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The Maximum Locus of the Bloch Norm

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In memory of Mohamed Zarrabi

ABSTRACT. For a Bloch function f in the unit ball in $\mathbb{C}^n$, we study the maximal locus of the Bloch norm of f ; namely, the set L_f where the Bergman length of the gradient vector field of f attains its maximum. We prove that for $n \geq 2$, the set L_f consists of a finite union of real analytic sets with dimensions at most $2n - 2$. This is not the case for $n = 1$ as was proved earlier by Cima and Wogen. We also give some rigidity properties of the set L_f . In particular, we give some sufficient criteria for constructing extreme functions in the Little Bloch ball.

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1. Introduction and statement of the main results

Bloch functions play an important role in the interaction between geometric function and operator theory. They are often viewed as symbols of bounded Hankel operators. See [7] and the references therein. Therefore, it is always interesting to study subclasses of Bloch functions and understand the spectral properties they reflect on their relevant Hankel operators. In [2], J. Cima and W. Wogen proved that if f is a holomorphic function in the little Bloch closed unit ball of the unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$, then the set $L_f := \{z \in \Delta : (1 - |z|^2)|f'(z)| = 1\}$ consists of finitely many points and closed simple curves. Subsequent proofs of this result were given in [1] and [5]. In this note we shall consider the higher dimensional setting of this problem. In this case there are several equivalent functional quantities which define the Bloch space. We

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shall consider herein the quantity given by the length of the Bergman gradient vector field of f with respect to the Bergman metric. Therefore, we start by recalling the basic elements of the Bergman geometry [4].

Let Ω be a bounded domain in $\mathbb{C}^n$. We denote by $\text{Aut}(\Omega)$ the group of all holomorphic automorphisms of Ω and let b_Ω be the (differential) Bergman metric. It is given by

$$b_\Omega^2(z, \zeta) = \sum_{i,j=1}^n b_{ij}(z) \zeta_i \bar{\zeta}_j, \quad \text{for } (z, \zeta) \in \Omega \times \mathbb{C}^n,$$

where

$$b_{ij}(z) := \frac{\partial^2 \log K(z, z)}{\partial z_i \partial \bar{z}_j}$$

and K is the Bergman kernel of Ω . This metric is automorphism invariant. Namely, for each $(z, \zeta) \in \Omega \times \mathbb{C}^n$

$$b_\Omega(\varphi(z), \varphi'(z)\zeta) = b_\Omega(z, \zeta), \quad \text{for all } \varphi \in \text{Aut}(\Omega),$$

where φ' denotes the complex Jacobian matrix of φ . If f is holomorphic function in Ω , then the Bergman gradient vector field of f is given by

$$\tilde{\nabla} f = 2 \sum_{i,j=1}^n b^{ij} \frac{\partial f}{\partial \bar{z}_i} \frac{\partial}{\partial z_j},$$

where $(b^{ij}(z))$ is the inverse of the matrix $(b_{ij}(z))$. The square of the length with respect to the Bergman metric of $\tilde{\nabla} f$ is given by

$$\|\tilde{\nabla} f\|^2 = 2 \sum_{i,j=1}^n b^{ij} \frac{\partial f}{\partial \bar{z}_i} \frac{\partial f}{\partial z_j}.$$

We set

$$(Qf)(z) := \frac{n+1}{2} \|\tilde{\nabla} f\|^2(z)$$

This quantity is invariant under $\text{Aut}(\Omega)$, due to the invariant property of the Bergman metric b_Ω . Namely,

$$Q(f \circ \varphi) = Q(f) \circ \varphi \quad \text{for all } \varphi \in \text{Aut}(\Omega).$$

We shall say that the holomorphic function on Ω is a Bloch function, if

$$\|f\|_{\mathfrak{B}}^2 = \sup_{z \in \Omega} (Qf)(z) < \infty.$$

We denote by $\mathfrak{B}(\Omega)$ the space of all holomorphic Bloch functions f on Ω . An element $f \in \mathfrak{B}(\Omega)$ for which $\lim_{z \rightarrow \partial\Omega} (Qf)(z) = 0$, is called a little Bloch function. The little Bloch space $\mathfrak{B}_0(\Omega)$ is the subspace of $\mathfrak{B}(\Omega)$ which consists of all little Bloch functions. The pair $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ is a Banach space which contains $\mathfrak{B}_0(\Omega)$ as a closed subspace. We denote by $\mathfrak{B}^1(\Omega)$ the unit ball in $(\mathfrak{B}(\Omega), \|\cdot\|_{\mathfrak{B}})$ and $\mathfrak{B}_0^1(\Omega)$ the unit ball in $(\mathfrak{B}_0(\Omega), \|\cdot\|_{\mathfrak{B}})$.

We set

$$L_f := \{z \in \Omega : (Qf)(z) = \|f\|_{\mathfrak{B}}^2\}$$

and call L_f the Bloch maximum locus of f .

If $\mathbb{B}$ is the unit ball in $\mathbb{C}^n$, it is known that for a holomorphic function on $\mathbb{B}$ we have

$$(Qf)(z) := (1 - |z|^2) [|\nabla f|^2(z) - |\mathcal{R}f|^2(z)]$$

where $\nabla f = (\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n})$ and $(\mathcal{R}f)(z) := \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$ is the radial derivative of f .

