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On the Cyclicity of Dilated Systems in Lattices: Multiplicative Sequences, Polynomials, Dirichlet-type Spaces and Algebras

NIKOLAI NIKOLSKI¹

To the memory of Mohamed Zarrabi, an unforgettable colleague and remarkable man

ABSTRACT. The aim of these notes is to discuss the completeness of the dilated systems in a most general framework of an arbitrary sequence lattice X , including weighted ℓ^p spaces. In particular, general multiplicative and completely multiplicative sequences are treated. After the Fourier–Bohr transformation, we deal with the cyclicity property in function spaces on the corresponding infinite dimensional Reinhardt domain $\mathbb{D}_X^\infty$. Functions with (weakly) dominating free term and (in particular) linearly factorable functions are considered. The most attention is paid to the cases of the polydiscs $\mathbb{D}_X^\infty | \mathbb{C}^N = \mathbb{D}^N$ and the ℓ^p -unit balls $\mathbb{D}_X^\infty | \mathbb{C}^N = \mathbb{B}_p^N$, in particular to Dirichlet-type and Dirichlet–Drury–Arveson-type spaces and algebras, as $X = \ell^p(\mathbb{Z}_+^N, (1 + |\alpha|)^s)$, $s = (s_1, s_2, \dots)$ and $X = \ell^p(\mathbb{Z}_+^N, (\frac{|\alpha|!}{|\alpha|!})^t (1 + |\alpha|)^s)$, $s, t \geq 0$, as well as to their infinite variables analogues. We privileged the largest possible scale of spaces and the most elementary instruments used.

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¹ *Université de Bordeaux, Institut de Mathématiques.*

e-mail: nikolski@math.u-bordeaux.fr (Corresponding Author).

CONTENTS

1. Introduction. 10th proof of a Haar Lemma	239
2. Invariant sequence lattices (ISL)	242
3. Totally multiplicative sequences are always cyclic	243
4. The Fourier–Bohr transform in complex domain	244
5. Cyclicity of (relatively) multiplicative sequences	249
6. Beurling’s necessary condition, and a few cases where it is sufficient	253
7. Factorable functions in Dirichlet-type spaces and algebras in polydiscs	260
8. Factorable functions in Dirichlet–Drury–Arveson spaces and algebras in ℓ^p -unit ball	266
9. Brief notes on the history of the periodic Dilation Completeness Problem (DCP).	271
References	272

1. Introduction. 10th proof of a Haar Lemma

Given a Banach space of complex sequences X , the *dilation completeness problem* (DCP) consists in a description of *cyclic elements* $x = (x_k)_{k \geq 1} \in X$ for the dilation semigroup $(D_n)_{n \geq 1}$, i.e. $x \in X$ such that $E_x = X$ where

$$E_x =: \text{span}_\sigma (D_n x : n = 1, 2, 3, \dots).$$

Below, we often suppose that the space X is an “**ISL**” — (Dilation) Invariant Sequence Lattice, i.e. a Banach space of complex sequences such that:

- (i) with every $x \in X$ it contains all sequences $y = (y_n)_{n \geq 1}$ having $|y_n| \leq |x_n|$, $\forall n \in \mathbb{N}$ and $\|y\|_X \leq \|x\|_X$,
- (ii) the dilations D_n

$$D_n \left(\sum_k x_k \delta_k \right) := \sum_k x_k \delta_{nk}$$

are well defined on X , $x \in X \Rightarrow D_n x \in X, \forall n \geq 1$, and

(iii) $\delta_1 := (1, 0, 0, \dots) \in X$ and so all *standard unit vectors* δ_k (and their linear hull X_{00}),

$$\delta_k = (\delta_{k,j})_{j \geq 1} \quad (\delta_{k,j} \text{ being the Kronecker delta}),$$

are contained in X .

We distinguish “normalized” **ISL** (or **N-ISL**) by the property $\|\delta_k\|_X = 1$, $\forall k$. In this case,

$$\ell^1(\mathbb{N}) \subset X \subset \ell^\infty(\mathbb{N}),$$

with the corresponding inequalities for the norms.

The notation $\text{span}_\sigma(\dots)$ means *the closed linear span of a set of vectors* $(\dots)$ in X , closed for the *weak topology* $\sigma(X, X')$ induced by the *dual lattice* X' with respect to the *standard duality*

$$\langle x, y \rangle = \sum_{n \geq 1} x_n y_n,$$

where X' stands for

$$X' := \left\{ y = (y_n) : \sum_{n \geq 1} |x_n y_n| < \infty, \forall x \in X \right\}.$$

Notice that always $X' \subset X^*$ (the Banach dual space), and if the set X_{00} of all finitely supported sequences (the linear span of δ_k , $k = 1, 2, \dots$) is dense in X , then $X' = X^*$. Clearly, if $X' = X^*$ then $\text{span}_\sigma = \text{span}_X$, the norm closed linear hull in X . In any case, the dual norm on X' is defined by $\|y\|_{X^*} = \sup\{|\langle x, y \rangle| : \|x\|_X \leq 1\}$.

As examples of **N-ISL**, we can mention the *rearrangement invariant sequence spaces*, **RIS** (see details below, as well as [31]), and in particular, $\ell^p(\mathbb{N})$ spaces ($1 \leq p \leq \infty$), Orlicz, Lorentz, Marcinkiewicz sequence spaces X . As **ISL**’s, one can consider some weighted lattices X_w ($x \in X_w$ iff $(x_n w_n) \in X$), see below. Quite recently, there were discovered many interesting facts on the DCP related to Dirichlet series and the analytic number theory, mostly on the Hilbert space $X = \ell^2(\mathbb{N})$, see for example [19, 50, 41, 45, 11], as well as the references in Sections 4.4, 5.4, 6.8, 6.10 and 9 below.

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Later on, we return to the general setting described above, but just now, consider the DCP in the space $X = \ell^\infty(\mathbb{N})$ of all bounded complex sequences, with the usual sup norm. The following statement is our departure point for this paper.

1.1. Haar's Lemma. *If $a_k \in \mathbb{C}$, $\sum_{k \geq 1} |a_k| < \infty$ and $\sum_{k \geq 1} a_{nk} = 0$ ($\forall n \geq 1$), then $a_j = 0$, $\forall j \geq 1$.*

It seems that the lemma first appeared in G. Polya and G. Szegő problem book [49], Part I, N°129 with a (rather involved) proof attributed to Alfred Haar (without detailed references). Much later, in [50], p. 294, the proposition is referred to as just “a simple classical lemma” and is proved making use some specific procedures (related to the sieve of Eratosthenes and the classical Möbius inversion formula — see below). In [50], the lemma is then used for the proof of a A. Wintner result quoted in Remark 3.2 below.

Clearly, by the duality, Lemma 1.1 means that *the constant sequence*

$$\mathbf{1} = (1, 1, \dots)$$

is a (D_n) -cyclic vector in $\ell^\infty(\mathbb{N})$ for the weak- $$ topology $\sigma(\ell^\infty, \ell^1) = \sigma(\ell^\infty, (\ell^\infty)')$.*

In fact, historically, similar cyclicity properties in the space ℓ^2 for more general *totally (or, completely) multiplicative sequences* x ,

$$x_{nk} = x_n x_k \quad (\forall n, k \in \mathbb{N})$$

were (re)discovered many times, under various hypotheses and forms, and usually with different proofs. I found 9 such proofs in the literature, starting from the initial Haar's lemma, and then in A. Wintner [60], N. P. Romanov [53], N. I. Ahiezer [1], P. Hartman [18] (the most complete treatment), V. Ya. Kozlov [27], H. Hedenmalm–P. Lindquist–K. Seip [19], N. Nikolski [41] (with [43]), H. Queffélec [50]. Below, a tenth proof of that fact is given in much more general framework, surely the shortest one and completely elementary. In subsequent sections, we similarly prove some more general statements.

Note that a lattice X always contains the set X_{00} of finitely supported sequences $x = \sum_k x_k \delta_k$, i.e. finite linear combinations of the standard unit vectors δ_k . The set X_{00} is dense in X for $\sigma(X, X')$ topology.

1.2. Extended Haar's Lemma. *Let $x = (x_k)_{k \geq 1}$, $x_1 = 1$, be a bounded (complex) totally multiplicative sequence. If $\sum_{k \geq 1} |a_k| < \infty$ and*

$$\sum_{k \geq 1} x_k a_{nk} = 0 \quad (\forall n \geq 1), \text{ then } a_j = 0, \forall j \geq 1.$$

Proof. Let $p_1 = 2, p_2 = 3, \dots, p_s, \dots$ be all integer primes. Setting $x(0) = x$, define $x(s)$, $s = 1, 2, \dots$, recurrently as

$$x(s) = (id - x_{p_s} D_{p_s})x(s-1), \quad s = 1, \dots$$

Obviously, $x(s) \in E_x$ for every $s = 0, 1, \dots$. By multiplicativity, all coordinates of $x(1)$ with indices in $2\mathbb{N}$ are zero, whereas all others are the same as for $x(0)$, and next, all coordinates of $x(2)$ with indices in $3\mathbb{N}$ are zero, whereas all others are the same as for $x(1)$, etc.

Since (clearly) $\|x(s)\|_\infty \leq \|x(s-1)\|_\infty$ and all coordinates of $x(s)$ with indices between 2 and p_s are zero, we obtain $(\sigma^*) \lim_{s \rightarrow \infty} x(s) = \delta_1$, where $\delta_1 = (1, 0, 0, \dots)$ and $\sigma^* = \sigma(\ell^\infty, \ell^1)$ stands for the $*$ -weak topology.

It follows that $\delta_1 \in E_x$, and then $\delta_n = D(n)\delta_1 \in E_x, \forall n$. Since $E_x \perp a$ (in the sense of the introduced duality), $a = 0$. $\square$

Below, we extend Lemma 1.2 to general **ISLs** and to (partially) multiplicative sequences, and list some more observations on the DCP in the D_n -invariant lattices (**ISL**).

2. Invariant sequence lattices (ISL)

As it was mentioned, *in this paper, the spaces* where the semigroup $(D_n)_{n \in \mathbb{N}}$ acts, are the D_n -invariant sequence lattices (**ISL**) defined in points (i)–(iii) of Section 1 above. If in addition, the norm in X is permutation invariant,

$$\|(x_n)\| = \|(x_{\sigma(n)})\| \quad (\text{for every bijection } \sigma : \mathbb{N} \rightarrow \mathbb{N}),$$

we speak on a *rearrangement invariant space* X (**RIS**). The classical examples of **RIS** are $X = \ell^p = \ell^p(\mathbb{N})$, as well as the Orlicz and Lorentz sequence spaces X (see [31]).

Clearly, a **RIS** is a **N-ISL**, a normalized **ISL** as it is defined in Section 1. As further examples of **ISL**, one can consider the corresponding weighted spaces: given a **N-ISL** X , let

$$X_w = \{(x_n) : (x_n w_n) \in X\},$$

where (w_n) is a positive weight satisfying

$$\sup_{k \geq 1} \frac{w_{nk}}{w_k} < \infty, \quad \forall n \in \mathbb{N}.$$

Besides its own norm topology, we consider on X a “weakened topology” $\sigma(X, X')$; it can be different from the standard weak topology $\sigma(X, X^*)$, especially in the case where X' is rather a predual space than the dual one X^* (as for $X = \ell^\infty(\mathbb{N})$). Notice that the subspace X_{00} is always $\sigma(X, X')$ dense in X (indeed, if $y \in X'$ and $\langle \delta_n, y \rangle = 0, \forall n$ then $y = 0$).

We are using the following obvious coordinate-wise meaning of the X' -weak convergence.

