal{A}}(u_n), (u_n - u) \rangle = \limsup_{n \rightarrow \infty} \langle \mathcal{I}(u_n), (u_n - u) \rangle \leq 0. \quad (3.5)$$

Since $\tilde{\mathcal{A}}$ satisfies the (S_+) -property, we deduce from (3.1) and (3.5) that $u_n \rightarrow u$ in $\mathcal{V}$. Lastly, because $\mathcal{I}$ is continuous, we conclude that $\mathcal{I}(u_n) \rightarrow \mathcal{I}(u)$ in $\mathcal{V}'$, which proves that $\mathcal{I}$ is pseudomonotone.

• $\mathcal{I}$ is coercive:

For $u = (u_1, u_2) \in \mathcal{V}$, we have

$$\begin{aligned}
\langle \tilde{\mathcal{A}}(u), u \rangle &= \left(\int_{\Omega} |\nabla u_1|^{p_1(x)} + \mu_1 |\nabla u_1|^{q_1(x)} dx + \int_{\Omega} |\nabla u_2|^{p_2(x)} + \mu_2 |\nabla u_2|^{q_2(x)} dx \right) \\
&\geq \|u_1\|_{W_0^{1,p_1(x)}(\Omega)}^{\tau_1} + \|u_2\|_{W_0^{1,p_2(x)}(\Omega)}^{\tau_2} + \mu_1 \|u_1\|_{W_0^{1,q_1(x)}(\Omega)}^{v_1} + \mu_2 \|u_2\|_{W_0^{1,q_2(x)}(\Omega)}^{v_2} \\
&\geq \|u\|_{\mathcal{V}}^{\tau} + \mu_0 \|u\|_{\mathcal{V}}^{v},
\end{aligned}$$

where for $i \in \{1, 2\}$, we have

$$\begin{cases} \tau_i = p_i^-, & \text{if } \int_{\Omega} |\nabla u_i|^{p_i(x)} dx \geq 1, \\ \tau_i = p_i^+, & \text{if } \int_{\Omega} |\nabla u_i|^{p_i(x)} dx \leq 1, \end{cases}$$

and

$$\begin{cases} v_i = q_i^-, & \text{if } \int_{\Omega} |\nabla u_i|^{q_i(x)} dx \geq 1, \\ v_i = q_i^+, & \text{if } \int_{\Omega} |\nabla u_i|^{q_i(x)} dx \leq 1, \end{cases}$$

with $\tau = \min \{\tau_1, \tau_2\} > 1$ and $\nu = \min \{\nu_1, \nu_2\} > 1$. Using (H_4) and (H_5) , for any $\epsilon > 0$, there exists $M_\epsilon \in L^1(\Omega)$ such that for all $s = (s_1, s_2) \in \mathbb{R}^2$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{2N}$ for a.e. $x \in \Omega$, we have

$$\mathbf{f}(x, s, \xi) \cdot s \leq (\lambda_1(p) + \epsilon) \left(|s_1|^{\alpha_1(x)} |s_2|^{\alpha_2(x)} \right) + c \left(|\xi_1|^{q_1(x)} + |\xi_2|^{q_2(x)} \right) + M_\epsilon(x).$$

Consequently, we obtain

$$\begin{aligned} \langle \mathcal{I}(u), u \rangle &= \langle \tilde{\mathcal{A}}(u), u \rangle - \int_{\Omega} \mathbf{f}(x, u, \nabla u) \cdot u dx - \langle h, u \rangle \\ &\geq \langle \tilde{\mathcal{A}}_p(u), u \rangle - (\lambda_1(p) + \epsilon) \int_{\Omega} |u_1|^{\alpha_1(x)} |u_2|^{\alpha_2(x)} dx + \langle \tilde{\mathcal{A}}_q(u), u \rangle \\ &\quad - c \int_{\Omega} \left(|\nabla u_1|^{q_1(x)} + |\nabla u_2|^{q_2(x)} \right) dx - \int_{\Omega} M_\epsilon(x) dx - \|h\|_{\mathcal{V}'} \|u\|_{\mathcal{V}}. \end{aligned}$$

On the other hand, we have

$$\mathcal{A}_q(u) \leq \theta_q \langle \tilde{\mathcal{A}}_q(u), u \rangle \quad \text{and} \quad \lambda_1(p) \mathcal{B}_p(u) \leq \mathcal{A}_p(u) \leq \theta_p \langle \tilde{\mathcal{A}}_p(u), u \rangle,$$

with

$$\theta_p := \max \left\{ \frac{1}{p_1^-}, \frac{1}{p_2^-} \right\} < 1 \quad \text{and} \quad \theta_q := \max \left\{ \frac{1}{q_1^-}, \frac{1}{q_2^-} \right\} < 1.$$