Finally, we fix some notations. For $z = (z_1, \dots, z_n), w = (w_1, \dots, w_n) \in \mathbb{C}^n$, and $r > 0$, we set $z \bullet w := \sum_{j=1}^n z_j w_j$, $\langle z, w \rangle := z \bullet \bar{w}$ and $\mathbb{B}(z, r) := \{w \in \mathbb{B} : |w - z| \leq r\}$.

Throughout the paper we shall assume that $n \geq 2$. Our main results are the following.

Theorem A Suppose that $f \in \mathfrak{B}_0^1(\mathbb{B})$. Then there are finitely many real analytic sets $E_1, \dots, E_N$ in $\mathbb{C}^{n-1}$ and for each $j = 1, \dots, N$, there is a real analytic injective mapping $\psi_j : \Omega_j \rightarrow \mathbb{C}^n$ defined on a neighborhood Ω_j of E_j in $\mathbb{C}^{n-1}$ such that

$$L_f = \cup_{j=1}^N \psi_j(E_j).$$

In particular, the d -dimensional Hausdorff measure $\mathcal{H}_d(L_f)$ of L_f is zero for any $d > 2n - 2$.

Theorem B Let Ω be a bounded domain in $\mathbb{C}^n$ and let $f \in \mathfrak{B}_0^1(\Omega)$.

(1) Suppose that for some $a \in L_f$ there are real analytic curves $\gamma_1, \dots, \gamma_n :]-1, 1[\rightarrow \mathbb{B}$ such that

$$\begin{cases} \gamma_1(0) = \dots = \gamma_n(0) = a \\ a \in \overline{L_f \cap \gamma_j(]-1, 1[) \setminus \{a\}}, \text{ for } j = 1, \dots, n \\ \mathbb{C}^n = \text{span}_{\mathbb{C}}(\gamma_1'(0), \dots, \gamma_n'(0)). \end{cases}$$

Then f is an extreme point of $\mathfrak{B}_0^1(\mathbb{B})$.

(2) Suppose that $f \in \mathfrak{B}_0^1(\mathbb{B})$ and the dimension of the real analytic set L_f is $\dim_{\mathbb{R}} L_f = 2n - 2$. Then one of the of the following holds.

(i) f is an extreme point of $\mathfrak{B}_0^1(\mathbb{B})$.

(ii) L_f contains an $(n - 1)$ -dimensional complex submanifold of $\mathbb{C}^n$.

Remark Let $f \in \mathfrak{B}(\mathbb{B})$ and $\mathbb{L}$ a complex line in $\mathbb{C}^n$. Then the following are equivalent.

(1) L_f contains $\mathbb{B} \cap \mathbb{L}$.

(2) L_f contains $\mathbb{B}(z, r) \cap \mathbb{L}$ for some $z \in \mathbb{L}$ and some $r > 0$.

2. Rigidity properties of the Bloch maximum locus

For $z \in \mathbb{C}^n$ and $r > 0$, let $\Delta(z, r) := \{\lambda z : \lambda \in \Delta, |\lambda| \leq r\}$.

In this section we begin with the following result.

Theorem 2.1 Suppose that $f \in \mathfrak{B}_0^1(\mathbb{B})$ satisfies $|(\nabla f)(0)| = 1$, and let $\nabla f = \sum_{k=0}^{\infty} P_k$, where each $P_k : \mathbb{B} \rightarrow \mathbb{C}^n$ is a holomorphic polynomial mapping with homogeneous components of degree k . If $z \in \mathbb{C}^n$, then the following are equivalent.

(1) L_f contains $\Delta(z, r)$ for all $r > 0$ such that $rz \in \mathbb{B}$.

(2) L_f contains $\Delta(z, r)$ for some $r > 0$.

(3) For each $j \in \mathbb{N}$, we have

$$|P_j(z)|^2 = |z|^{2j} |P_0|^2 + |z \bullet P_{j-1}(z)|^2.$$

Proof That (1) implies (2) is trivial. To prove that (2) implies (3), suppose that L_f contains $\Delta(z, r)$ for some $r > 0$. If $w \in \mathbb{B}$, then

$$(\mathcal{R}(Qf))(w) = \sum_{j,k} [(j - (j+1)|w|^2) \langle P_j(w), P_k(w) \rangle - (j+1 - (j+2)|w|^2) (w \bullet P_j(w)) \overline{(w \bullet P_k(w))}] .$$

Since $(\mathcal{R}(Qf))$ vanishes identically on L_f it follows that

$$\begin{aligned} 0 &= \frac{1}{2\pi} \int_0^{2\pi} (\mathcal{R}(Qf))(re^{i\theta}z) d\theta \\ &= \sum_j [(j - (j+1)|rz|^2) |P_j(rz)|^2 - (j+1 - (j+2)|rz|^2) |rz \bullet P_j(rz)|^2] . \end{aligned}$$