2.1. Lemma. Let X be an **ISL**, and $\sigma = \sigma(X, X')$.

(1) (Coordinate dominated convergence) If $x, x^{(n)} \in X$ and if $|x_j^{(n)}| \leq |x_j|$, $\forall n, j$ and $\lim_n x_j^{(n)} = 0$, $\forall j$, then $(\sigma) \lim_n x^{(n)} = 0$.

(2) Assume that the finitely supported sequences X'_{00} are norm dense in X' (for example, X_{00} is dense in X and X is reflexive), and let $(x^{(n)})_{n \geq 1}$ be a sequence in X . Then, $(\sigma) \lim_n x^{(n)} = x$ if and only if $\sup_n \|x^{(n)}\|_X < \infty$ and $\lim_n x_j^{(n)} = x_j$, $\forall j \geq 1$.

Proof. In fact, the proofs of the both claims are classical exercises, the first one with a use of the Lebesgue dominated convergence: $\lim_n \sum_j x_j^{(n)} y_j = \sum_j x_j y_j$. $\square$

Notice that without any hypothesis, the coordinatewise convergence and norm boundedness are not sufficient for the weak $\sigma(X, X')$ convergence (for instance, for $X = \ell^1$).

3. Totally multiplicative sequences are always cyclic

A sequence of complex numbers $x = (x_k)_{k \in \mathbb{N}}$ is *totally multiplicative* if $x_{nk} = x_n x_k$ for all $n, k \in \mathbb{N}$ (so, $x_1 = 1$, if $x \neq 0$); x is called also a semicharacter of $\mathbb{N}$.

3.1. Proposition. Any nonzero semicharacter x , $x \neq 0$, lying in an **ISL** X , is $\sigma(X, X')$ -cyclic for the dilation semigroup (D_n) , i.e. $E_x = X$, where

$$E_x = \text{span}_\sigma(D_n x : n \in \mathbb{N}).$$

(So, in the case when $X' = X^*$, it is norm cyclic).

In particular (an extended form of the above Haar's Lemma), if such a sequence x is in an **ISL** X and $a \in X'$, and

$$\sum_{k \geq 1} a_{nk} x_k = 0 \quad (\forall n \geq 1), \text{ then } a_j = 0, \quad \forall j \geq 1.$$

Proof. We use the sequences $x(s) \in E_x$ considered in the proof of “Extended Haar's Lemma” (1.2 of Section 1),

$$x(s) =: (id - x_{p_s} D_{p_s})x(s-1), \quad s = 1, \dots, x(0) = x.$$

Then, $|x(s)_j| \leq |x_j|$ for every s and j , so that — by Lemma 2.1 — $(\sigma) \lim_s x(s) = \delta_1 \in E_x$, and hence $\delta_n \in E_x$ ($\forall n \geq 1$). But as it was mentioned, $\mathcal{L}in(\delta_n : n \geq 1)$ is $\sigma(X, X')$ -dense in X . $\square$

3.2. Remark. Notice, that for $X = \ell^p$, $1 \leq p < \infty$ and for every reflexive **ISL**, Proposition 3.1 gives D_n -cyclicity also for the norm convergence.

Moreover, if $x \in X$ is $\sigma(X, X')$ -cyclic in a **ISL** X , it is $\sigma(Y, Y')$ -cyclic in any larger **ISL** $Y \supset X$.

As an example to 3.1, a sequence $(k^{-\gamma})_{k \geq 1}$ is dilation-cyclic in any **ISL** X where it is contained. In particular, forestalling a short discussion in Section 9 below, it follows that a function sequence $\varphi(nx)$, $n = 1, 2, \dots$ is complete in $L^2(0, 1)$ if $\varphi = \sum_{k \geq 1} \frac{\sin(k\pi x)}{k^\gamma}$, $\gamma > 1/2$ (A. Wintner [60]; later on, this fact was repeated in many other sources).

Now, we extend our language introducing the standard additive writing for the semigroup $\mathbb{N}$, and recalling the *Fourier–Bohr transform*.

4. The Fourier–Bohr transform in complex domain

First, we recall the language of the dilation semigroup. Consider the set of positive integers $\mathbb{N}$ as a subsemigroup of a (multiplicative) group of positive rationals $\mathbb{Q}_+$ endowed with the discrete topology. The latter is canonically isomorphic to the (additively written) group $\mathbb{Z}(\infty)$ — a subgroup of $\mathbb{Z}^\infty = \mathbb{Z} \times \mathbb{Z} \times \dots$ consisting of finitely supported sequences: $\alpha \in \mathbb{Z}(\infty)$ if and only if $\alpha = (\alpha_j)_{j \geq 1}$, $\alpha_j \in \mathbb{Z}$ and $\text{supp}(\alpha) = \{j \in \mathbb{N} : \alpha_j \neq 0\}$ is finite. The isomorphism is given by the Euclid prime decompositions:

$$r \mapsto \alpha(r) \quad (r \in \mathbb{Q}_+), \text{ where } r = p^{\alpha(r)} := \prod p_j^{\alpha_j},$$

$p = (p_1, p_2, \dots)$ stands for the sequence of the consecutive primes ($p_1 = 2$, $p_2 = 3$, etc.) and $\alpha(r) = (\alpha_1, \alpha_2, \dots)$, $\alpha_j \in \mathbb{Z}$. The α -range of the subsemigroup $\mathbb{N} \subset \mathbb{Q}_+$ is $\alpha(\mathbb{N}) = \mathbb{Z}_+(\infty) =: \mathbb{Z}(\infty) \cap \mathbb{Z}_+^\infty$.

The (compact) dual group of $\mathbb{Z}(\infty)$ (and so of $\mathbb{Q}_+$) is $\mathbb{T}^\infty = \mathbb{T} \times \mathbb{T} \times \dots$ (with the infinite product topology), where the duality is written in a natural way:

$$\{\alpha, \zeta\} = \zeta^\alpha := \prod \zeta_j^{\alpha_j}, \quad \alpha = (\alpha_j) \in \mathbb{Z}(\infty) \text{ and } \zeta = (\zeta_j)_{j \geq 1} \in \mathbb{T}^\infty,$$

and respectively,

$$\{r, \zeta\} = \zeta^{\alpha(r)} := \prod \zeta_j^{\alpha_j}, \quad \alpha(r) = (\alpha_j) \in \mathbb{Z}(\infty), r = p^{\alpha(r)} \text{ and } \zeta \in \mathbb{T}^\infty.$$

The corresponding Fourier (Laplace) transformation is called *the Bohr transform* (or *the Bohr lift*, [8]). For a finitely supported function h on $\mathbb{Q}_+$ (or, respectively, h on $\mathbb{Z}(\infty)$), it is defined as

$$Bh(\zeta) = \sum_{r \in \mathbb{Q}_+} h(r) \zeta^{\alpha(r)}; \text{ resp., } Bh(\zeta) = \sum_{\alpha \in \mathbb{Z}(\infty)} h(\alpha) \zeta^\alpha, \quad \zeta \in \mathbb{T}^\infty.$$

The shift operations for functions on $\mathbb{Q}_+$ and $\mathbb{Z}(\infty)$ are defined, respectively, as

$$D_s h(r) = h(rs^{-1}) \quad (r, s \in \mathbb{Q}_+); \quad S_\beta h(\alpha) = h(\alpha - \beta) \quad (\alpha, \beta \in \mathbb{Z}(\infty)).$$

The Bohr transformation transforms both shifts into the multiplication M_{ζ^α} by the corresponding character, $M_{\zeta^\alpha} f = \zeta^\alpha f(\zeta)$, so that the following lemma holds.

4.1. Lemma.

$$BD_s = M_{\zeta^\alpha} B \quad (\alpha = \alpha(s)) \text{ and } BS_\beta = M_{\zeta^\beta} B. \square$$

4.2. Domain where the Bohr transform exists. For functions having support in $\mathbb{N} \subset \mathbb{Q}_+$, the Bohr transform of $x \in X(\mathbb{N})$ is naturally defined on the corresponding X' -circular domain $\mathbb{D}_{X'}^\infty$,

$$\mathbb{D}_{X'}^\infty = \{ \lambda = (\lambda_s)_{s \geq 1} : (\lambda^{\alpha(n)})_{n \geq 1} \in X(\mathbb{N})' \},$$

or, in an additive writing, for $x \in X(\mathbb{Z}_+(\infty))$,

$$\mathbb{D}_{X'}^\infty = \{ \lambda = (\lambda_s)_{s \geq 1} : (\lambda^\alpha)_{\alpha \in \mathbb{Z}_+(\infty)} \in X(\mathbb{Z}_+(\infty))' \}.$$

Precisely,

$$Bx(\lambda) = \sum_{\alpha \in \mathbb{Z}_+(\infty)} x_\alpha \lambda^\alpha, \quad \lambda \in \mathbb{D}_{X'}^\infty, \quad \forall x \in X(\mathbb{Z}_+(\infty)).$$

Since the above series are absolutely convergent, they represent functions holomorphic on the interior part of the corresponding domain. In fact, since X and X' are lattices, $\lambda = (\lambda_s)_{s \geq 1} \in \mathbb{D}_{X'}^\infty$ and $|\mu_s| \leq |\lambda_s|$ ($\forall s \geq 1$) entail $\mu \in \mathbb{D}_{X'}^\infty$, so that — following the standard complex analysis terminology — $\mathbb{D}_{X'}^\infty$ is a *complete Reinhardt domain* in $\mathbb{C}^\infty$.

We often identify a lattice $X = X(\mathbb{N})$ on $\mathbb{N}$ with its transplantation to $\mathbb{Z}_+(\infty) = \alpha(\mathbb{N})$, always with the help of the above defined map $n \mapsto \alpha(n) = (\alpha_1, \dots, \alpha_s, 0, \dots)$. So, given a sequence space X , we formally should write $X \circ \alpha^{-1}$ for its transplanted image on $\mathbb{Z}_+(\infty)$,

$$X \circ \alpha^{-1} = \{ (x_\alpha) : x_\alpha = x_n, \text{ if } \alpha = \alpha(n) \text{ and } (x_n) \in X \}.$$

For short, omitting α^{-1} , we hope that the writings like $X(\mathbb{Z}_+(\infty)) = X(\mathbb{N})$ (formally, $= X(\mathbb{N}) \circ \alpha$) would not lead to confusions.

For a power series realization, we speak on

$$X_A, \text{ the space of power series } f = \sum_{n \geq 1} \hat{f}(n) z^n,$$

with $(\hat{f}(n))_{n \geq 1} \in X = X(\mathbb{N})$, (A stands for 'analytic'). Respectively,

$$X_A(\mathbb{D}_{X'}^\infty) \text{ denotes the space of power series } f = \sum_{\alpha \in \mathbb{Z}_+(\infty)} \hat{f}(\alpha) \lambda^\alpha,$$

defined (with the absolute convergence) for $\lambda \in \mathbb{D}_{X'}^\infty$, and $(\hat{f}(\alpha)) \in X(\mathbb{Z}_+(\infty)) = X \circ \alpha^{-1}$. In fact, the latter one is the Bohr transform of the former, $X_A(\mathbb{D}_{X'}^\infty) = BX(\mathbb{Z}_+(\infty)) = BX(\mathbb{N})$. The spaces $X_A(\mathbb{D}_{X'}^\infty)$ are equipped with the standard X -norm.