Hence, it follows

$$\begin{aligned} \langle \mathcal{I}(u), u \rangle &\geq \left(1 - \theta_p \frac{\lambda_1(p) + \epsilon}{\lambda_1(p)} \right) \langle \tilde{\mathcal{A}}_p(u), u \rangle + (1 - c\mu_0\theta_q) \langle \tilde{\mathcal{A}}_q(u), u \rangle \\ &\quad - \|M_\epsilon\|_{L^1(\Omega)} - \frac{\alpha_0}{\tau} \|u\|_{\mathcal{V}}^\tau - \frac{1}{\alpha_0\tau'} \|h\|_{\mathcal{V}'}^{\tau'}, \end{aligned}$$

where $0 < \alpha_0 < \tau$, with $\tau' = \frac{\tau}{\tau - 1}$. Therefore, we get

$$\begin{aligned} \frac{\langle \mathcal{I}(u), u \rangle}{\|u\|_{\mathcal{V}}} &\geq \gamma \frac{\langle \tilde{\mathcal{A}}(u), u \rangle}{\|u\|_{\mathcal{V}}} - \frac{\|M_\epsilon\|_{L^1(\Omega)}}{\|u\|_{\mathcal{V}}} - \frac{\alpha_0}{\tau} \|u\|_{\mathcal{V}}^{\tau-1} - \frac{1}{\alpha_0\tau'} \frac{\|h\|_{\mathcal{V}'}^{\tau'}}{\|u\|_{\mathcal{V}}}, \\ &\geq \left(\gamma - \frac{\alpha_0}{\tau} \right) \|u\|_{\mathcal{V}}^{\tau-1} - \frac{\|M_\epsilon\|_{L^1(\Omega)}}{\|u\|_{\mathcal{V}}} - \frac{1}{\alpha_0\tau'} \frac{\|h\|_{\mathcal{V}'}^{\tau'}}{\|u\|_{\mathcal{V}}}, \end{aligned}$$

where $\gamma = \min \left\{ 1 - \theta_p \frac{\lambda_1(p) + \epsilon}{\lambda_1(p)}, 1 - c\mu_0\theta_q \right\}$. Now, by choosing c, ϵ and α_0 such that

$$\begin{cases} c < \frac{1}{\mu_0\theta_q}, \\ 0 < \epsilon < \frac{(1 - \theta_p) \lambda_1(p)}{\theta_p}, \\ 0 < \alpha_0 < \tau\gamma, \end{cases}$$

then, we obtain $\lim_{\|u\|_{\mathcal{V}} \rightarrow \infty} \frac{\langle \mathcal{I}(u), u \rangle}{\|u\|_{\mathcal{V}}} = \infty$. Therefore, $\mathcal{I}$ is coercive.

Thus, all the assumptions of Theorem 2.2 are satisfied. Therefore, there exists $u_0 \in X$ such that $\mathcal{I}(u_0) = 0$. This completes the proof.

4. Uniqueness result

Next, we consider the uniqueness of the solutions of system (PS). We consider the following hypotheses

- (F₁) For all $s = (s_1, s_2)$ and $t = (t_1, t_2) \in \mathbb{R}^2$, and for all $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2) \in \mathbb{R}^N$,
- $$(\mathbf{f}(x, s, \xi_1, \xi_2) - \mathbf{f}(x, t, \eta_1, \eta_2)) \cdot (s - t) \leq \rho_1 |s_1 - t_1| |s_2 - t_2| + \rho_2 (|\xi_1 - \eta_1|^2 + |\xi_2 - \eta_2|^2),$$
- for a.e $x \in \Omega$,
- (F₂) $\rho_2 < 1$ and $\rho_1 < 2(1 - \rho_2)\lambda_1(2, 2)$,
- (F₃) For all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^N$ and for a.e $x \in \Omega$, $s_j \rightarrow \mathbf{f}_i(x, s_1, s_2, \xi_1, \xi_2)$ is decreasing, for $i, j \in \{1, 2\}$,
- (F₄) For $i, j \in \{1, 2\}$, for all $s = (s_1, s_2) \in \mathbb{R}^2$ and for a.e $x \in \Omega$, we have $f_i(x, s_1, s_2, \xi_1, \xi_2) = f_i(x, s_1, s_2, |\xi_1|, |\xi_2|)$ and $\eta_j \rightarrow f_i(x, s_1, s_2, \eta_1, \eta_2)$ is decreasing.
- (F₅) For $i \in \{1, 2\}$, we have $q_i = p_i$.

Theorem 4.1. Suppose that the hypotheses of Theorem 3.1 alongside (F₁) and (F₂) are verified. In addition, if $p_i(x) = 2$ and $\alpha_i(x) = 1$, for all $x \in \bar{\Omega}$ and for $i \in \{1, 2\}$, then problem (PS) has a unique solution.

Proof. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two weak solutions to (PS). Considering the weak formulation of (u_1, u_2) and (v_1, v_2) , by choosing $\phi = u_1 - v_1$ and $\psi = u_2 - v_2$ as test functions, and adding the obtained equations to each other, we get

$$\begin{aligned} & \int_{\Omega} |\nabla (u_1(x) - v_1(x))|^2 dx + \mu_1 \int_{\Omega} \left(\mathcal{F}_{q_1(x)}(\nabla u_1) - \mathcal{F}_{q_1(x)}(\nabla v_1) \right) \cdot (\nabla u_1 - \nabla v_1) dx \\ & + \int_{\Omega} |\nabla (u_2(x) - v_2(x))|^2 dx + \mu_2 \int_{\Omega} \left(\mathcal{F}_{q_2(x)}(\nabla u_2) - \mathcal{F}_{q_2(x)}(\nabla v_2) \right) \cdot (\nabla u_2 - \nabla v_2) dx \\ & = \int_{\Omega} (\mathbf{f}(x, u, \nabla u) - \mathbf{f}(x, v, \nabla v)) \cdot (u - v) dx. \end{aligned}$$