By induction we get that for $j = 1, 2, \dots$,

$$|P_j(z)|^2 = |z|^{2j} |P_0|^2 + |z \bullet P_{j-1}(z)|^2,$$

showing that (3) holds. Finally, if (3) holds, let $r > 0$ such that $rz \in \mathbb{B}$. Then a little computing shows that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} Qf(re^{i\theta}z) d\theta &= \sum_{k=0}^{\infty} |(1 - |rz|^2)[P_k(rz)|^2 - |rz \bullet P_{k-1}(rz)|^2] \\ \text{by (3)} &= (1 - |rz|^2) \sum_{k=0}^{\infty} |rz|^{2k} |P_0|^2 = |P_0|^2 = |(\nabla f)(0)| = 1. \end{aligned}$$

Since $\frac{d\theta}{2\pi}$ is a probability measure and $f \in \mathfrak{B}^1(\mathbb{B})$ it follows that

$$(Qf)(re^{i\theta}z) = 1 \text{ for almost all } \theta,$$

so that by the continuity of Qf we see that

$$(2.1) \quad (Qf)(re^{i\theta}z) = 1, \text{ for all } \theta \in [0, 2\pi].$$

Hence (1) follows. □

Corollary 2.2 Suppose that $f \in \mathfrak{B}_0^1(\mathbb{B})$ satisfies $|(\nabla f)(0)| = 1$, and let $z \in \mathbb{C}^n$. Then $\Delta(z, r)$ is not contained in L_f for any $r > 0$.

Proof Suppose that L_f contains $\Delta(z, r)$, for some $r > 0$, and let r_n be an increasing sequence of positive real numbers such that $|z| \lim_{n \rightarrow \infty} r_n = 1$. By Theorem 2.1 we see that L_f contains $\Delta(z, r_n)$ for any n . By virtue of (2.1) we have that $(Qf)(r_n z) = 1$ for all n , which contradicts that f is a little Bloch function. □

3. Extreme points of the unit ball of the little Bloch space

We begin this section with the following

Proposition 3.1 Let Ω be a bounded domain in $\mathbb{C}^n$ and $f, f_1, f_2 \in \mathfrak{B}^1(\Omega)$ with $f = \frac{1}{2}(f_1 + f_2)$. Then

$$(\nabla f)(z) = (\nabla f_1)(z) = (\nabla f_2)(z), \text{ for all } z \in L_f.$$

If, in addition, L_f is a uniqueness set, then $f = f_1 = f_2$, and thus f is an extreme point in the Bloch ball.

Proof First, it is easy to see that the hypothesis $f, f_1, f_2 \in \mathfrak{B}^1(\Omega)$ and $f = \frac{1}{2}(f_1 + f_2)$ implies that

$$(Qf)(z) = (Qf_1)(z) = (Qf_2)(z), \text{ for all } z \in L_f.$$

This equality implies that $|\tilde{\nabla} f|^2 = |\tilde{\nabla} f_1|^2 = |\tilde{\nabla} f_2|^2$ and hence

$$|\tilde{\nabla}(f_1 - f_2)|^2 = 4 \sum_{i,j=1}^n b^{ij}(z) \frac{\partial}{\partial z_i}(f_1 - f_2) \overline{\frac{\partial}{\partial z_j}(f_1 - f_2)} = 0,$$

for $z \in L_f$. Since $(b^{ij}(z))$ is positive definite, the proposition follows. $\square$

The following result is a real analytic version of the Weierstrass Preparation Theorem. It can be found in [3].

Lemma 3.2 Suppose that $g(x, y, w)$ is a convergent power series where the variables (x, y, w) are in a neighborhood of the origin in $\mathbb{R} \times \mathbb{R} \times \mathbb{C}^{n-1}$. Suppose that $g(0, 0, 0) = 0$ and

$$(3.1) \quad g(0, y, 0) = \sum_{k=l}^{\infty} a_k y^k, \quad l \geq 1, \quad a_l \neq 0.$$

Then there are power series $v(x, y, w)$ with $v(0, 0, 0) \neq 0$, and $u_k(x, w), 0 \leq k \leq l-1$, such that

$$g(x, y, w) = (y^l + u_{l-1}(x, w)y^{l-1} + \cdots + u_0(x, w))v(x, y, w).$$

Lemma 3.3 Let $g(x, y, w)$ be a real valued function of the form

$$(3.2) \quad g(x, y, w) = y^2 + u_1(x, w)y + u_0(x, w),$$

where the variables (x, y, w) are in a neighborhood of the origin in $\mathbb{R} \times \mathbb{R} \times \mathbb{C}^{n-1}$, and $u_1(x, w)$ and $u_0(x, w)$ are real analytic functions in some neighborhood of the origin. Suppose that $g(0, 0, 0) = 0$ and there is $\varepsilon > 0$ such that

$$(3.3) \quad g(x, y, w) > 0 \text{ for } 0 < |x|^2 + |y|^2 + |w|^2 < \varepsilon.$$

Then there are positive integer m and $\delta > 0$ such that

$$(3.4) \quad (|x|^2 + |y|^2 + |w|^2)^m < g(x, y, w) \text{ for } 0 < |x|^2 + |y|^2 + |w|^2 < \delta.$$

Proof Since $g(0, 0, 0) = 0$ we see that $u_0(0, 0) = 0$. Also, by (3.3) we see that $u_1(0, 0) = 0$. Hence u_1 has a power series expansion of the form

$$u_1(x, w) = p_j(x, w) + \sum_{k=j+1}^{\infty} p_k(x, w), \text{ with } j \geq 1.$$

where each p_k is a homogeneous polynomial of degree k .