Example. For $X = \ell^p$, $1 \leq p \leq \infty$, it is easy to see that the corresponding domains are *multidisks*:

$$\mathbb{D}_{X'}^\infty = \{ \lambda = (\lambda_s)_{s \geq 1} : |\lambda_s| < 1; \lambda \in \ell^{p'} \} \text{ for } 1 < p \leq \infty,$$

(since, by the Euler product/summation formula, $\lambda \in \ell^{p'}(\mathbb{N}) \Leftrightarrow (\lambda^\alpha)_{\alpha \in \mathbb{Z}_+(\infty)} \in \ell^{p'}(\mathbb{Z}_+(\infty))$), where $\frac{1}{p'} + \frac{1}{p} = 1$, and for $X = \ell^1$

$$\mathbb{D}_{X'}^\infty = \overline{\mathbb{D}}^\infty =: \{ \lambda = (\lambda_s)_{s \geq 1} : |\lambda_s| \leq 1 \}.$$

For short,

$$\mathbb{D}_{(\ell^p)'}^\infty = \mathbb{D}_{p'}^\infty.$$

(For $p = 2$, $\mathbb{D}_2^\infty$ is the *Hilbert multidisc* considered already in 1909 in [21]). Bohr transforms of $X = \ell^p$ are power series with ℓ^p coefficients

$$B\ell^p = \ell_A^p(\mathbb{D}_{p'}^\infty) = \left\{ f : f(\lambda) = \sum_{\alpha \in \mathbb{Z}_+(\infty)} a_\alpha \lambda^\alpha : a \in \ell^p(\mathbb{Z}_+(\infty)); \lambda \in \mathbb{D}_{p'}^\infty \right\}.$$

Clearly, $\mathbb{D}_{p'}^\infty = \mathbb{D}^\infty \cap \ell^{p'}(\mathbb{N})$ and $\mathbb{D}_\infty^\infty = \overline{\mathbb{D}}^\infty$.

More general, for an **N-ISL** X the following inclusions hold

$$\mathbb{D}_1^\infty \subset \mathbb{D}_{X'}^\infty \subset \overline{\mathbb{D}}_\infty^\infty,$$

so that $\mathbb{D}_{X'}^\infty$ is in between of two above multidiscs.

The following is an immediate consequence of Lemma 4.1.

4.3. Lemma. *Let X be an ISL. A subspace $E \subset X$ is $(D_n)_{n \in \mathbb{N}}$ invariant if and only if its Bohr image BE is $M_\zeta =: \{M_{\zeta^\alpha} : \alpha \in \mathbb{Z}(\infty)\}$ -invariant.*

In particular, $x \in X$ is $(D_n)_{n \in \mathbb{N}}$ -cyclic if and only if Bx is M_ζ -cyclic in $BX(\mathbb{D}_{X'}^\infty)$, i.e.

$$BX(\mathbb{D}_{X'}^\infty) = \text{clos} (pBx : p \text{ runs over all polynomials})$$

(clos with respect to the corresponding topology).

4.4. Comments. (1) M_ζ -cyclic functions are also called *weakly invertible*.

(2) In the above notation, a semicharacter (totally multiplicative sequence) $x = (x_n)_{n \geq 1}$, $x_n = \omega^{\alpha(n)}$, lying in $\ell^2(\mathbb{N})$, has as its Bohr transform a *reproducing kernel* $k_{\bar{\omega}} = Bx$ of the Hardy space $\ell_A^2(\mathbb{D}_2^\infty) = H^2(\mathbb{D}_2^\infty)$ at the point $\bar{\omega} = (\bar{\omega}_j)_{j \geq 1} \in \mathbb{D}_2^\infty$ of the Hilbert multidisc $\mathbb{D}_2^\infty$,

$$k_{\bar{\omega}}(\zeta) = \sum_{\alpha \in \mathbb{Z}_+(\infty)} \omega^\alpha \zeta^\alpha = \prod_{j \in \mathbb{N}} \frac{1}{1 - \zeta_j \omega_j}, \quad \zeta \in \mathbb{D}_2^\infty.$$

In this case, Proposition 3.1 is classical, and at least 7 independent proofs are known, as in the first 7 references quoted in Section 1. In fact, the matter is on the cyclicity of some (various from paper to paper) totally multiplicative sequences in the space ℓ^2 ; paper [50] deals with the space ℓ^∞ . All these proofs, excepting [41], essentially use certain number theoretic identities and inequalities, in particular the following Möbius inversion formula which holds for every complex sequences $a = (a_j)_{j \geq 1}$ and $b = (b_j)_{j \geq 1}$:

$$a_n = \sum_{d|n} b_d \quad (n = 1, 2, \dots) \Leftrightarrow b_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_d \quad (n = 1, 2, \dots),$$

where μ stands for the *Möbius function*: $\mu(p) = -1$, $\mu(p^k) = 0$ for $k \geq 2$ and every prime p , and $\mu(p_1^{\alpha_1} \dots p_s^{\alpha_s}) = \mu(p_1^{\alpha_1}) \dots \mu(p_s^{\alpha_s})$ for any integer $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$. In fact, the mentioned Möbius inversion formula has a very simple form in terms of the Bohr transforms:

$$Ba = B\mathbf{1} \cdot Bb \Leftrightarrow Bb = \frac{1}{B\mathbf{1}} \cdot Ba,$$

where $\mathbf{1} = (1, 1, \dots)$, $B\mathbf{1}(\zeta) = \prod_{k \geq 1} \frac{1}{1 - \zeta_k}$ for $\sum_k |\zeta_k| < \infty$, $|\zeta_k| < 1$ ($k \geq 1$).

The tenth proof of the cyclicity given in Sections 1 and 3 is the simplest one — it uses nothing special but the definition of the cyclicity itself and the basic remarks on the duality.

(3) In the Hilbert space case of $\ell^2(\mathbb{N})$, the Parseval theorem says that

$$\|x\|_{\ell^2(\mathbb{N})}^2 = \int_{\mathbb{T}^\infty} |Bx(\zeta)|^2 dm_\infty(\zeta), \quad \forall x \in \ell^2(\mathbb{N}).$$

Moreover, at least for functions f continuous on $\mathbb{T}^\infty$ (for instance, for $f = Bx$, $x \in \ell^1(\mathbb{N})$), the following Bohr's formula holds

$$\int_{\mathbb{T}^\infty} f(\zeta) dm_\infty(\zeta) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(K^t) dt,$$

where $t \mapsto K^t := (K_1^{-it}, K_2^{-it}, \dots) \in \mathbb{T}^\infty$, $t \in \mathbb{R}$, stands for an arbitrary *Kronecker solenoid* on $\mathbb{T}^\infty$; the latter means the following property: the numbers $\log K_j$, $j = 1, 2, \dots$ are *rationally independent* (K_j are real and positive). In particular, for $K_j = p_j$ (consecutive primes) and for $x \in \ell^2(\mathbb{N})$, we deal with *Dirichlet series*

$$Bx(K^t) = \sum_{n \in \mathbb{N}} \frac{x_n}{n^{it}}; \quad t \in \mathbb{R}.$$

Multiplication in $BX(\mathbb{D}_{X'}^\infty)$ (in particular, multiplication of Dirichlet series) corresponds to the so called “Dirichlet convolution” in $X(\mathbb{N})$. The choice $x_n = n^{-\gamma}$, $\gamma > 1/2$ (a completely multiplicative sequence!), gives the *Euler-Riemann ζ -function* on a vertical line, $Bx(K^t) = \sum_{n \in \mathbb{N}} \frac{1}{n^{\gamma+it}}$, $t \in \mathbb{R}$.

(4) Finite dimensional sections $\mathbb{D}_{X'}^J = \mathbb{D}_{X'}^\infty \cap \mathbb{C}^J$ (J is a set of indices). Given a (finite) set of indices $J \subset \mathbb{N}$, let $\mathbb{C}^J = \{\zeta \in \mathbb{C}^\infty : \zeta = (\zeta_j), \text{supp}(\zeta) \subset J\}$. Obviously,

$$\ell_{\|\delta_\alpha\|}^1(\mathbb{Z}_+(\infty)) \subset X \subset \ell_{\|\delta_\alpha\|}^\infty(\mathbb{Z}_+(\infty)),$$

and hence

$$\left\{ \lambda = (\lambda_j) : \sum_{\alpha \in \mathbb{Z}_+(\infty)} \frac{|\lambda^\alpha|}{\|\delta_\alpha\|_X} < \infty \right\} \subset \mathbb{D}_{X'}^\infty \subset \left\{ \lambda = (\lambda_j) : \sup_\alpha \frac{|\lambda^\alpha|}{\|\delta_\alpha\|_X} < \infty \right\}.$$

In particular, if X is an **N-ISL** (as any ℓ^p space), we simply have

$$\{\lambda \in \ell^1 : |\lambda_j| < 1\} \subset \mathbb{D}_{X'}^\infty \subset \{\lambda : |\lambda_j| \leq 1\},$$

and so

$$\mathbb{D}^J \subset \mathbb{D}_{X'}^J \subset \overline{\mathbb{D}}^J \text{ for every finite set } J \subset \mathbb{N}.$$

In general, we can only claim $\lambda \in \mathbb{D}_{X'}^\infty \Rightarrow \sup_{k \in \mathbb{N}} |\lambda_j^k| / \|\delta_{ke_j}\|_X < \infty$, where $e_j = (\delta_{j,k})_{k \geq 1} \in \mathbb{Z}_+(\infty)$, so that

$$|\lambda_j| \leq \varliminf_k \|\delta_{ke_j}\|_X^{1/k} =: R_j < \infty \quad (\forall j).$$

Sometimes, the latter necessary condition for $\lambda \in \mathbb{D}_{X'}^\infty$ is not far from being sufficient, as for example for $X = \ell_{w_n}^p(\mathbb{N})$ with a totally multiplicative weight $w_n = n^\gamma$, $n \geq 1$ (in this case, $R_j = p_j^\gamma$, p_j being the j -th consecutive prime). For some other examples this is not the case. For example, for an important space $X = \ell_{w_\alpha}^2(\mathbb{Z}_+(\infty))$ with

$$w_\alpha^2 = \frac{\alpha!}{|\alpha|!} = \frac{\alpha_1! \alpha_2! \dots \alpha_s!}{(\alpha_1 + \alpha_2 + \dots + \alpha_s)!}, \quad \alpha \in \mathbb{Z}_+(\infty),$$

$R_j = 1$ for every j , but $\mathbb{D}_{X'}^J$ are rather far from the polydiscs $\mathbb{D}^J$; in fact, $\mathbb{D}_{X'}^J$ are the *open unit balls* $\mathbb{B}^J = \{\lambda = (\lambda_j)_{j \in J} : \sum_j |\lambda_j|^2 < 1\}$ (see Lemma 8.1). The Bohr transform of X is known as the *Drury-Arveson space*, important

for several variables interpolation theory and several operators theory (the row contractions), see for example [3, 51], and references therein. We return to this space in Sections 6 and 8.