Moreover, for $r(x) > 1$ for all $x \in \bar{\Omega}$ and $w_1, w_2 \in W_0^{1, r(x)}(\Omega)$, we have

$$0 \leq \left(\mathcal{F}_{r(x)}(\nabla w_1) - \mathcal{F}_{r(x)}(\nabla w_2) \right) \cdot \nabla (w_1 - w_2) dx.$$

Therefore, using assumption (F₁) leads to

$$\begin{aligned} \int_{\Omega} |\nabla (u_1(x) - v_1(x))|^2 + |\nabla (u_2(x) - v_2(x))|^2 dx & \leq \rho_1 \int_{\Omega} |u_1 - v_1| |u_2 - v_2| \\ & + \rho_2 \left(|\nabla (u_1(x) - v_1(x))|^2 + |\nabla (u_2(x) - v_2(x))|^2 \right) dx. \end{aligned}$$

Hence, we have

$$(1 - \rho_2) \left(\int_{\Omega} |\nabla (u_1(x) - v_1(x))|^2 dx + \int_{\Omega} |\nabla (u_2(x) - v_2(x))|^2 dx \right) \leq \rho_1 \int_{\Omega} |u_1 - v_1| |u_2 - v_2| dx.$$

Thus, we obtain

$$2(1 - \rho_2) \mathcal{A}_{(2,2)}(u - v) \leq \rho_1 \mathcal{B}_{(2,2)}(u - v). \quad (4.1)$$

On the other hand, by using the definition of $\lambda_1(2, 2)$, we get

$$\mathcal{B}_{(2,2)}(u - v) \leq \frac{1}{\lambda_1(2, 2)} \mathcal{A}_{(2,2)}(u - v).$$

By using (4.1), we obtain

$$\mathcal{A}_{(2,2)}(u - v) \leq \frac{\rho_1}{2(1 - \rho_2) \lambda_1(2, 2)} \mathcal{A}_{(2,2)}(u - v).$$

If $\mathcal{A}_{(2,2)}(u - v) \neq 0$, we obtain $\rho_1 \geq 2(1 - \rho_2) \lambda_1(2, 2)$, which contradicts hypothesis (F_2) .

Therefore, we have $\mathcal{A}_{(2,2)}(u - v) = 0$, which means that $u = v$, since $(u, v) \in \mathcal{V}$. This completes the proof.

Theorem 4.2. Suppose that the hypotheses of Theorem 3.1 and $(F_3) - (F_5)$ are verified, then problem (PS) has a unique solution.

Proof. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two weak solutions to problem (PS). Considering the weak formulation of (u_1, u_2) and (v_1, v_2) , by choosing $\phi = (u_1 - v_1)_+$ and $\psi = (u_2 - v_2)_+$ as test functions, and adding the obtained equations to each other, we have as in the proof of Theorem 4.1

$$\begin{aligned} 0 &\leq \int_{\{u_1(x) > v_1(x)\}} \left(\mathcal{F}_{p_1(x)}(\nabla u_1) - \mathcal{F}_{p_1(x)}(\nabla v_1) \right) \cdot \nabla (u_1 - v_1) dx \\ &\quad + \int_{\{u_2(x) > v_2(x)\}} \left(\mathcal{F}_{p_2(x)}(\nabla u_2) - \mathcal{F}_{p_2(x)}(\nabla v_2) \right) \cdot \nabla (u_2 - v_2) dx \\ &= \int_{\Omega} \left(\mathcal{F}_{p_1(x)}(\nabla u_1) - \mathcal{F}_{p_1(x)}(\nabla v_1) \right) \cdot \nabla (u_1 - v_1)_+ dx \\ &\quad + \int_{\Omega} \left(\mathcal{F}_{p_2(x)}(\nabla u_2) - \mathcal{F}_{p_2(x)}(\nabla v_2) \right) \cdot \nabla (u_2 - v_2)_+ dx \\ &= \int_{\Omega} (\mathbf{f}_1(x, u, \nabla u) - \mathbf{f}_1(x, v, \nabla v)) \cdot (u_1 - v_1)_+ dx \\ &\quad + \int_{\Omega} (\mathbf{f}_2(x, u, \nabla u) - \mathbf{f}_2(x, v, \nabla v)) \cdot (u_2 - v_2)_+ dx. \end{aligned}$$