Writing g in the form

$$g(x, y, w) = \left(y + \frac{u_1(x, w)}{2} \right)^2 + \left(u_0(x, w) - \frac{(u_1(x, w))^2}{4} \right),$$

yields that for $0 < |x|^2 + |y|^2 + |w|^2 < \varepsilon$, we have

$$(3.5) \quad g(x, -\frac{u_1(x, w)}{2}, w) > 0.$$

Therefore, for some positive integer m and some $\delta > 0$ we have

$$u_0(x, w) - \frac{(u_1(x, w))^2}{4} = \sum_{k=2m}^{\infty} q_k(x, w), \quad \text{for } |x|^2 + |w|^2 < \delta,$$

where $m \geq 1$ and q_k are homogeneous polynomials of degree k such that $q_{2m}(x, w) > 0$ for $0 < |x|^2 + |w|^2 < \delta$. By taking δ smaller if necessary we may assume that for some $c > 0$

$$(3.6) \quad \begin{cases} \frac{1}{2}q_{2m}(x, w) \leq u_0(x, w) - \frac{(u_1(x, w))^2}{4} \\ (u_1(x, w))^2 \leq 4(p_j(x, w))^2 \leq 4c(|x|^2 + |w|^2) \end{cases}$$

Set

$$A := \{(x, y, w) : y^2 \leq 4c(|x|^2 + |w|^2), \ 0 < |x|^2 + |w|^2 < \delta\}.$$

From (3.6) we see that

$$(3.7) \quad g(x, y, w) \geq u_0(x, w) - \frac{(u_1(x, w))^2}{4} \geq \frac{1}{2}q_{2m}(x, w).$$

For for $(x, y, w) \in A$ we have

$$(|x|^2 + |y|^2 + |w|^2)^m \leq (1 + 4c^2)(|x|^2 + |w|^2).$$

Thus for some $a > 0$,

$$q_{2m}(x, w) \geq 2(|x|^2 + |y|^2 + |w|^2)^m, \quad \text{for all } (x, y, w) \in A,$$

so that by (3.7) we obtain that

$$(3.8) \quad g(x, y, w) \geq a(|x|^2 + |y|^2 + |w|^2)^m, \quad \text{for all } (x, y, w) \in A.$$

Next, set

$$B := \{(x, y, w) : y^2 \geq 4c(|x|^2 + |w|^2), \ 0 < |x|^2 + |w|^2 < \delta\}.$$

By (3.6) there is $b > 0$

$$(3.9) \quad g(x, y, w) \geq \left(y + \frac{u_1(x, w)}{2}\right)^2 \geq \frac{y^2}{4} \geq b(|x|^2 + |y|^2 + |w|^2)^m,$$

for all $(x, y, w) \in B$. From (3.8) and (3.9) we obtain that for constant constant C

$$(3.10) \quad g(x, y, w) \geq C(|x|^2 + |y|^2 + |w|^2)^m, \quad \text{for } 0 < |x|^2 + |y|^2 + |w|^2 < \delta.$$

By choosing δ smaller and m larger if necessary, we may assume that $C = 1$. This completes the proof. $\square$

Lemma 3.4 *Let $f \in \mathfrak{B}^1(\mathbb{B})$ such that $|\nabla f(0)| = 1$. Then there is a unit vector $\xi \in \mathbb{C}^n$ such that if $z \in \mathbb{C}^n$ has the orthogonal decomposition below*

$$(3.11) \quad z = (x + iy)\xi + w, \quad x, y \in \mathbb{R}, w \in \mathbb{C}^n, \quad \text{with } \langle w, \xi \rangle = 0.$$

and

$$g(x, y, w) := 1 - (Qf)((x + iy)\xi + w),$$

then there are power series $v(x, y, w), u(x, w)$ with $v(0, 0, 0) \neq 0$ and $u(0, 0) \neq 0$, $v_2(x, w)$ and $v_k(w)$, $k = 0, 1$, such that for some $\delta > 0$

$$(3.12) \quad g(x, y, w) = \left(\left(y + \frac{v_2(x, w)}{2} \right)^2 + \left(x + \frac{v_1(w)}{2} \right)^2 + v_0(w)u(x, w) \right) v(x, y, w)$$

and $v_0(w) \geq 0$, $u(x, w) > 0$, and $v(x, y, w) > 0$ for $|z| < \delta$.