4.5. Multipliers. Given an ISL $X = X(\mathbb{N})$, a (formal) power series $F = \sum_{\alpha \in \mathbb{Z}_+(\infty)} \hat{F}(\alpha) \zeta^\alpha$ in $\mathbb{D}_{X'}^\infty$ is called *multiplier* (or *X-multiplier*) if

$$f \in X_A(\mathbb{D}_{X'}^\infty) \Rightarrow Ff \in X_A(\mathbb{D}_{X'}^\infty),$$

where $X_A(\mathbb{D}_{X'}^\infty) = BX(\mathbb{D}_{X'}^\infty)$. The set of multipliers, denoted $Mult(X_A) = Mult(X_A(\mathbb{D}_{X'}^\infty))$ and endowed with the multiplier norm and pointwise multiplication, is a Banach algebra containing polynomials and such that $Mult(X_A(\mathbb{D}_{X'}^\infty)) \subset X_A(\mathbb{D}_{X'}^\infty) \cap H^\infty(\mathbb{D}_{X'}^\infty)$. It is clear that

$$\sum_{\alpha \in \mathbb{Z}_+(\infty)} |\hat{F}(\alpha)| \cdot \|\zeta^\alpha\|_{Mult} < \infty \Rightarrow F \in Mult(X_A),$$

and $F \in Mult(X_A) \Rightarrow \sup_\alpha |\hat{F}(\alpha)| \cdot \|\zeta^\alpha\|_{Mult} < \infty$. Indeed, the latter property follows from the rotation invariance of $Mult(X_A)$: if $F_t(\zeta) = F(t\zeta)$, where $t\zeta = (t_i \zeta_i)_{i \geq 1}$, $t_i \in \mathbb{T}$, then $F_t f = (F f_t)_t$ and hence $\|F_t\|_{Mult} = \|F\|_{Mult}$,

$$\|\hat{F}(\alpha) \zeta^\alpha\|_{Mult} = \left\| \int_{\mathbb{T}^\infty} \bar{t}^\alpha F_t dm_\infty(t) \right\|_{Mult} \leq \|F\|_{Mult}.$$

It follows, in particular, that

$$\ell_A^1(\mathbb{D}_{X'}^\infty) \subset Mult(X_A(\mathbb{D}_{X'}^\infty)) \Leftrightarrow \sup_\alpha \|\zeta^\alpha\|_{Mult} = \sup_{n \geq 1} \|D_n\|_{X \rightarrow X} < \infty.$$

The following lemma is obvious.

4.6. Lemma. *Let X be an ISL, and suppose $f, g \in X_A$ are cyclic and at least one of them is in $Mult(X_A)$. Then $F = fg$ is cyclic.*

Proof. Suppose $g \in Mult(X_A)$, and let (p_n) be polynomials such that $\lim_n \|p_n f - 1\|_X = 0$. Then $\lim_n p_n F = g \in E_F$, and since g is cyclic, $E_F = X_A$. $\square$

5. Cyclicity of (relatively) multiplicative sequences

A sequence $x = (x_k)_{k \in \mathbb{N}}$ (of complex numbers) is called *(relatively) multiplicative* if $x_{nk} = x_n x_k$ for coprime n and k : $\text{GCD}(n, k) = 1$. For example, the Möbius function μ is multiplicative (but not totally multiplicative). A long list of arithmetic multiplicative functions can be found in [34].

5.1. Lemma. (1) A sequence $x = (x_k)_{k \in \mathbb{N}}$ is multiplicative if and only if, for every $N \in \mathbb{N}$, its Bohr transform Bx is of the form

$$Bx(\zeta) = f_1(\zeta_1) \dots f_N(\zeta_N) F_N(\zeta), \quad \zeta = (\zeta_1, \dots),$$

where f_j are functions (formal power series) in one variable and F_N depends on $\zeta_{N+1}, \zeta_{N+2}, \dots$ only. The representation is unique.

(2) A sequence $x = (x_k)_{k \in \mathbb{N}}$ with finite support is multiplicative if and only if, for a certain N and some polynomials f_j ,

$$Bx(\zeta) = f_1(\zeta_1) \dots f_N(\zeta_N), \quad \zeta = (\zeta_1, \dots).$$

Any function of this Lemma, say f , satisfies $\hat{f}(0) = 1$, $0 = (0, 0, \dots)$.

Proof. (1) If x is multiplicative and $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, then $x_n = a_1(\alpha_1) \dots a_s(\alpha_s)$ where $a_s = (a_s(j))_{j \geq 1}$, $a_s(j) = x_{p_s^j}$. Making the Euler partial summation

$$Bx(\zeta) = \sum_{\alpha_1 \geq 0} a_1(\alpha_1) \zeta_1^{\alpha_1} \sum_{\alpha_2 \geq 0} a_2(\alpha_2) \zeta_2^{\alpha_2} \dots \sum_{\alpha_{N+1} \geq 0} a_{N+1}(\alpha_{N+1}) \zeta_{N+1}^{\alpha_{N+1}} \dots,$$

we obtain the claimed expression. The converse is also obvious.

(2) This is an automatic consequence of (1) since Bx depends on a finite family of variables only. $\square$

Below, we make use the following *restriction operation* (and the corresponding notation): given a set of indices $\sigma \subset \mathbb{N}$, an **ISL** X , and $x \in X$, define $R_\sigma x$ by

$$B(R_\sigma x) = \sum_{\alpha, \text{supp}(\alpha) \subset \sigma} \hat{B}x(\alpha) \zeta^\alpha,$$

or equivalently, $(R_\sigma x)_n = x_n$ if $\text{supp}(\alpha(n)) \subset \sigma$, and $(R_\sigma x)_n = 0$ for any other $n \in \mathbb{N}$. We also are using the notation

$$R_\sigma F = B(R_\sigma x) \text{ for } F = Bx.$$

Clearly, $R_\sigma x \in X$ and $\|R_\sigma x\| \leq \|x\|$. If a sequence $\zeta = (\zeta_j)_{j \in \mathbb{N}}$ is considered as a function on $\mathbb{N}$ and χ_σ stands for the indicator (characteristic) function of σ , then $R_\sigma F(\zeta) = F(\chi_\sigma \zeta)$ where $F = Bx$.

The norm $\|\cdot\|_X$ is called (relatively) *multiplicative norm* if $\|fg\|_X = \|f\|_X \|g\|_X$ for $f \in R_\sigma X$, $g \in R_{\sigma'} X$ with $\sigma \cap \sigma' = \emptyset$ (for sequences, the product means a convolution on $\mathbb{Z}_+(\infty)$). One can easily check the following lemma.

5.2. Lemma. Let X be an **ISL**. (1) If $x \in X$ is a multiplicative sequence, then the functions f_j ($j \in \mathbb{N}$) and F_N from Lemma 5.1 satisfy $f_j = B(R_{\{j\}} x)$ and $F_N = B(R_\sigma x)$, where $\sigma = \{N+1, N+2, \dots\}$.

(2) For $X = \ell^p$, $1 \leq p \leq \infty$, the norm $\|\cdot\|_X$ is multiplicative; the same property holds for the weighted space $X = \ell_w^p$ where $w = (w_n)_{n \geq 1}$ is a (relatively) multiplicative weight.

The following Proposition partially reduces the cyclicity of a multiplicative sequence to the cyclicity of its one-variable factors f_j (in the sense of Lemma 5.1).

5.3. Proposition. *Let X be an ISL. (1) If $x \in X$ is D_n -cyclic then $R_\sigma x$ is D_n -cyclic in $R_\sigma X$ for every $\sigma \subset \mathbb{N}$, $\sigma \neq \emptyset$.*

In particular, if x is cyclic and multiplicative, the functions f_j of Lemma 5.1 are z -cyclic in

$$X_j(\mathbb{D}) =: BR_{\{j\}}X = \left\{ f = \sum_{k \geq 0} a_k z^k : a_k = y_{p_j^k}, y \in X \right\},$$

X_j is equipped with the norm induced from X .

(2) Assume the norm $\|\cdot\|_X$ is multiplicative. Then the converse to (1) holds: if $x \in X$ is multiplicative and every f_j of Lemma 5.1 is z -cyclic in X_j then x is D_n -cyclic in X .

(3) If $x \in X$ is multiplicative and all f_j and F_N of Lemma 5.1 are in $\text{Mult}(X)$ (see 4.5 above), then the converse to (1) holds: z -cyclicity of f_j in X_j implies the D_n -cyclicity of x in X .

Proof. (1) The restriction map R_σ is multiplicative, $R_\sigma(FQ) = R_\sigma(F)R_\sigma(Q)$, and if Q is a polynomial in ζ , $R_\sigma Q$ is a polynomial in ζ_j , $j \in \sigma$. If polynomials Q_n are such that $\lim_n \|FQ_n - 1\|_X = 0$, $F = Bx$, then $\lim_n \|R_\sigma(F)R_\sigma(Q_n) - 1\|_X = 0$, and the principal claim follows.

Since $f_j = R_{\{j\}}F$, $F = Bx$, the stated cyclicity of f_j is also proved.

(2) In the notation of Lemma 5.1, $F(\zeta) = f_1(\zeta_1) \cdots f_N(\zeta_N)F_N(\zeta)$, $F = Bx$, let $(p_{1,n})$ be a sequence of polynomials in ζ_1 satisfying $\lim_n \|p_{1,n}f_1 - 1\|_X = 0$. Then

$$\lim_n \left\| p_{1,n}F - \left(\prod_2^N f_j \right) F_N \right\|_X = \lim_n \|p_{1,n}f_1 - 1\|_X \left\| \left(\prod_2^N f_j \right) F_N \right\|_X = 0,$$

and so $(\prod_2^N f_j)F_N \in BE_x$. Repeating with other variables, we obtain $(\prod_m^N f_j)F_N \in BE_x$ for all $m \leq N$, and hence $F_N \in BE_x$ for all N .

Moreover, since $F_N = BR_\sigma x$, $\sigma = \{N+1, N+2, \dots\}$ and X is a lattice, we have $\|F\|_X \geq \|F_N\|_X$. But all Fourier–Taylor coefficients $\hat{F}_N(\alpha)$ of F_N , excepting $\hat{F}_N(0) = 1$, become zero starting from a certain N , so, by Lemma 2.1(1), $(\sigma) \lim_N F_N = 1 \in BE_x$. This implies the cyclicity of x : $BE_x = BX$.

(3) The same reasoning as in (2), but with a use of the multiplier norm, gives

$$\lim_n \left\| p_{1,n} F - \left(\prod_2^N f_j \right) F_N \right\|_X \leq \lim_n \|p_{1,n} f_1 - 1\|_X \left\| \left(\prod_2^N f_j \right) F_N \right\|_{Mult(X)} = 0.$$

This implies $F_N \in E_F$. The weak convergence $(\sigma) \lim_N F_N = 1$ follows by the same reason as in (2) above. $\square$

5.4. Comments. (1) Historically, the case $X = \ell^2$ was considered earlier at least twice. In this case $X_j(\mathbb{D}) = H^2(\mathbb{D})$ (the standard Hardy space in the disc $\mathbb{D}$) for every j , the norm of X is multiplicative in the sense of above definition, and $Mult(\ell^2) = H^\infty(\mathbb{D})$ (the algebra of bounded holomorphic functions in $\mathbb{D}$). The main result of [18] shows the equivalence of D_n -cyclicity of multiplicative sequences to the cyclicity of functions f_j in $H^2(\mathbb{D})$, but in a slightly weaker form than in points (2)–(3) above, assuming additionally the Wiener algebra condition $f_j, F_N \in \ell_A^1$. Our proof of Proposition 5.3 is much less involved than the Hartman's one. Paper [27] claims the discussed equivalence for $X = \ell^2$ in full generality (as it follows from 5.3(2)), but the paper contains no proofs (and since then, they never appeared elsewhere; the paper also contains a number of misprints).

(2) Proposition 5.3 gives yet another proof to 3.1, the eleventh one... (but under the hypotheses of 5.3(2) on the space X , or the hypotheses of 5.3(3) on the sequence x): indeed, for completely multiplicative $x \in X$, the functions f_j are just rational fractions $f_j(z) = \frac{1}{1-\lambda z}$ whose cyclicity in any **ISL** is immediate: $(1 - \lambda z)f_j = 1$.