Then,

$$\int_{\Omega} (\mathbf{f}_1(x, u, \nabla u) - \mathbf{f}_1(x, v, \nabla v)) \cdot (u_1 - v_1)_+ dx = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4,$$

where

$$\begin{aligned}
\mathcal{I}_1 &= \int_{\Omega} (\mathbf{f}(x, u_1, u_2, |\nabla u_1|, |\nabla u_2|) - \mathbf{f}(x, v_1, u_2, |\nabla u_1|, |\nabla u_2|)) \cdot (u_1 - v_1)_+ dx, \\
\mathcal{I}_2 &= \int_{\Omega} (\mathbf{f}(x, v_1, u_2, |\nabla u_1|, |\nabla u_2|) - \mathbf{f}(x, v_1, v_2, |\nabla u_1|, |\nabla u_2|)) \cdot (u_1 - v_1)_+ dx, \\
\mathcal{I}_3 &= \int_{\Omega} (\mathbf{f}(x, v_1, v_2, |\nabla u_1|, |\nabla u_2|) - \mathbf{f}(x, v_1, v_2, |\nabla v_1|, |\nabla u_2|)) \cdot (u_1 - v_1)_+ dx, \\
\mathcal{I}_4 &= \int_{\Omega} (\mathbf{f}(x, v_1, v_2, |\nabla v_1|, |\nabla u_2|) - \mathbf{f}(x, v_1, v_2, |\nabla v_1|, |\nabla v_2|)) \cdot (u_1 - v_1)_+ dx.
\end{aligned}$$

According to the hypothesis (F_3) , we have $\mathcal{I}_1 \leq 0$. Similarly, we have

$$\mathcal{I}_2 = \int_{\Omega_2} (\mathbf{f}(x, v_1, u_2, |\nabla u_1|, |\nabla u_2|) - \mathbf{f}(x, v_1, v_2, |\nabla u_1|, |\nabla u_2|)) \cdot (u_2 - v_2)_+ \frac{(u_1 - v_1)_+}{(u_2 - v_2)_+} dx,$$

where $\Omega_2 = \{x, (u_2 - v_2)_+(x) \neq 0\}$. Using hypothesis (F_3) , we have $\mathcal{I}_2 \leq 0$. On the other hand, we get

$$\begin{aligned}
\mathcal{I}_3 &= \int_{\Omega_3} (\mathbf{f}(x, v_1, v_2, |\nabla u_1|, |\nabla u_2|) - \mathbf{f}(x, v_1, v_2, |\nabla v_1|, |\nabla u_2|)) \cdot \\
&\quad (|\nabla u_1| - |\nabla v_1|)_+ \frac{(u_1 - v_1)_+}{(|\nabla u_1| - |\nabla v_1|)_+} dx,
\end{aligned}$$

where $\Omega_3 = \{x, (|\nabla u_1| - |\nabla v_1|)_+(x) \neq 0\}$. According to the hypothesis (F_3) , we have $\mathcal{I}_3 \leq 0$. On the other hand, we get

$$\begin{aligned}
\mathcal{I}_4 &= \int_{\Omega_4} (\mathbf{f}(x, v_1, v_2, |\nabla v_1|, |\nabla u_2|) - \mathbf{f}(x, v_1, v_2, |\nabla v_1|, |\nabla v_2|)) \cdot \\
&\quad (|\nabla u_2| - |\nabla v_2|)_+ \frac{(u_1 - v_1)_+}{(|\nabla u_2| - |\nabla v_2|)_+} dx,
\end{aligned}$$

where $\Omega_4 = \{x, (|\nabla u_2| - |\nabla v_2|)_+(x) \neq 0\}$. According to the hypothesis (F_3) , we have $\mathcal{I}_4 \leq 0$. Therefore, we get

$$\int_{\Omega} (\mathbf{f}_1(x, u, \nabla u) - \mathbf{f}_1(x, v, \nabla v)) \cdot (u_1 - v_1)_+ dx \leq 0.$$

Using the same arguments, we obtain

$$\int_{\Omega} (\mathbf{f}_2(x, u, \nabla u) - \mathbf{f}_2(x, v, \nabla v)) \cdot (u_2 - v_2)_+ dx \leq 0,$$

which implies that

$$\int_{\{u_1(x) > v_1(x)\}} (\mathcal{F}_{p_1(x)}(\nabla u_1) - \mathcal{F}_{p_1(x)}(\nabla v_1)) \cdot \nabla (u_1 - v_1) dx = 0,$$

and

$$\int_{\{u_2(x) > v_2(x)\}} (\mathcal{F}_{p_2(x)}(\nabla u_2) - \mathcal{F}_{p_2(x)}(\nabla v_2)) \cdot \nabla (u_2 - v_2) dx = 0.$$

respectively. From this, it follows $\nabla u_1(x) = \nabla v_1(x)$ for a.e x on $\mathcal{U}_1 = \{x \in \Omega : u_1(x) > v_1(x)\}$ and $\nabla u_2(x) = \nabla v_2(x)$ for a.e x on $\mathcal{U}_2 = \{x \in \Omega : u_1(x) > v_1(x)\}$. Now, for $x \in \mathcal{U}_1$, we have $(u_1 - v_1)_+(x) = (u_1 - v_1)(x)$ and $\nabla (u_1 - v_1)_+(x) = \nabla u_1(x) - \nabla v_1(x) = 0$, while for $x \in \Omega \setminus \mathcal{U}_1$, we have $(u_1 - v_1)_+(x) = 0$ and $\nabla (u_1 - v_1)_+(x) = 0$.