Proof As before, we write $\nabla f = \sum_{k=0}^{\infty} P_k$, where each $P_k : \mathbb{B} \rightarrow \mathbb{C}^n$ is a holomorphic polynomial mapping with homogeneous components of degree k . After a unitary change of coordinates, we may assume that $P_0 = e_1 = (1, 0, \dots, 0)$, and write

$$P_j(z) := \begin{pmatrix} P_{j1}(z) \\ \vdots \\ P_{jn}(z) \end{pmatrix}, \text{ for } j \geq 1.$$

Thus

$$(3.13) \quad \begin{cases} \frac{\partial f}{\partial z_1} = 1 + \sum_{m=1}^{\infty} P_{m1}(z) \\ \frac{\partial f}{\partial z_j} = \sum_{m=1}^{\infty} P_{mj}(z) \text{ for } j \geq 2. \end{cases}$$

Moreover, for $z \in \mathbb{B}$ we have

$$\begin{aligned} (Qf)(z) &= (1 - |z|^2) \sum_{j,k=0}^{\infty} \left[\langle P_j(z), P_k(z) \rangle - (z \bullet P_j(z)) \overline{(z \bullet P_k(z))} \right] \\ &= (1 - |z|^2) [1 + 2\operatorname{Re} P_{11}(z) + O(|z|^2)]. \end{aligned}$$

If $P_{11}(\zeta) \neq 0$ for some $\zeta \in S$ then we choose such ζ so that $P_{11}(\zeta) = |P_{11}(\zeta)|$, and thus for $0 < r < 1$

$$\begin{aligned} (Qf)(r\zeta) &= (1 - r^2) [1 + 2rP_{11}(\zeta) + O(r^2)] \\ &= 1 + 2rP_{11}(\zeta) + O(r^2) > 1 \text{ for } r \text{ close to } 0. \end{aligned}$$

This contradicts the fact that $f \in \mathfrak{B}^1(\mathbb{B})$. Therefore, P_{11} vanishes identically on $\mathbb{C}^n$, and thus

$$\frac{\partial^2 f}{\partial z_1 \partial z_j}(0) = 0 \text{ for } j = 1, \dots, n.$$

from which it follows that

$$(3.14) \quad P_1(e_1) = 0.$$

Setting

$$H(z) = |P_1(z)|^2 - |z_1|^2 + 2\operatorname{Re} P_{21}(z), \text{ for } z = (z_1, \dots, z_n) \in \mathbb{C}^n,$$

we see from (3.13) that

$$(Qf)(z) = (1 - |z|^2) [1 + H(z) + O(|z|^3)].$$

If $|H(\zeta)| > 1$ for some $\zeta \in S$ then we choose such ζ so that $H(\zeta) = |H(\zeta)|$, and thus for $0 < r < 1$

$$\begin{aligned} (Qf)(r\zeta) &= (1 - r^2) [1 + r^2 H(\zeta) + O(r^3)] \\ &= 1 + r^2(H(\zeta) - 1) + O(r^3) > 1 \text{ for } r \text{ close to } 0. \end{aligned}$$

This contradicts the fact that $f \in \mathfrak{B}^1(\mathbb{B})$, and hence $|H(z)| \leq |z|^2$, for all $z \in \mathbb{C}^n$.

Next we set

$$(3.15) \quad \begin{cases} P_{21}(e_1) := |P_{21}(e_1)|e^{i\theta}, \text{ with } \theta \in [0, 2\pi] \\ \xi = \xi(t) := e^{\frac{i}{2}(t-\theta)}e_1, \text{ for } t \in [0, 2\pi]. \end{cases}$$

Thus

$$(3.16) \quad P_{21}(\xi) = |P_{21}(e_1)|e^{it}, \text{ for all } t \in [0, 2\pi].$$

Now every $z \in \mathbb{C}^n$ has a unique orthogonal decomposition of the form (3.11), namely,

$$z = (x + iy)\xi + w, \quad x, y \in \mathbb{R}, w \in \mathbb{C}^n, \text{ with } \langle w, \xi \rangle = 0.$$

Using this we define

$$g(x, y, w) := 1 - (Qf)((x + iy)\xi + w).$$

Then we have $g(0, 0, 0) = 1 - |\nabla f(0)|^2 = 0$. We shall show that for some $\alpha > 0$ we

$$g(0, y, 0) = \alpha y^2 + O(y^3).$$

Indeed we have that,

$$\begin{cases} g(0, y, 0) = 1 - (Qf)(iy\xi) \\ = 1 - (1 - y^2) [1 + y^2 H(i\xi) + O(y^3)] . \end{cases}$$

By (3.14) and (3.15) we see that $P_1(\xi) = 0$. Therefore

$$H(i\xi) = -1 - 2|P_{21}(e_1)| \cos t$$

so that

$$(3.17) \quad g(0, y, 0) = 2(1 + |P_{21}(e_1)| \cos t)y^2 + O(y^3) = \alpha y^2 + O(y^3).$$

where

$$(3.18) \quad \alpha = \alpha(t) := 2(1 + |P_{21}(e_1)| \cos t) > 0, \text{ for } t \neq \pi.$$