(3) *On the cyclicity property of functions of one variable in $\mathbb{D}$* , such as functions f_j above. In general, the property is still not yet well studied for many lattices X . The classical lattice cases are the following:

— *the Hardy space $X = H^2$* (f is cyclic if and only if it is (Beurling) outer);

— *the Wiener algebra $X = \ell_A^1(\mathbb{D})$* (where f is cyclic if and only if it is invertible, i.e. $f(z) \neq 0, \forall z \in \overline{\mathbb{D}}$). Of course, the latter description is still true for any function Banach algebra for which $\overline{\mathbb{D}}$ is the space of its maximal ideals.

In principle, it is not known whether there exists a lattice (or just a Banach space) X of holomorphic functions such that

i) $f \in X \Rightarrow zf \in X$,

ii) the disc $\mathbb{D}$ is its natural definition domain, i.e.,

$$\mathbb{D} = \{\lambda \in \mathbb{C} : f \mapsto f(\lambda) \text{ is bounded on } X\},$$

and

iii) a function $f \in X$ is z -cyclic in X if and only if $f(z) \neq 0$ ($\forall z \in \mathbb{D}$). The problem was discussed, for example, already in [39].

For some particular lattices $X_A(\mathbb{D})$, there exists quite extensive literature on the cyclicity problem. Without any attempt to give an account on the known facts, let us mention a line of research started from A. Beurling paper [7]. Namely, following Beurling's ideas, the following problem is considered in [38]: given a class of holomorphic functions in $\mathbb{D}$, to find a "minimal" Banach space $X_A(\mathbb{D})$ where they are cyclic (weakly invertible). It is shown that given two "regular" (log-convex, and more) weights $0 < w_n \leq W_n$, $n \geq 0$, a necessary and sufficient condition for the cyclicity in $\ell_A^p(w_n)$ of every $f \in \ell_A^p(W_n)$, $f(z) \neq 0$ ($\forall z \in \mathbb{D}$) is $\sum_{n \geq 1} n^{-3/2} \log \frac{W_n}{w_n} = \infty$. For more recent results see [9].

Many deep *bi-cyclicity* results for "Bergman-type" and "Dirichlet-type" spaces $\ell^p(\mathbb{Z}, w_n)$ with power-like weights are classical (A. Beurling, L. Carleson, R. Salem, D. Newman, L. Brown, A. Shields, ...), many others are obtained quite recently. They deal with the size of the boundary zero sets of functions measured with capacities, Hausdorff dimensions and h -Hausdorff measures. Just for giving some sample examples and sources for further references, we mention [10, 15, 9, 30, 29] (the latter text contains also a comprehensive description of a large part of previous results).

Some additional information on (simple) cyclicity of polynomials (and not only) and a general approach to the question can be found in Subsection 6.10(2) below.

6. Beurling's necessary condition, and a few cases where it is sufficient

Let X be an **ISL**. Following the definition of $\mathbb{D}_X^\infty$, the functional $x \mapsto Bx(\lambda)$ is linear and continuous on X for every $\lambda \in \mathbb{D}_X^\infty$, so that the following lemma is obvious. For the case $X = \ell^2$, the lemma is usually called 'Beurling's lemma', [6]; in fact, it was known in the 1940s independently to many: to A. Wintner, A. Beurling, Ph. Hartman, D. Bourgin, V. Ya. Kozlov...

6.1. Lemma. *If $x \in X$ is a D_n -cyclic element, then $Bx(\lambda) \neq 0$ for every $\lambda \in \mathbb{D}_X^\infty$.*

Even if the converse to the property of Lemma 6.1 is probably never true in the Banach space setting (see a remark in 5.4(3)), for some special elements $x \in X$, it however can be valid. Below, we show a few examples of such x 's in case of some **ISLs** X of interest.

6.2. D_n -cyclicity in $\ell^1(\mathbb{N}) = \ell^1(\mathbb{Z}_+(\infty))$. It is well-known and easy to check that $\ell_A^1(\overline{\mathbb{D}}^\infty) = B\ell^1(\mathbb{N})$ is a Banach algebra (with the usual pointwise multiplication, which corresponds to the convolution of sequences on the additive semigroup $\mathbb{Z}_+(\infty)$). Its maximal ideals are defined by the point evaluation functionals on the closed infinite dimensional polydisc: $f \mapsto f(\lambda)$, $\lambda \in \overline{\mathbb{D}}^\infty = \mathbb{D}_{(\ell^1)}^\infty$. This implies the following fact.

6.3. Lemma. *An element $x \in \ell^1(\mathbb{N})$ is D_n -cyclic if and only if $Bx(\lambda) \neq 0$, $\forall \lambda \in \overline{\mathbb{D}}^\infty$.*

Proof. The necessity is already mentioned above. For the sufficiency, assume $f := Bx(\lambda) \neq 0$, $\forall \lambda \in \overline{\mathbb{D}}^\infty$. Then $1/f \in \ell_A^1(\overline{\mathbb{D}}^\infty)$, and so $1/f$ is the norm limit in $\ell_A^1(\overline{\mathbb{D}}^\infty)$ of a sequence of polynomials (p_n) . Since $p_n f \in E_{Bx}$, we have $1 = \lim_n (p_n f) \in E_{Bx}$, and so $E_{Bx} = \ell_A^1(\overline{\mathbb{D}}^\infty)$. $\square$

6.4. Free term dominating functions are cyclic. Lemma 6.3 implies that a function $f \in \ell_A^1(\overline{\mathbb{D}}^\infty)$ having $|\hat{f}(0)| > \sum_{\alpha \in \mathbb{Z}_+(\infty), \alpha \neq 0} |\hat{f}(\alpha)|$ is cyclic in $\ell_A^1(\overline{\mathbb{D}}^\infty)$, and so in any **ISL** containing this algebra. Considering cyclicity for the weakened topology $\sigma(\ell^1, c_0)$, where $c_0 = c_0(\mathbb{Z}_+(\infty))$ stands for the predual space of sequences tending to zero at infinity, $(c_0)^* = \ell^1$, or the cyclicity in larger **ISL** $X \supset \ell^1$ satisfying the conditions of Lemma 2.1(2), we can include the equality case in the domination condition.

6.5. Proposition. *The following “free term domination condition”*

$$|\hat{f}(0)| \geq \sum_{\alpha \in \mathbb{Z}_+(\infty), \alpha \neq 0} |\hat{f}(\alpha)|, \quad f \neq 0,$$

*implies the D_n - $\sigma(X, X')$ -cyclicity of $f \neq 0$ in any **ISL** X from Lemma 2.1(2) containing $\ell^1(\mathbb{Z}_+(\infty))$ (and the D_n - $\sigma(\ell^1, c_0)$ -cyclicity in ℓ^1 itself).*

In particular, the conclusion holds for any polynomial f with a dominating free term.

Proof. Multiplying by a constant, we can always think that $\hat{f}(0) = 1$. Let $0 < r < 1$ and $f_r = 1 + r(f - 1)$, $0 < r < 1$. Then $1/f_r \in \ell_A^1(\overline{\mathbb{D}}^\infty)$, and

$$\frac{f}{f_r} = 1 + \frac{f - f_r}{f_r}.$$

Moreover, $\|f - f_r\|_1 = (1 - r) \sum_{\alpha \in \mathbb{Z}_+(\infty), \alpha \neq 0} |\hat{f}(\alpha)| \leq 1 - r$, and

$$f_r = 1 + r(f - 1), \text{ where } \|r(f - 1)\|_1 \leq r < 1.$$

Hence,

$$\|1/f_r\|_1 \leq \frac{1}{1 - r}, \text{ and } \left\| \frac{f}{f_r} \right\|_1 \leq 1 + \left\| (f - f_r) \frac{1}{f_r} \right\|_1 \leq 2 \quad (0 < r < 1).$$

Moreover, $f/f_r \in E_f^{\ell^1} \subset E_f^X$, and by Lemma 2.1 $1 = (\sigma) \lim_{r \rightarrow 1} \frac{f}{f_r} \in E_f^X$ (and the same for $\sigma(\ell^1, c_0)$ convergence in ℓ^1), and so $E_f^X = X$, and f is $\sigma(X, X')$ -cyclic. $\square$

6.6. Comments. (1) As we will see, for a special class of polynomials in $\mathbb{C}^N$ (factorable polynomials of theorem 6.9 below), the cyclicity holds under the only condition $f(\zeta) \neq 0$, $\zeta \in \mathbb{D}^N$ in any **ISL** X_A containing ℓ_A^1 and satisfying conditions of Lemma 2.1(2). Since every polynomial in one variable is linearly factorable, we keep that claim for z -cyclicity of any polynomial in this kind **ISL**'s in $\mathbb{D}$.

As to generic polynomials in several variables, the situation is more involved; we discuss it briefly in Comments 6.10 below.

(2) Proposition 6.5 is applied to ℓ_A^p , $1 < p < \infty$, and to any reflexive **ISL** $X_A \supset \ell_A^1$ where X_{00} is dense. For $p = \infty$ and $p = 1$, it concerns the cyclicity for weak* topologies $\sigma(\ell^\infty, \ell^1)$, resp. $\sigma(\ell^1, c_0)$ (but not $\sigma(\ell^1, \ell^\infty)$; look on $f = 1 - z$).

6.7. Linearly factorable polynomials are cyclic if $\mathbb{D}_{X'}^N = \mathbb{D}^N$. Whatever is an **ISL** X , it contains the linear subset X_{00} consisting of all finitely supported sequences, and its Bohr transform BX_{00} is the set of all polynomials in $\mathbb{C}^\infty$. Every polynomial $p = Bx$, $x \in X_{00}$, $p(\zeta) = p(\zeta_1, \zeta_2, \dots) = \sum_{n=1}^N x_n \zeta^{\alpha(n)}$, depends on a finite number of variables ζ_j , and one can consider the minimal set of such essential variables

$$J(x) = \bigcup_{n, x_n \neq 0} \text{supp}(\alpha(n)).$$

Given a subset of indices $J \subset \mathbb{N}$, consider the restriction BX^J of the **ISL** BX to the variables ζ_j , $j \in J$, namely,

$$BX^J = \left\{ f \in BX : f = \sum_{\alpha \in \mathbb{Z}_+(\infty)} \hat{f}(\alpha) \zeta^\alpha, \text{supp}(\hat{f}(\alpha)) \subset (\alpha \in \mathbb{Z}_+(\infty) : \text{supp}(\alpha) \subset J) \right\}.$$

It was observed in [41], that for $X = \ell^2$, a polynomial Bx is cyclic in $B\ell^2$ (which is the Hardy space on $\mathbb{D}_2^\infty$) if and only if it is (already) cyclic in $(B\ell^2)^J$, where J is any (finite) subset of $\mathbb{N}$, $J(x) \subset J$; if $J = [1, N]$, $(B\ell^2)^J = H^2(\mathbb{D}^N)$. The same reasoning as in [41] shows the following lemma.

6.8. Lemma. *A polynomial Bx is M_ζ cyclic in BX if and only if it is (already) cyclic in $(BX)^J$, where J is any (finite) subset of $\mathbb{N}$, $J(x) \subset J$.*

Applying Lemma 6.8 with Lemma 6.1, we see that if a polynomial $p = Bx$ is cyclic in $(BX)^J$, then $p(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_{X'}^J$, where

$$\mathbb{D}_{X'}^J := \mathbb{D}_{X'}^\infty \cap \mathbb{C}^J.$$

If X is an **N-ISL**, for every finite set J , $\mathbb{D}^J \subset \mathbb{D}_{X'}^J \subset \overline{\mathbb{D}}^J$ (but for a weighted X_{w_n} these finite dimensional sections can be different from the polydiscs).