Consequently, we have $\nabla (u_1 - v_1)_+(x) = 0$, a.e in Ω and then $(u_1 - v_1)_+ = 0$ because $(u_1 - v_1)_+ \in W_0^{1,p_1(x)}(\Omega)$. Then, we get $u_1 \leq v_1 = 0$, for a.e x in Ω . Similarly, we obtain $u_2 \leq v_2 = 0$ for a.e x in Ω . Similarly, we obtain $u_1 \geq v_1 = 0$, for a.e x in Ω and $u_2 \geq v_2 = 0$, for a.e x in Ω . Therefore, $u = v$ and the solution is unique.

Now, we provide some examples that verify the hypotheses of the existence and uniqueness theorems.

Example 1. Let the function $f = (f_1, f_2)$ be defined by

$$f_1(x, s_1, s_2, \xi_1, \xi_2) = |s_2|^{\alpha_2(x)} |s_1|^{\alpha_1(x)-2} s_1 \left(a_1(x) + \frac{d_1(x)}{1 + |s|^{\alpha(x)}} |\xi_1|^{q_1(x)} \right),$$

and

$$f_2(x, s_1, s_2, \xi_1, \xi_2) = |s_1|^{\alpha_1(x)} |s_2|^{\alpha_2(x)-2} s_2 \left(a_2(x) + \frac{d_2(x)}{1 + |s|^{\alpha(x)}} |\xi_2|^{q_2(x)} \right),$$

where the functions $\alpha_1(x)$ and $\alpha_2(x)$ are defined in the first Section and a_1, a_2, d_1 , and d_2 are in $L^\infty(\Omega)$ where they satisfy the condition $\|a_1 + a_2\|_\infty < \lambda_1(p)$. We have

$$\mathbf{f}(x, s, \xi) \cdot s = (a_1(x) + a_2(x)) |s|^{\alpha(x)} + \frac{|s|^{\alpha(x)}}{1 + |s|^{\alpha(x)}} \left(d_1(x) |\xi_1|^{q_1(x)} + d_2(x) |\xi_2|^{q_2(x)} \right).$$

Then, f satisfies all hypotheses of Theorem 3.1.

Example 2. Let the function $f = (f_1, f_2)$ be defined by

$$f_1(x, s_1, s_2, \xi_1, \xi_2) = a_1(x) \sin s_2 + d_1(x) \xi_1.$$

and

$$f_2(x, s_1, s_2, \xi_1, \xi_2) = a_2(x) \sin s_1 + d_2(x) \xi_2,$$

where the functions $\alpha_1(x)$ and $\alpha_2(x)$ are defined in the first Section and a_1, a_2, d_1 and d_2 are in $L^\infty(\Omega)$ where they satisfy the condition $\|a_1\|_\infty + \|a_2\|_\infty < \lambda_1(p)$. We have

$$\mathbf{f}(x, s, \xi) \cdot s = a_1(x) s_1 \sin s_2 + a_2(x) s_2 \sin s_1 + d_1(x) s_1 \xi_1 + d_2(x) s_2 \xi_2.$$

Then, f satisfies all hypotheses of Theorem 4.1.

Example 3. Let the function $f = (f_1, f_2)$ be defined by

$$f_1(x, s_1, s_2, \xi_1, \xi_2) = -a_1(x) s_2 - \frac{2}{\pi} \left(\arctan s_2 + \frac{s_2}{1 + s_2^2} \right) \left(1 + d_1(x) \frac{|\xi_2|^2}{1 + |\xi_2|^2} \right),$$

and

$$f_2(x, s_1, s_2, \xi_1, \xi_2) = -a_2(x) s_1 - \frac{2}{\pi} \left(\arctan s_1 + \frac{s_1}{1 + s_1^2} \right) \left(1 + d_2(x) \frac{|\xi_1|^2}{1 + |\xi_1|^2} \right),$$

where a_1, a_2, d_1 and d_2 are positive functions in $L^\infty(\Omega)$ that satisfy the condition $\|a_1\|_\infty + \|a_2\|_\infty < \lambda_1(p)$. Then, f satisfies all hypotheses of Theorem 4.2.

5. Case of a gradient-type system

In this section, we focus on gradient-type systems. Hence, we assume that $f_i(x, u) = \frac{\partial F}{\partial u_i}(x, u)$ for $i \in \{1, 2\}$. The system (PS) becomes

$$\begin{cases} -\Delta_{p_i(x)} u_i - \mu_i \Delta_{q_i(x)} u_i = \frac{\partial F}{\partial u_i}(x, u) + h_i(x) & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega. \end{cases} \quad (\text{GS})$$

For $u = (u_1, u_2) \in \mathcal{V}$, define $\mathcal{A}_0(u_1, u_2)$ and $\mathcal{B}(u_1, u_2)$ as in Section 2 by

$$\begin{cases} \mathcal{A}_0(u) = \int_{\Omega} \frac{1}{p_1(x)} |\nabla u_1|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u_2|^{p_2(x)} dx, \\ \mathcal{A}(u) = \mathcal{A}_0(u) + \int_{\Omega} \frac{\mu_1}{q_1(x)} |\nabla u_1|^{q_1(x)} dx + \int_{\Omega} \frac{\mu_2}{q_2(x)} |\nabla u_2|^{q_2(x)} dx, \\ \mathcal{B}(u) = \int_{\Omega} |u_1|^{\alpha_1(x)} |u_2|^{\alpha_2(x)} dx = \int_{\Omega} |u|^{\alpha(x)} dx, \end{cases}$$

and denote

$$F_{\infty}(x) \equiv \limsup_{|s| \rightarrow +\infty} \frac{F(x, s)}{|s|^{\alpha(x)}}.$$