Applying Lemma 3.2 to g we can find real analytic functions $v(x, y, w)$ with $v(0, 0, 0) \neq 0$, and $u_k(x, w)$, $k = 0, 1$, such that

$$g(x, y, w) = (y^2 + u_1(x, w)y + u_0(x, w)) v(x, y, w).$$

It follows that,

$$(3.19) \quad \begin{cases} v(0, 0, 0) = \lim_{y \rightarrow 0} \frac{g(0, y, 0)}{y^2} = \alpha \\ u_0(x, w) - \frac{(u_1(x, w))^2}{4} \geq 0 \text{ and } u_0(0, 0) = u_1(0, 0) = 0. \end{cases}$$

Choose δ' small enough so that

$$x^2 + |w|^2 + \frac{|u_1(x, w)|^2}{4} < \delta^2, \text{ for } x^2 + |w|^2 < \delta'.$$

For $x^2 + |w|^2 < \delta'$, set

$$g_1(x, w) := g\left(x, -\frac{u_1(x, w)}{2}, w\right) = \left(u_0(x, w) - \frac{(u_1(x, w))^2}{4}\right) v\left(x, -\frac{u_1(x, w)}{2}, w\right).$$

From (3.19) we see that $g_1(0, 0) = 0$ and $g_1(x, w) \geq 0$ for $x^2 + |w|^2 < \delta'$. Since $u_1(0, 0) = 0$, one can write $u_1(x, 0) = 2\beta x + O(x^2)$, for some $\beta \in \mathbb{R}$. Thus,

$$\begin{aligned} g_1(x, 0) &= g\left(x, -\frac{u_1(x, 0)}{2}, 0\right) \\ &= 1 - (Qf)\left[\left(x - \frac{i}{2}u_1(x, 0)\right)\xi\right] \\ &= 1 - (1 - x^2 - \beta^2 x^2) \left[1 + H\left[\left(x - \frac{i}{2}u_1(x, 0)\right)\xi\right] + O(x^3)\right]. \end{aligned}$$

On the other hand,

$$\begin{aligned} &H\left[\left(x - \frac{i}{2}u_1(x, 0)\right)\xi\right] \\ &= -\left(x^2 + \frac{(u_1(x, 0))^2}{4}\right) + 2|P_{21}(e_1)| \operatorname{Re} e^{it} \left(x - \frac{i}{2}u_1(x, 0)\right)^2 \\ &= -\left(x^2 + \frac{(u_1(x, 0))^2}{4}\right) \\ &\quad + 2|P_{21}(e_1)| \left[\cos t \left(x^2 - \frac{1}{4}(u_1(x, 0))^2\right) + x(\sin t)(u_1(x, 0))\right] \\ &= -x^2|P_{21}(e_1)| \left[1 + \beta^2 - 2(1 - \beta^2)\cos t + 4\beta \sin t\right] + O(x^3). \end{aligned}$$

Therefore,

$$\begin{aligned} g_1(x, 0) &= 2x^2 \left[1 - |P_{21}(e_1)| \cos t + \beta^2(1 + |P_{21}(e_1)| \cos t) + 2\beta|P_{21}(e_1)| \sin t\right] \\ &\quad + O(x^3) \end{aligned}$$

If we choose $t > 0$ so that $\beta^2(1 + |P_{21}(e_1)| \cos t) + 2\beta|P_{21}(e_1)| \sin t \geq 0$, then

$$g_1(x, 0) = \nu x^2 + O(x^3),$$

where

$$\nu = \nu(t) := 1 - \cos t + \beta^2(1 + \cos t) + \beta \sin t > 0,$$

so that by Lemma 3.2 there are real analytic functions $v_1(w), v_3(w), v_4(x, w)$ such that

$$g_1(x, w) = (x^2 + v_1(w)x + v_3(w))v_4(x, w), \text{ for } x^2 + |w|^2 < \delta'.$$

Hence for $x^2 + y^2 + |w|^2 < \delta'$, we have

$$g(x, y, w) = \left[\left(y + \frac{1}{2}u_1(x, w) \right)^2 + \left(x + \frac{1}{2}v_1(w) \right)^2 + v_3^2(w) - \frac{v_2^2(w)}{4}v_4(x, w) \right] v(x, y, w).$$

If we set

$$v_2(x, w) := u_1(x, w), \quad u(x, w) := \frac{v_4(x, w)}{v(x, -\frac{u_1(x, w)}{2}, w)}$$

$$v_0(w) := v_3^2(w) - \frac{v_1^2(w)}{4}$$

then the lemma follows. $\square$

If φ_a is the biholomorphic involution of $\mathbb{B}$ which exchanges 0 and a , we denote by $\varrho : \mathbb{B} \times \mathbb{B} \rightarrow [0, 1[$ the pseudo-hyperbolic distance in $\mathbb{B}$. It is defined by

$$\varrho(z, a) := |\varphi_a(z)|, \text{ for } z, a \in \mathbb{B}.$$

It is known [6] that ϱ is distance in $\mathbb{B}$ which defines the topology of $\mathbb{B}$. We denote $\mathbb{B}(a, r)$ the ϱ -ball centered at a with radius r .

Proposition 3.5 Let $f \in \mathfrak{B}(\mathbb{B})$ such that $\|f\|_{\mathfrak{B}} = 1$.