A “polynomial converse” to above Beurling’s necessary condition 6.1 claims that, given a polynomial $p = Bx$, $(Bx(\lambda) \neq 0, \forall \lambda \in \mathbb{D}_{X'}^J) \Rightarrow (x \text{ is } D_n\text{-cyclic})$. Historically, this property was first claimed for the Hilbert space $X = \ell^2(\mathbb{N})$ (for which $\mathbb{D}_{X'}^{[1,N]} = \mathbb{D}^N$) by V. Ya. Kozlov in [26]; however, the first published proof appeared only 20 years later, in [35] of J. Neuirth, J. Ginsberg, and D. Newman. For a simplified proof and generalizations, see [41], [45] (also, [44] for details of Kozlov’s personal story).

It follows (of course) that the condition $p(\lambda) \neq 0, \forall \lambda \in \mathbb{D}^J$ is still necessary and sufficient for cyclicity in every larger **ISL** $X_A \supset \ell_A^2$.

In the next Proposition, we consider **ISL**’s X containing ℓ^1 , so that

$$\mathbb{D}_{X'}^{[1,N]} \subset \overline{\mathbb{D}}^N, \quad \forall N \geq 1,$$

and will restrict ourselves to **ISL**’s $X = X(\mathbb{Z}_+^N)$ on $\mathbb{Z}_+^N$. The Bohr transforms Bx of finitely supported sequences $x \in X_{00}$ (i.e., polynomials), in principle, are defined on the whole $\mathbb{C}^N$; as above we use the notation

$$X_A(\mathbb{D}^N) := BX(\mathbb{Z}_+^N).$$

A polynomial f in $\mathbb{C}^N$ is said *linearly factorable* if

$$f = \prod_{j=1}^n f_j,$$

where f_j are affine linear functions, $f_j(\zeta) = f_j(\zeta_1, \dots, \zeta_N) := a + \sum_{i=1}^N a_i \zeta_i$ ($a, a_i \in \mathbb{C}$). If f is linearly factorable then the zero set of f in $\mathbb{C}^N$ is a finite union of (complex) hyperplanes. For $N = 1$, every polynomial is linearly factorable, but for $N \geq 2$ this is already not the case (for example, $f(\zeta) = h(\zeta_1, \dots, \zeta_{N-1}) + \lambda \zeta_N$ is not factorable, if $\lambda \neq 0$, h a polynomial of degree ≥ 2).

6.9. Proposition. *Let $X(\mathbb{Z}_+^N)$ be an **ISL** containing $\ell^1(\mathbb{Z}_+^N)$, and f a linearly factorable polynomial in $\mathbb{C}^N$, such that $f(\zeta) \neq 0$ for $\zeta \in \mathbb{D}^N$. Then f is cyclic in $X_A(\mathbb{D}^N)$ (for weak $\sigma(X, X')$ topology).*

*In particular, for $N = 1$, every polynomial with $f(\zeta) \neq 0, \zeta \in \mathbb{D}$ is cyclic in any **ISL** of the above type (for example, it is norm cyclic in any $\ell_A^p(\mathbb{D})$, $1 < p < \infty$, and, generally, in any reflexive **ISL** containing ℓ^1 (think on $X = \ell^p(\ell^q)$ type spaces)).*

Proof. Let $f = \prod_{j=1}^n f_j$, where f_j are affine linear in $\mathbb{C}^N$. With Lemma 4.6, it suffices to show that every f_j is cyclic in X_A . Since f_j is an affine linear

polynomial, $f_j(\zeta) = a + \sum_{i=1}^N a_i \zeta_i$, and $f_j(\zeta) \neq 0$ for $\zeta \in \mathbb{D}^N$, it follows $|a| \geq \sum_{i=1}^N |a_i|$. So f_j is (σ) -cyclic in X by Proposition 6.5. $\square$

Remark. We return later on (Section 8) to the cyclicity of factorable polynomials in some ISL's $X(\mathbb{Z}_+^N)$ with $\mathbb{D}_{X'}^N = \mathbb{B}^N$ (the unit ball in $\mathbb{C}^N$).

6.10. Comments. (1) By the same reason as in the last lines, one can see that a sequence $x = (x_k) \in X$ supported by the set of prime integers and such that $Bx(\zeta) \neq 0$ for $\zeta \in \mathbb{D}^N$, $\forall N \geq 1$ satisfies $|x_1| \geq \sum_{k \geq 2} |x_k|$, and hence x is D_n -cyclic in X , an ISL from Proposition 6.5.

(2) *Cyclicity of polynomials in a single variable.* This elementary case plays a special role due to Proposition 5.3. Here, some (folklore) details are presented since it appears impossible to find written references.

Let X be a shift invariant sequence Banach space,

$$X_A = \left\{ f = \sum_{n \geq 0} \hat{f}(n) z^n, (\hat{f}(n))_{n \geq 0} \in X \right\},$$

where the set of all polynomials is contained and dense in X_A . Keeping the above notation, denote

$$\mathbb{D}_{X'} = \{\lambda \in \mathbb{C} : \varphi_\lambda : f \mapsto f(\lambda) \text{ is bounded on polynomials in } X_A\}$$

the domain of definition of X_A .

Claim. *Under the above hypothesis, a polynomial f is z -cyclic in X_A if and only if $f(\lambda) \neq 0$, $\lambda \in \mathbb{D}_{X'}$.*

Indeed, if $f(\lambda) = 0$ for some $\lambda \in \mathbb{D}_{X'}$, f is clearly non cyclic. Conversely, if $f(z) \neq 0$, $z \in \mathbb{D}_{X'}$, f is a finite product $f = c \prod (z - \lambda)$, $c \neq 0$ of factors $z - \lambda$, $\lambda \in \mathbb{C} \setminus \mathbb{D}_{X'}$ (or $f = \text{const} = c \neq 0$). Since φ_λ , $\lambda \in \mathbb{C} \setminus \mathbb{D}_{X'}$ is unbounded on polynomials, the kernel $\text{Ker}(\varphi_\lambda) = \{p : p \text{ is a polynomial, } p(\lambda) = 0\}$ is dense in X_A . The z -cyclicity of $z - \lambda$ follows. Since $f = (z - \lambda)q(z)$ and the polynomial q is (obviously) a multiplier of X_A , $q \in E_f$ (see Lemma 4.6). Repeating the reasoning with the resting zeroes of f , we get $c \in E_f$ and so $E_f = X_A$. This means that f is cyclic. $\square$

Example. Let X be a shift invariant sequence lattice on $\mathbb{Z}_+$. Then there exists R , $0 \leq R \leq \infty$, such that $\mathbb{D}_{X'} = \{\lambda \in \mathbb{C} : (\lambda^n)_{n \geq 0} \in X'\}$ is either $\mathbb{D}_R = \{\lambda \in \mathbb{C} : |\lambda| < R\}$ or $\overline{\mathbb{D}}_R = \{\lambda \in \mathbb{C} : |\lambda| \leq R\}$. If $R = \infty$, the only cyclic polynomial is $f = \text{const} \neq 0$. If $R = 0$, every polynomial f with $f(0) \neq 0$ is cyclic.

If $X = \ell^p(\mathbb{Z}_+, w_n)$, $w_n > 0$, we have $R = \overline{\lim}_n (w_n^{1/n})$ (and $\mathbb{D}_{X'} = \overline{\mathbb{D}}_R \Leftrightarrow (R^n/w_n) \in \ell^{p'}(1/p + 1/p' = 1)$). In particular, for Dirichlet-type weights $w_n = (1 + n)^\gamma$, $\mathbb{D}_{X'} = \mathbb{D}$ for $\gamma \leq 1/p'$, and $\mathbb{D}_{X'} = \overline{\mathbb{D}}$ for $\gamma > 1/p'$. By the

way, in the latter case, X_A is a Banach algebra having $\overline{\mathbb{D}}$ as the space of maximal ideals (see [16]), and so the cyclicity criterion is $f(\lambda) \neq 0$, $\lambda \in \overline{\mathbb{D}}$ for any function $f \in X$, not only for polynomials.

(3) z -cyclicity problem for generic functions in one variable. It is far from to be trivial and still mostly not solved, even for the Dirichlet-type lattices $X = \ell^p(\mathbb{Z}_+, w_n)$, $w_n = (1 + n)^\gamma$, despite many known deep results. For these spaces, the question is in the size of the boundary zero set $Z(f) = \{\lambda \in \mathbb{T} : f(\lambda) = 0\}$ measured with the corresponding capacity (as it is the case for bi-cyclic sequences on $\mathbb{Z}$), see Comments 5.4(3). The use of the potential theory forces to work mostly with the Hilbert space cases $\ell^2(\mathbb{Z}_+, w_n)$. See for example, a recent paper [22] where many other researches are quoted. Notice that usually the z -cyclicity questions in the framework of holomorphic spaces X_A are more involved than the “real-analysis” problems for bi-cyclicity on $\mathbb{Z}$, but not always — for example, a deep construction of N. Lev–A. Olevskii (2011) of two $\ell^1(\mathbb{Z})$ -functions f and g with the same zero sets on $\mathbb{T}$, $Z(f) = Z(g)$, one of which is $\ell^p(\mathbb{Z})$ -bicyclic ($1 < p < 2$), and the second is not, contrasts with a rather obvious such example for analytic z -cyclicity: $f = (1 - z)^N$ is z -cyclic in all $\ell^p(\mathbb{Z}_+)_A$ ($1 < p \leq \infty$), and $g = (1 - z)^N \exp \frac{z+1}{z-1}$ is not in any $\ell^p(\mathbb{Z}_+)_A$ ($1 < p \leq \infty$) (known already from an “abstract Keldysh approach” of [38]), but $Z(f) = Z(g) = \{1\}$.

(4) Polynomials in several variables. The matter is about the ζ^α -cyclicity in spaces $X(\mathbb{Z}_+^N)_A$ in $\mathbb{D}_{X'}^N$, where X is a lattice. Usually, X is a “Dirichlet-type” space $X = \ell_{w_\alpha}^p(\mathbb{Z}_+^N)$. The “Beurling necessary condition” for ζ^α -cyclicity (if F is cyclic then $F(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_{X'}^N$) is still the departure point. For the moment, all known results are either for the case $\mathbb{D}_{X'}^N = \mathbb{D}^N$ or $\mathbb{D}_{X'}^N = \mathbb{B}^N$, the unit ball of $\mathbb{C}^N$: $\mathbb{B}^N = \{\zeta \in \mathbb{C}^N : \sum_1^N |\zeta_j|^2 < 1\}$.

The case when $\mathbb{D}_{X'}^N = \mathbb{D}^N$. As is mentioned above, for the Hardy space $X_A = \ell^2(\mathbb{Z}_+^N)_A$ in the polydisc $\mathbb{D}^N$, the sufficiency of Beurling’s condition for polynomials (F does not vanish in $\mathbb{D}^N \Rightarrow F$ is cyclic) was claimed by V. Ya. Kozlov [26] and proved in [35] (then independently, in [19] and [41], where two proofs are given, including the one based on a Łojasiewicz inequality).

A power-like weighted Dirichlet-type space $X = \ell_{w_\alpha}^p(\mathbb{Z}_+^N)$,

$$w_\alpha = \alpha^s =: (1 + \alpha_1)^{s_1} \dots (1 + \alpha_N)^{s_N}, \quad \alpha \in \mathbb{Z}_+^N,$$

is a Banach algebra with $\overline{\mathbb{D}}^N$ as the maximal ideal space if and (for $p > 1$) only if $s_j > 1/p'$ for every j , $1 \leq j \leq N$ (for $p = 1$, it is an algebra for

$s_j \geq 0$), see 7.1 below. Therefore, in this case, the cyclicity criterion for $F \in X_A$, $X = \ell_{w_\alpha}^p(\mathbb{Z}_+^N)$ is $F(\zeta) \neq 0$, $\zeta \in \overline{\mathbb{D}}^N$ (not only for polynomials).