Suppose that function F satisfies the following hypotheses

- (h_1) $\frac{\partial F}{\partial s_1}$ and $\frac{\partial F}{\partial s_2} : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions such that there exist positive continuous functions α_i^* , nonnegative functions $e_i \in L^1(\Omega)$ with $1 < \alpha_i^*(x) < p_i^*(x)$, and nonnegative constants a_i for $i \in \{1, 2\}$, such that for a.e. $x \in \Omega$ and all $(s_1, s_2) \in \mathbb{R}^2$, we have

$$\left| \frac{\partial F}{\partial s_1}(x, s_1, s_2) \right| \leq a_1 \alpha_1(x) |s_1|^{\alpha_1^*(x)-1} |s_2|^{\alpha_2^*(x)} + e_1(x),$$

and

$$\left| \frac{\partial F}{\partial s_2}(x, s_1, s_2) \right| \leq a_2 \alpha_2(x) |s_1|^{\alpha_1^*(x)} |s_2|^{\alpha_2^*(x)-1} + e_2(x).$$

- (h_2) There exist a nonnegative constant A_1 and a nonnegative function $A_2 \in L^1(\Omega)$ such that, for all $x \in \Omega$ we have

$$|F(x, s)| \leq A_1 |s|^{\alpha(x)} + A_2(x).$$

- (h_3) $F_{\infty}(x) < \lambda_1$ uniformly for a.e. $x \in \Omega$.

In this section, we state and prove a nonresonance result below the positive infimum λ_1 . Define

$$\mathcal{I}(u) = \mathcal{A}(u) - \int_{\Omega} F(x, u(x)) dx - \langle h, u \rangle,$$

where $\langle u, \varphi \rangle = \langle u, \varphi \rangle_{\mathcal{V}, \mathcal{V}'}$ denotes the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$.

Theorem 5.1. *Under hypotheses (P_1) or (P_2) and (h_1) – (h_3), problem (GS) admits at least one solution $v \in \mathcal{V}$, for any $h \in \mathcal{V}'$. Moreover, suppose that (F_1) and (F_2) are verified and $p_i(x) \geq 2$, for all $x \in \Omega$ and $i \in \{1, 2\}$, then the solution is unique.*

Proof. We want to minimize the following energy functional $\mathcal{I}$ associated with problem (GS)

$$\mathcal{I}(u) \equiv \mathcal{A}(u) - \int_{\Omega} F(x, u(x)) dx - \langle h, u \rangle, \forall u \in \mathcal{V}.$$

To this end, we show that $\mathcal{I}$ is weakly lower semicontinuous and coercive.

• **$\mathcal{I}$ is weakly lower semicontinuous:**

Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{V}$ be a sequence such that

$$u_n \rightharpoonup u \text{ in } \mathcal{V}. \quad (5.1)$$

Then, taking the compact embedding $W_0^{1,p_i(x)}(\Omega) \hookrightarrow L^{p_i(x)}(\Omega)$ into consideration gives

$$u_{i,n} \rightarrow u \text{ in } L^{p_i(x)}(\Omega). \quad (5.2)$$

Moreover, there exist $\theta \in L^{p_i(x)}(\Omega)$ such that

$$|u_{i,n}(x)| \leq \theta_i(x), \text{ for a.e } x \in \Omega.$$

Therefore, by hypothesis (h_2) , we have

$$\begin{aligned} F(x, u_n(x)) &\leq A_1 |u_n|^{\alpha(x)} + A_2(x) \\ &\leq A_1 |\theta(x)|^{\alpha(x)} + A_2(x). \end{aligned}$$

Using the fact that $F(x, u_n(x)) \rightarrow F(x, u(x))$, for a.e $x \in \Omega$ and applying Fatou's lemma, we obtain

$$\limsup_{n \rightarrow +\infty} \int_{\Omega} F(x, u_n(x)) dx \leq \int_{\Omega} F(x, u(x)) dx.$$

Hence

$$\begin{aligned} \mathcal{I}(u) &= \mathcal{A}(u) - \int_{\Omega} F(x, u(x)) dx - \langle u, h \rangle \\ &\leq \liminf_{n \rightarrow +\infty} \mathcal{A}(u_n) - \limsup_{n \rightarrow +\infty} \int_{\Omega} F(x, u_n(x)) dx - \lim_{n \rightarrow +\infty} \langle h, u_n \rangle \\ &\leq \liminf_{n \rightarrow +\infty} \left\{ \mathcal{A}(u_n) - \int_{\Omega} F(x, u_n(x)) dx - \langle h, u_n \rangle \right\} \\ &\leq \liminf_{n \rightarrow +\infty} \mathcal{I}(u_n). \end{aligned}$$

Consequently, $\mathcal{I}$ is weakly lower semicontinuous.