(1) If a is an isolated point in L_f , then there are a positive integer m , and $\delta > 0$ such that

$$(3.20) \quad (Qh)(z) + (\varrho(z, a))^{2m} < 1 \text{ for } 0 < \varrho(z, a) < \delta.$$

(2) If L_f is finite, then f is not an extreme point in $\mathfrak{B}_0^1$.

Proof First we prove part (1). First, we assume $a = 0$. Then the hypothesis of (1) means that $|\nabla f(0)| = 1$ and for some $\varepsilon > 0$

$$(Qf)(z) < 1, \text{ for all } 0 < |z| < \varepsilon.$$

Using the same notations as in Lemma 3.4, we see that the hypothesis in part (1) means that 0 is an isolated point in L_f . Thus, if ε is chosen sufficiently small, then for $0 < x^2 + y^2 + |w|^2 < \varepsilon$, we have $g(x, y, w) > 0$. This fact, combined with Lemmas 3.3 and 3.4, implies that there are a positive integer m and a real number $\delta > 0$ such that for $z = (x + iy)\zeta + w$ we have

$$(3.21) \quad |z|^{2m} < g(x, y, w) = 1 - (Qf)(z) \text{ for } 0 < |z| < \delta.$$

Now if a is an isolated point in L_f , and if φ_a is the biholomorphic involution of $\mathbb{B}$ which exchanges 0 and a then 0 is an isolated point in $L_{(f \circ \varphi_a)}$. By (3.21), one can find $\delta > 0$ and a positive integer m such that

$$(Qh)(z) + (\varrho(z, a))^{2m} < 1 \text{ for } 0 < \varrho(z, a) < \delta.$$

This proves part (1).

To prove part (2), suppose that $L_f = \{a_1, \dots, a_p\}$. Applying (3.20) to each a_j , we see that there are a positive integer m_j and $\delta_j > 0$ such that

$$(3.22) \quad (Qf)(z) + (\varrho(z, a_j))^{2m_j} < 1 \text{ for } 0 < \varrho(z, a_j) < \delta_j.$$

Let $m = \max\{m_j : 1 \leq j \leq p\}$ and for $N \geq 2m$ set

$$\eta(z) := z_1^{2N+1}, \quad h_j(z) := (\eta \circ \varphi_{a_j})(z)$$

$$h(z) := \prod_{j=1}^p h_j(z), \text{ for } z \in \mathbb{B}.$$

Then for $z \in \mathbb{B}$ and $j = 1, \dots, p$,

$$(Q\eta)(z) = (2N+1)|z_1|^{2N}(1-|z|^2)(1-|z_1|^2) \leq (2N+1)|z|^{2N}$$

$$(Qh_j)(z) \leq (2N+1)(1-|\varphi_{a_j}(z)|^2)(\varrho(z, a_j))^{2N} \leq (2N+1)(\varrho(z, a_j))^{2N}$$

from which it follows that $\eta \in \mathfrak{B}_0^1(\mathbb{B})$, and $h_j \in \mathfrak{B}_0^1(\mathbb{B})$ for $j = 1, \dots, p$. On the other hand, for $z \in \mathbb{B}$,

$$\begin{aligned} \sqrt{(Qh)(z)} &\leq \sum_{j=1}^p \sqrt{(Qh_j)(z)} \prod_{k \neq j} |h_k(z)| \\ &\leq \sqrt{2N+1} \sum_{j=1}^p (\varrho(z, a_j))^N \prod_{k \neq j} (\varrho(z, a_k))^{2N+1} \\ &\leq p\sqrt{2N+1} \left[\prod_{j=1}^p \varrho(z, a_j) \right]^N. \end{aligned}$$

This implies that if $\lambda \in \Delta$ and $z \in \mathbb{E}(a_j, \delta_j)$, then

$$\begin{aligned} Q(f + \lambda h)(z) &\leq Q(f)(z) + 2|\lambda|\sqrt{(Qh)(z)} + |\lambda|^2(Qh)(z) \\ &\leq (Qf)(z) + 2p|\lambda|\sqrt{2N+1} \left[\prod_{k=1}^p \varrho(z, a_k) \right]^N \\ &\quad + p^2|\lambda|^2(2N+1) \left[\prod_{k=1}^p \varrho(z, a_k) \right]^{2N} \\ &\leq (Qf)(z) + |\lambda| \left(2p\sqrt{2N+1} + p^2(2N+1) \right) [\varrho(z, a_j)]^N. \end{aligned}$$

This proves that if $z \in \cup_j \mathbb{E}(a_j, \delta_j)$, then by (3.22) we see that

$$(3.23) \quad Q(f + \lambda h)(z) \leq 1 \text{ for } |\lambda| \leq \frac{1}{2p\sqrt{2N+1} + p^2(2N+1)}.$$

On the other hand if we set

$$\begin{cases} M := \sup\{\sqrt{(Qf)(z)} : z \in \mathbb{B} \setminus \cup_j E(a_j, \delta_j)\} \\ M' := \sup\{\sqrt{(Qh)(z)} : z \in \cup_j E(a_j, \delta_j)\} \end{cases}$$

then $M < 1$ and $M' \leq p^2(2N+1)$. Thus for $z \in \mathbb{B} \setminus \cup_j E(a_j, \delta_j)$, we have

$$(3.24) \quad \sqrt{(Q(f + \lambda h))(z)} \leq \sqrt{(Qf)(z)} + |\lambda|\sqrt{(Qh)(z)} \leq M + |\lambda|p^2(2N+1).$$