For Dirichlet-type spaces which are not algebras, the question is practically settled for the case of bidisc $\mathbb{D}^2$ ($N = 2$) and always for $p = 2$ by G. Knese, L. Kosinski, T. Ransford, A. Sola, see [24] and references therein. The answer depends on the set of boundary zeroes $Z(F) \cap \mathbb{T}^2$ and says, roughly speaking, the following: a polynomial F , $F(\zeta) \neq 0$ ($\forall \zeta \in \mathbb{D}^N$) is cyclic if $Z(F) \cap \mathbb{T}^2$ is finite, and non cyclic if it is infinite (with some restrictions on the exponents s_j). For the case of tridisc $\mathbb{D}^3$ ($N = 3$), for the moment, the answers are already partial and depend on the dimension of $Z(F) \cap \mathbb{T}^3$ and its “orientation”, see [5] where some observations on the general case of $\mathbb{D}^N$, $N > 2$ are also given (see also 7.2, 7.3 below). In particular, some results (with rather technical statements) depend on geometric characteristics as non-vanishing Gaussian curvature of certain submanifolds of $Z(F) \cap \mathbb{T}^N$.

The case when $\mathbb{D}_{X'}^N = \mathbb{B}^N$. Here, a series of weighted Hilbert spaces $X = \mathfrak{D}_s = \ell_{w_\alpha(s)}^2(\mathbb{Z}_+(\infty))$, $s \in \mathbb{R}$ is considered, all around the famous Drury-Arveson space,

$$\mathfrak{D}_s = \{x = (x_\alpha)_{\alpha \in \mathbb{Z}_+(\infty)} : (x_\alpha w_\alpha) \in \ell^2\}, \quad w_\alpha = \left(\frac{\alpha!}{|\alpha|!} \right)^{1/2} \cdot (|\alpha| + 1)^s,$$

where $\alpha! = \alpha_1! \alpha_2! \dots$, $|\alpha| = \alpha_1 + \alpha_2 + \dots$ ($\alpha \in \mathbb{Z}_+(\infty)$), $s \in \mathbb{R}$. The Drury-Arveson space $X = \mathfrak{D}_0$ is the subject of numerous deep studies due to its role in multidimensional Nevanlinna-Pick theory, the multi-operator theory (the so called row contractions), and more; it is recognized as the most genuine multivariable analogue of the Hardy space of the disc. See [51, 56, 17] for outlines of the theory. The reproducing kernel formula for Bohr transforms $F \in B\mathfrak{D}_0$,

$$F(\zeta) = (F, k_\zeta)_{\mathfrak{D}_0} \text{ where } k_\zeta(w) = \frac{1}{1 - (w, \zeta)_{\mathbb{C}^N}},$$

shows that $\mathbb{D}_{X'}^N = \mathbb{B}^N$ for every $N \in \mathbb{N}$, and these equalities still hold for all $\mathfrak{D}_s$. It is known [51] that in two and three variables ($N = 2, 3$), the Beurling condition $Z(F) \cap \mathbb{B}^N = \emptyset$ is sufficient for a polynomial F to be cyclic in $B\mathfrak{D}_0$, but for $N \geq 4$ this is already not the case. In particular, $F(\zeta) = 1 - N^{N/2} \zeta_1 \zeta_2 \dots \zeta_N$, $N \geq 4$ are not cyclic in $B\mathfrak{D}_0$.

On the other hand, it is shown, [52], that $F \in B\mathfrak{D}_0$ and $1/F \in B\mathfrak{D}_0$ imply the cyclicity of F in $B\mathfrak{D}_0$ (in contrast to the Bergman-type spaces $B\mathfrak{D}_s$, $s = -1$). This result contains, for example, that if $F \in \text{Mult}(B\mathfrak{D}_0)$

and F depends on a finitely many variables (in particular, if F is a polynomial) and is separated from zero $|F(\zeta)| \geq c > 0$ for $\zeta \in \mathbb{B}^N$, then F is cyclic in $B\mathfrak{D}_0$ (proved early in 2011 by S. Cortea, E. Sawyer, B. Wick). Below, we show (Section 8) that, for every $N \geq 1$, factorable polynomials non-vanishing on $\mathbb{B}^N$ are cyclic in $\mathfrak{D}_s$ for $s \leq 1/2$, and for $s > 1/2$ the cyclicity criterion for every function F is $F(\zeta) \neq 0$ for all $\zeta \in \overline{\mathbb{B}}^N$.

In [25], the case of polynomials in two variables in $B\mathfrak{D}_s(\mathbb{B}^2)$ is considered. It is shown that for $s \leq 1/4$, a polynomial F is cyclic if and only if $F(\zeta) \neq 0$ for $\zeta \in \mathbb{B}^2$, and for $1/4 < s \leq 1/2$ — if and only if $Z(F) \cap \partial\mathbb{B}^2$ is finite. For $s > 1/2$, the cyclicity criterion is $F(\zeta) \neq 0$ for all $\zeta \in \overline{\mathbb{B}}^2$.

7. Factorable functions in Dirichlet-type spaces and algebras in polydiscs

A kind of power-like weighted ℓ^2 spaces are known as “Dirichlet-type spaces”, referring to the classical Dirichlet space $\ell_w^2(\mathbb{Z}_+)_A$, $w_n = (1+n)^{1/2}$. Here and in the next Section, we give an elementary treatment to the problem of cyclicity, first, for factorable polynomials, and then to more general factorable functions in the framework of the Dirichlet-type spaces $\ell_w^p(\mathbb{Z}_+(\infty))_A$. It is already treated, in particular, in [5, 24], even for any polynomials but only for the Hilbert space case $p = 2$ and the Dirichlet-type weights (as it is the case, in general, for the most part of the existing DCP literature). Speaking on polynomials, we can restrict ourselves on the sequence spaces over a finite dimensional subsemigroup $\mathbb{Z}_+^N$. A special situation occurs when the space $\ell_w^p(\mathbb{Z}_+(\infty))_A$ is an algebra with the point-wise multiplication and a well identified space of maximal ideals. In this Section we consider the weights w_α factored by variables (and so, oriented to the polydisc case $\mathbb{D}^N$), and in Section 8 — the ball oriented weights of Drury–Arveson type.

Let $s \in \mathbb{R}^N$, and let $w(s)$ be a power-like weight function on $\mathbb{Z}_+^N$ defined by

$$w_\alpha(s) = (1 + \alpha_1)^{s_1} \dots (1 + \alpha_N)^{s_N}, \quad \alpha \in \mathbb{Z}_+^N.$$

For short, the notation $\alpha^s := w_\alpha(s)$ is used as well. Consider a lattice

$$X_w^N = \ell^p(\mathbb{Z}_+^N, \alpha^s) := \{x = (x_\alpha) : (x_\alpha w_\alpha) \in \ell^p(\mathbb{Z}_+^N)\}, \quad 1 \leq p \leq \infty,$$

and its Bohr transform $BX_w^N = \ell^p(\mathbb{Z}_+^N, \alpha^s)_A$ defined on the domain $\mathbb{D}_{(X_w^N)}^N$. It is easy to show that for $X = \ell^p(\mathbb{Z}_+^N, \alpha^s)$,

$$\mathbb{D}_{X'}^N = \mathbb{D}^N \text{ if } 1 < p \leq \infty, s_j \leq \frac{1}{p'} = \frac{p-1}{p} \ (\forall j); \text{ or } p = 1 \text{ and } s_j < 0 \ (\forall j),$$

and

$$\mathbb{D}_{X'}^N = \overline{\mathbb{D}}^N \text{ if } 1 < p \leq \infty, s_j > \frac{1}{p'} = \frac{p-1}{p} \ (\forall j); \text{ or } p = 1 \text{ and } s_j \geq 0 \ (\forall j);$$

mixed products of $\mathbb{D}$'s and $\overline{\mathbb{D}}$'s are also can be easily identified (but we will not enter in these details). In the generic infinite variable case

$$X_w = \ell^p(\mathbb{Z}_+(\infty), \alpha^s) := \{x = (x_\alpha) : (x_\alpha w_\alpha) \in \ell^p(\mathbb{Z}_+(\infty))\}, \quad 1 \leq p \leq \infty,$$

one can similarly show that, for $p > 1$,

$$\mathbb{D}_{X'_w}^\infty = \left\{ \lambda = (\lambda^\alpha) : \sum_{j \geq 1} f_j(|\lambda_j|^{p'}) < \infty \right\}, \text{ where } f_j(x) = \sum_{n \geq 1} \frac{x^n}{(n+1)^{p's_j}}.$$

In particular, (1) always, $\mathbb{D}_{X'_w}^\infty \subset \overline{\mathbb{D}}^\infty$; (2) if $s_j \leq \frac{1}{p'} \ (\forall j)$ then $\mathbb{D}_{X'_w}^\infty = \mathbb{D}_{p'}^\infty$ (i.e., $\lambda \in \mathbb{D}_{X'_w}^\infty \Leftrightarrow |\lambda_j| < 1 \ (\forall j)$ and $\lambda \in \ell^{p'}(\mathbb{N})$); (3) for the case where $1 < c \leq p's_j \leq C < \infty \ (\forall j)$, one has $\lambda \in \mathbb{D}_{X'_w}^\infty \Leftrightarrow |\lambda_j| \leq 1 \ (\forall j)$ and $\lambda \in \ell^{p'}(\mathbb{N})$; and finally, (4) in order to have simply $\mathbb{D}_{X'_w}^\infty = \overline{\mathbb{D}}^\infty$, it is sufficient that $p's_j \geq c \log(j+1)$, $c > 1/\log 2$.

For $p = 1$, we always have $\mathbb{D}_{X'_w}^\infty = \overline{\mathbb{D}}^\infty$.

The Hilbert space case $p = 2$ in the bidisc $\mathbb{D}^2$ ($N = 2$) of theorem 7.2 below is contained in [24], and already for arbitrary polynomials (not only for the factorable ones) but with more involved proofs. See also comments in 7.3.

We start with the following known (at least for sequences on the additive semigroup $\mathbb{Z}_+$ related to holomorphic spaces in $\mathbb{D}$) general lemma on weighted ℓ^p spaces which are convolution algebras. In fact, the question has quite a long history: it started with J. Wermer's result [59] for the continuous case of convolution algebras $L^p(\mathbb{R}, w)$. It could be a sample theorem for repeating evrywhere, but was overlooked and rediscovered independently, and in a different language, for the sequence spaces $\ell^p(\mathbb{Z}_+, w_n)$ much later in [36, 37, 23, 57]. It is also known that the sufficient condition of Lemma 7.1 for a weighted ℓ^p space to be a convolution algebra is also necessary under some natural regularity conditions on the weight (a kind of log concavity, but not in the full generality [23]; see for example, comments in [16]). It is clear that a similar to 7.1 claim can be shown for weighted L^p norms on any locally compact abelian group (or semigroup). For completeness, we give a short proof to Lemma 7.1.

7.1. Lemma. *Let $X = \ell^p(\mathbb{Z}_+(\infty), w_\alpha)$. If*

$$C_\gamma =: \sup_{\gamma \in \mathbb{Z}_+(\infty)} W_\gamma < \infty, \text{ where } W_\gamma = \left(\sum_{\alpha+\beta=\gamma} \left(\frac{w_\gamma}{w_\alpha w_\beta} \right)^{p'} \right)^{1/p'},$$

$\frac{1}{p} + \frac{1}{p'} = 1$, then X_A is an algebra with pointwise multiplication, and the complex homomorphisms of X_A are of the form $\varphi_\lambda : F \mapsto F(\lambda)$, $F \in X_A$ where $\lambda \in \mathbb{D}_X^\infty$, (in this case, the domain $\mathbb{D}_X^\infty$ is compact with respect to a norm bounded coordinate-wise convergence in $X' = \ell^{p'}(\mathbb{Z}_+(\infty), 1/w_\alpha)$).