• **$\mathcal{I}$ is coercive:**

Using Remark 2.1 for $u = (u_1, u_2)$, we have

$$\begin{aligned} \mathcal{A}(u) &\geq \mathcal{A}_0(u) \geq \eta \left(\|u_1\|_{\mathcal{V}_1}^{\sigma_1} + \|u_2\|_{\mathcal{V}_2}^{\sigma_2} \right) \\ &\geq \eta \|u\|_{\mathcal{V}}^{\sigma}, \end{aligned}$$

where $\eta = \min \left\{ \frac{1}{p_1^+}, \frac{1}{p_2^+} \right\}$, and for $i \in \{1, 2\}$, we have

$$\begin{cases} \sigma_i = p_i^- & \text{if } \int_{\Omega} |\nabla u_i|^{p_i(x)} dx \geq 1, \\ \sigma_i = p_i^+ & \text{if } \int_{\Omega} |\nabla u_i|^{p_i(x)} dx \leq 1, \end{cases}$$

where $\sigma = \min(\sigma_1, \sigma_2) > 1$. By (h_3) , for any $\epsilon > 0$ there exist $\delta_\epsilon > 0$ and $M_\epsilon \in L^1(\Omega)$ such that

$$F(x, s) \leq (F_\infty(x) + \epsilon) |s|^{\alpha(x)} + M_\epsilon(x), \quad \forall s : |s| > \delta_\epsilon, \text{ for a.e } x \in \Omega.$$

By the embedding theorem, there exist $\gamma_0, \gamma_1 > 0$ such that

$$\int_{\Omega} \frac{\mu_1}{q_1(x)} |\nabla u_1|^{q_1(x)} dx + \int_{\Omega} \frac{\mu_2}{q_2(x)} |\nabla u_2|^{q_2(x)} dx \geq \gamma_0 \mathcal{A}_0(u) - \gamma_1.$$

Now, using hypothesis (h_1) , we have

$$F(x, s) \leq (F_\infty(x) + \epsilon) |s|^{\alpha(x)} + M_\epsilon(x), \quad \forall s \in \mathbb{R}^2, \text{ for a.e } x \in \Omega.$$

Let $\epsilon > 0$ be fixed. We then have

$$\begin{aligned} \mathcal{I}(u) &= \mathcal{A}(u) - \int_{\Omega} F(x, u) dx - \langle u, h \rangle \\ &\geq (1 + \gamma_0) \mathcal{A}_0(u) - (\lambda_1 + \epsilon) \int_{\Omega} |u|^{\alpha(x)} dx - \int_{\Omega} M_\epsilon(x) - \|h\|_{\mathcal{V}'} \|u\|_{\mathcal{V}} - \gamma_1. \end{aligned}$$

On the other hand, we have

$$\mathcal{B}(u) = \int_{\Omega} |u|^{\alpha(x)} dx \leq \frac{1}{\lambda_1} \mathcal{A}_0(u).$$

Hence,

$$\mathcal{I}(u) \geq \left(1 + \gamma_0 - \frac{\lambda_1 + \epsilon}{\lambda_1}\right) \mathcal{A}_0(u) - \|M_\epsilon\|_{L^1(\Omega)} - \frac{\gamma}{p^-} \|u\|_{\mathcal{V}}^{p^-} - \frac{1}{\gamma (p^-)'} \|h\|_{\mathcal{V}'}^{(p^-)'} - \gamma_1,$$

where $0 < \gamma < p^-$ and $(p^-)' = \frac{p^-}{p^- - 1}$. Now, we have $\mathcal{A}_0(u) \geq \eta \|u\|_{\mathcal{V}}^{p^-}$, then

$$\lim_{\|u\|_{\mathcal{V}} \rightarrow \infty} \frac{\mathcal{A}_0(u)}{\|u\|_{\mathcal{V}}} = +\infty,$$

and

$$\mathcal{I}(u) \geq \left(\gamma_0 - \frac{\epsilon}{\lambda_1} - \frac{\gamma}{\eta p^-}\right) \mathcal{A}_0(u) - \|M_\epsilon\|_{L^1(\Omega)} - \frac{1}{\gamma p^-} \|h\|_{\mathcal{V}'}^{p'}.$$

By choosing γ and ϵ such that $\gamma_0 > \frac{\epsilon}{\lambda_1} + \frac{\gamma}{\eta p^-}$, we obtain

$$\lim_{\|u\|_{\mathcal{V}} \rightarrow \infty} \frac{\mathcal{I}(u)}{\|u\|_{\mathcal{V}}} = +\infty.$$

Hence the coerciveness of $\mathcal{I}$. Now, $\mathcal{I}$ being lower semicontinuous and coercive, there exists $u \in \mathcal{V}$ such that $\mathcal{I}(u) = \inf_{v \in \mathcal{V}} \mathcal{I}(v)$. Owing to hypothesis (h_1) , $\mathcal{I}$ is a C^1 -functional and $u \in \mathcal{V}$ is a weak solution of problem (GS) if and only if it is a critical point of $\mathcal{V}$.

The proof of uniqueness of the solution is similar to that of Theorem 4.1. This completes the proof.