Combining (3.23) and (3.24) yields that

$$\sup_{z \in \mathbb{B}} (Q(f + \lambda h)(z) \leq 1, \text{ for } |\lambda| \leq \min \left[\frac{1 - M}{p^2(2N + 1)}, \frac{1}{2p\sqrt{2N + 1} + p^2(2N + 1)} \right]$$

which shows that f is not an extreme point. $\square$

4. Proof of Theorems A and B

We shall need the following

Lemma 4.1 Let $f \in \mathfrak{B}^1(\mathbb{B})$. If $a \in L_f$, then there is $\delta > 0$ and a real analytic injective mapping $\psi = \psi_a : \delta\mathbb{B}_{n-1} \rightarrow \mathbb{B}_n$ and a real analytic function $u : \delta\mathbb{B}_{n-1} \rightarrow \mathbb{R}$ such that

$$(4.1) \quad L_f \cap \{z \in \mathbb{B} : \varrho(z, a) < \delta\} = \{\psi(w) : |w| < \delta \text{ and } u(w) = 0\}.$$

Proof If $a = 0$, then the lemma is an immediate consequence of Lemma 3.4. If a is an arbitrary element of L_f , then $0 \in L_{(f \circ \varphi_a)}$, and thus there exists $\delta > 0$ and a real analytic function $\eta : \delta\mathbb{B}_{n-1} \rightarrow \mathbb{B}_n$ and a real analytic function $u : \delta\mathbb{B}_{n-1} \rightarrow \mathbb{R}$ such that

$$L_{(f \circ \varphi_a)} \cap \{z \in \mathbb{B} : |z| < \delta\} = \{(\eta(w), w) : w \in \delta\mathbb{B}_{n-1} \text{ and } u(w) = 0\}.$$

Defining $\psi : \delta\mathbb{B}_{n-1} \rightarrow \mathbb{B}_n$ by

$$\psi(w) := \varphi_a(\eta(w), w), \text{ for } w \in \delta\mathbb{B}_{n-1}.$$

then it is easy to check that (4.1) holds. $\square$

Proof of Theorem A If $f \in \mathfrak{B}_0^1(\mathbb{B})$ then L_f is compact. By Lemma 4.1 there is a finite number of points $a_1, \dots, a_p$ in L_f , positive real numbers $\delta_1, \dots, \delta_p$, real analytic injective mappings $\psi_j : \delta_j\mathbb{B}_{n-1} \rightarrow \mathbb{B}_n$ and real analytic functions $u_j : \delta_j\mathbb{B}_{n-1} \rightarrow \mathbb{R}$, for $j = 1, \dots, p$, such that

$$L_f = \cup_{j=1}^p \{\psi_j(w) : |w| < \delta_j \text{ and } u_j(w) = 0\}.$$

This completes the proof of Theorem A. $\square$

Proof of Theorem B First we prove part (1). Since L_f is a real analytic set the hypothesis (1.1) implies that for some $\epsilon > 0$ we have that $\gamma_j(t) \in L_f$ for $|t| < \epsilon$ for all $j = 1, \dots, n$. We shall prove that L_f is a uniqueness set and by Proposition 3.1 part (1) of Theorem B will follow. The condition

$$\text{Span}_{\mathbb{C}}(\gamma'_1(0), \dots, \gamma'_n(0)) = \mathbb{C}^n$$

implies that dimension of any analytic set which contains L_f is greater than or equal to n , and so L_f is a uniqueness set.

Next, we will prove part (2). In view of Lemma 4.1 there are $\delta > 0$, and a real analytic injective mapping $\psi = \psi_a : \delta\mathbb{B}_{n-1} \rightarrow \mathbb{B}_n$ and a real analytic function $u : \delta\mathbb{B}_{n-1} \rightarrow \mathbb{R}$ such that $\psi(0) = a$ and

$$L_f \cap \{z \in \mathbb{B} : \varrho(z, a) < \delta\} = \{\psi(w) : |w| < \delta \text{ and } u(w) = 0\}.$$

Since $\dim L_f = 2n - 2$ and a is a regular point of L_f it follows that u must vanish identically on $\delta\mathbb{B}_{n-1}$. Hence $\psi(w) \in L_f \cap \mathbb{E}(a, \delta)$ for any $w \in \delta\mathbb{B}_{n-1}$. If f is not an extreme point in the little Bloch ball then there must exist an analytic set $A \subset \mathbb{C}^n$ with codimension 1 that contains L_f . The mapping $\psi : \delta\mathbb{B}_{n-1} \rightarrow A$ has maximum rank near 0. Therefore $\psi(\delta\mathbb{B}_{n-1})$ a neighborhood of a , and thus L_f contains an open subset of the set A^r of regular points of A . This completes the proof of part (2). $\square$

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