6. Conclusions and perspectives

In this paper, we proved the existence result for the problem (PS) and, under some conditions specified in the paper, we established the uniqueness of the solution. In addition, we provided some examples that verify both results. We also dealt with a system of gradient-type and presented a proof of existence using a variational method. Our plan for future research involves investigating resonance and nonresonance issues when considering the interaction of some terms in the equations with the variational spectrum of some associated eigenvalue systems when it can be defined. Positive solutions and multiplicity questions are also included in our research plan.

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References

- [1] E. ACERBI, G. MINGIONE, *Regularity results for stationary electrorheological fluids*, *Archive for Rational Mechanics and Analysis*, **164**, 213-259, (2002).
- [2] C.O. ALVES, A. MOUSSAOUI, *Existence and regularity of solutions for a class of singular $(p(x), q(x))$ -Laplacian systems*, *Complex Variables and Elliptic Equations*, **63**(2), 188-210, (2017).
- [3] S. N. ANTONTSEV, J. F. RODRIGUES, *On stationary thermorheological viscous flows*, *Annali dell'Universita di Ferrara, Sezione VII - Scienze Matematiche*, **52**, 19-36, (2006).
- [4] S. N. ANTONTSEV, S. I. SHMAREV, *Handbook of Differential Equations: Stationary Partial Differential Equations, Chapter 1*, **3**, (2006).
- [5] L. BOCCARDO, G. DE FIGUEIREDO, *Some remarks on a system of quasilinear elliptic equations*, *Nonlinear Differential Equations and Applications* **9**, 309-323, (2002).
- [6] F. BOZORGNIA, S.A. MOHAMMADI, T. VEJCHODSKÝ, *The first eigenvalue and eigenfunction of a nonlinear elliptic system*, *Applied Numerical Mathematics* **145**, 159-174, (2019).
- [7] Y. CHEN, S. LEVINE, M. RAO, *Variable exponent, linear growth functionals in image restoration*, *SIAM Journal on Applied Mathematics*, **66**(4), 1383-1406, (2006).
- [8] F. DE THÉLIN, *Première valeur propre d'un système elliptique non linéaire*, *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, **311**(10), 603-606, (1990).
- [9] L. DIENING, P. HARJULEHTO, P. HÄSTÖ, M. RUŽIČKA, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Heidelberg, (2011).
- [10] A. EL HACHIMI, F. DE THÉLIN, *Nonrésonance près de la première valeur propre d'un système elliptique quasilinéaire de type potentiel*, *Publicaciones Matemáticas*, **39**(2), 393-404, (1995).
- [11] A. EL KHALIL, S. EL MANOUNI, M. OUANAN, *Simplicity and stability of the first eigenvalue of a nonlinear elliptic system*, *International Journal of Mathematics and Mathematical Sciences*, **10**, Article ID 328340, 1555-1563, (2005).

- [12] X. FAN, Q.-H. ZHANG, *Existence of solutions for $p(x)$ -Laplacian Dirichlet problem*, *Nonlinear Analysis: Theory, Methods and Applications*, **52**(8), 1843-1852, (2003).
- [13] X. FAN, D. ZHAO, *On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$* , *Journal of Mathematical Analysis and Applications*, **263**(2), 424-446, (2001).
- [14] L. FARIA, O. MIYAGAKI, M. TANAKA, *Existence of a positive solution for problems with (p,q) -Laplacian and convection term in $\mathbb{R}^N$* , *Boundary Value Problems*, **158**, (2016).
- [15] P. FELMER, R. MANASEVICH, F. DE THÉLIN, *Existence and uniqueness of positive solutions for certain quasilinear elliptic systems*, *Communications in Partial Differential Equations*, **1**(11-12), 2013-2029, (1992).
- [16] T.-S. HSU, *Multiple positive solutions for a quasilinear elliptic system involving concave-convex nonlinearities and sign-changing weight functions*, *International Journal of Mathematics and Mathematical Sciences*, **2012**, Article ID 109214, 1-21, (2012).
- [17] D.A. KANDILAKIS, M. MAGIROPOULOS, N. ZOGRAPHOPOULOS, *The first eigenvalue of p -Laplacian systems with nonlinear boundary conditions*, *Boundary Value Problems*, **2005**(3), Article number: 251350, 307-321, (2005).
- [18] D. MOTREANU, C. VETRO, F. VETRO, *A parametric Dirichlet problem for systems of quasilinear elliptic equations with gradient dependence*, *Numerical Functional Analysis and Optimization*, **37**(12), 1551-1561, (2016).
- [19] A. MOUSSAOUI, J. VÉLIN, *On the first eigenvalue for a $(p(x), q(x))$ -Laplacian elliptic system*, *Electronic Journal of Qualitative Theory of Differential Equations*, **66**, 1-22, (2019).
- [20] D. NABAB, J. VÉLIN, *On a nonlinear elliptic system involving the $(p(x) - q(x))$ -Laplacian operator with gradient dependence*, *Complex Variables and Elliptic Equations*, **67**(7), 1554-1578, (2022).
- [21] M. RUŽIČKA, *Electrorheological fluids: modeling and mathematical theory*, *Lecture Notes in Mathematics*, Springer, Berlin, (2000).
- [22] N.B. ZOGRAPHOPOULOS, *On the principal eigenvalue of degenerate quasilinear elliptic systems*, *Mathematische Nachrichten*, **281**(9), 1351-1365, (2008).