Proof. Let $1 < p < \infty$ (modifications for $p = 1$ or $p = \infty$ will be obvious) and $\gamma \in \mathbb{Z}_+(\infty)$. It suffices to have an estimate $\|FG\| \leq C\|F\| \cdot \|G\|$ for the product of two polynomials $F, G \in X_A$. For $d = \deg(FG)$, $H = FG$ and $\hat{H}(\gamma) = \sum_{\alpha+\beta=\gamma} \hat{F}(\alpha)\hat{G}(\beta)$, one has

$$\begin{aligned} |\hat{H}(\gamma)|^p w_\gamma^p &= \left| \sum_{\alpha+\beta=\gamma} \hat{F}(\alpha)w_\alpha \hat{G}(\beta)w_\beta \cdot \frac{1}{w_\alpha w_\beta} \right|^p w_\gamma^p \leq \\ &\leq \left(\sum_{\alpha+\beta=\gamma} |\hat{F}(\alpha)w_\alpha|^p |\hat{G}(\beta)w_\beta|^p \right) \left(\sum_{\alpha+\beta=\gamma} \left(\frac{w_\gamma}{w_\alpha w_\beta} \right)^{p'} \right)^{p/p'}. \end{aligned}$$

Hence,

$$\begin{aligned} \|H\|^p &= \sum_{\gamma \in \mathbb{Z}_+^N} |\hat{H}(\gamma)|^p w_\gamma^p \leq C_\gamma^p \sum_{\gamma \in \mathbb{Z}_+^N} \left(\sum_{\alpha+\beta=\gamma} |\hat{F}(\alpha)w_\alpha|^p |\hat{G}(\beta)w_\beta|^p \right) = \\ &= C_\gamma^p \left(\sum_{\alpha \in \mathbb{Z}_+^N} |\hat{F}(\alpha)w_\alpha|^p \right) \left(\sum_{\beta \in \mathbb{Z}_+^N} |\hat{G}(\beta)w_\beta|^p \right) = C_\gamma^p \|F\|^p \|G\|^p, \end{aligned}$$

with a constant $C < \infty$ independent of F, G . It follows that X_A is a Banach algebra (with an equivalent norm). If $\varphi : X_A \rightarrow \mathbb{C}$ is a homomorphism of X_A , and $\lambda_j = \varphi(F_j)$, where $F_j(\zeta) = \zeta_j$ ($\forall \zeta \in \mathbb{D}_X$), then $\varphi(\zeta^\alpha) = \lambda^\alpha$ for every $\alpha \in \mathbb{Z}_+(\infty)$, where $\lambda = \lambda_\varphi =: (\lambda_1, \lambda_2, \dots)$, and so $\varphi(F) = \varphi_\lambda(F)$ for every polynomial F . Since φ is bounded, $\lambda \in \mathbb{D}_X^\infty$, and vice-versa, every such λ defines a homomorphism φ_λ . Clearly, $\varphi \mapsto \lambda_j$ is continuous for $\sigma(X', X)$ topology, and since the space $\{\varphi\}$ is compact, $\mathbb{D}_X^\infty$ is homeomorphic to the space of all $\{\varphi\}$ (the Gelfand space of X_A). $\square$

7.2. Proposition. Let $X = \ell^p(\mathbb{Z}_+^N, \alpha^s)$.

(1) If $s_j > 1/p'$ ($\forall j$), then X_A is an algebra with pointwise multiplication, and the cyclicity criterion for any function $F \in X_A$ is $F(\zeta) \neq 0$, $\forall \zeta \in \overline{\mathbb{D}}^N$.

(2) If $|s| = \sum_{j=1}^N s_j \leq 1/p'$, and f is a linearly factorable polynomial, then f is $\sigma(X, X')$ -cyclic in X_A if and only if $f(\zeta) \neq 0$ for $\zeta \in \mathbb{D}^N$.

Proof. (1) It suffices to check the condition of Lemma 7.1. Let $1 < p < \infty$ (modifications for $p = 1$ or $p = \infty$ are obvious) and $\gamma \in \mathbb{Z}_+^N$. Then (with

$$d = |\gamma|),$$

$$\sum_{\alpha+\beta=\gamma} \left(\frac{w_\gamma}{w_\alpha w_\beta} \right)^{p'} = \sum_{\alpha+\beta=\gamma} \frac{(1+\gamma_1)^{s_1 p'} \dots (1+\gamma_N)^{s_N p'}}{(1+\alpha_1)^{s_1 p'} (1+\beta_1)^{s_1 p'} \dots} \leq$$

(summation on $1 \leq \alpha_j \leq d-1$)

$$\begin{aligned} &\leq \left(B \sum_{\alpha_1} \frac{d^{s_1 p'}}{\alpha_1^{s_1 p'} (d - \alpha_1)^{s_1 p'}} \right) \dots \left(B \sum_{\alpha_N} \frac{d^{s_N p'}}{\alpha_N^{s_N p'} (d - \alpha_N)^{s_N p'}} \right) \leq \\ &\leq \left(B \sum_{k=1}^{d-1} \frac{d^m}{k^m (d-k)^m} \right)^N \leq C^{p'}, \end{aligned}$$

where $m = \min_j(s_j p') > 1$ and $B = B(s) < \infty$ is a constant, and

$$\begin{aligned} \left(\sum_{k=1}^{d-1} \frac{d^m}{k^m (d-k)^m} \right)^N &\leq \left(2 \sum_{k=1}^{d/2} \frac{d^m}{k^m (d-k)^m} \right)^N \leq \\ &\leq \left(B \sum_{k=1}^{d/2} \frac{1}{k^m} \right)^N \leq \left(B \sum_{k=1}^{\infty} \frac{1}{k^m} \right)^N =: C^{p'}. \end{aligned}$$

It follows that X_A is a Banach algebra (with an equivalent norm), and its maximal ideal space is $\mathbb{D}^N$ (Lemma 7.1). The cyclicity claim also follows.

(2) Let $f = \prod_{j=1}^n f_j$, where f_j are affine linear in $\mathbb{C}^N$. As in Proposition 6.9, it suffices to show that every f_j is cyclic in X_A . For that, we repeat the reasoning of 6.5, with a different estimate. Since f_j is an affine linear polynomial, $f_j(\zeta) = a - \sum_{i=1}^N a_i \zeta_i =: a - l(\zeta)$, and $f_j(\zeta) \neq 0$ for $\zeta \in \mathbb{D}^N$, it follows $|a| \geq \sum_{i=1}^N |a_i|$.

Simplifying the notation, let $a = 1$, $f(\zeta) = 1 - l(\zeta)$, $l(\zeta) = \sum_{i=1}^N a_i \zeta_i$, $1 \geq \sum_{i=1}^N |a_i|$ and

$$f_r = 1 - r l(\zeta).$$

Then (pointwise in $\mathbb{D}^N$),

$$\frac{1}{f_r} = \sum_{k \geq 0} r^k l(\zeta)^k = \sum_{k \geq 0} r^k \sum_{|\alpha|=k} a^\alpha \zeta^\alpha \frac{|\alpha|!}{\alpha!},$$

where $a = (a_1, \dots, a_N)$, $\alpha \in \mathbb{Z}_+^N$, $\alpha! = \alpha_1! \dots \alpha_N!$, and hence

$$\left\| \frac{1}{f_r} \right\|^p = \sum_{k \geq 0} r^{kp} \left\| \sum_{|\alpha|=k} a^\alpha \zeta^\alpha \frac{|\alpha|!}{\alpha!} \right\|^p,$$

where

$$\left\| \sum_{|\alpha|=k} a^\alpha \zeta^\alpha \frac{|\alpha|!}{\alpha!} \right\|^p = \sum_{|\alpha|=k} |a^\alpha|^p \left(\frac{|\alpha|!}{\alpha!} \right)^p w_\alpha^p.$$

Since $M_k =: \max_\alpha w_\alpha^p = \max_\alpha \left(\prod_{j=1}^N (1 + \alpha_j)^{s_j p} \right)$ over all $1 + \alpha_j \geq 0$ having $\sum_{j=1}^N (1 + \alpha_j) = N + k$ is

$$M_k = \prod_{j=1}^N \left((k + N) \frac{s_j}{|s|} \right)^{s_j p} = O(k^{|s|p}) \text{ as } k \rightarrow \infty,$$

and

$$\sum_{|\alpha|=k} |a^\alpha| \frac{|\alpha|!}{\alpha!} = \left(\sum_{i=1}^N |a_i| \right)^k \leq 1,$$

we get

$$\left\| \frac{1}{f_r} \right\|^p \leq C \sum_{k \geq 0} r^{kp} k^{|s|p}, \quad 0 < r < 1.$$

Hence

$$\left\| \frac{1}{f_r} \right\|^p = O \left(\frac{1}{(1 - r^p)^{1+|s|p}} \right), \quad \left\| \frac{1}{f_r} \right\| = O \left(\frac{1}{(1 - r)^{1/p+|s|}} \right) \quad (\text{as } r \rightarrow 1),$$

and so $\|1/f_r\| = O(\frac{1}{1-r})$. Getting $f/f_r \in E_f$ and $\|f/f_r\| = O(1)$ (as $r \rightarrow 1$), we finish the proof with the same argument as in 6.5 above. $\square$

7.3. Comments. If a weighted **ISL** $X = \ell^p(\mathbb{Z}_+^N, w_\alpha)$ satisfies

$$\max\{w_\alpha : |\alpha| = k\} = O(k^S), \quad S \geq 0 \text{ (as } k \rightarrow \infty)$$

and $S \leq 1/p'$, the same reasoning as in 7.2(2) shows that a factorable polynomial f is cyclic in X_A if $f(\zeta) \neq 0$, $\zeta \in \mathbb{D}^N$.

It is also of interests to compare Proposition 7.2 with some results in [5], where it is shown (Theorem 1 of that paper) that if $p = 2$ and the zero set $Z(f)$ of a polynomial f is contained in the union of *coordinate hyperplanes* $\zeta_j = c_j$, then the cyclicity holds under the weaker condition $0 < s_j \leq 1/2$ ($\forall j$). However, for a similar assumptions but with *non-coordinate hyperplanes*, this is already not the case, and the cyclicity is guaranteed only for the tridisc case $\mathbb{D}^3$, $N = 3$ and under condition $|s| \leq 1/2$ (coinciding with condition 7.2(2) for this case), see Theorem 5 of the paper cited.

In fact, under certain convergence assumptions, part 7.2(2) of the above Proposition can be done in the infinite variables setting of $X = \ell^p(\mathbb{Z}_+(\infty), w_\alpha)$, as follows.

7.4. Proposition. *Let $X = \ell^p(\mathbb{Z}_+(\infty), w_\alpha(s))$, $s = (s_1, \dots) \in \mathbb{R}_+^\infty$, and*

$$w_\alpha(s) = \prod_{j=1}^{\infty} (1 + \alpha_j)^{s_j}, \alpha \in \mathbb{Z}_+(\infty).$$

Then,

(1) $X = \ell^p(\mathbb{Z}_+(\infty), w_\alpha(s))$ is an **ISL**, and an elementary dilation D_{e_j} , $e_j = (\delta_{j,k})_{k \geq 1}$ has the norm $\|D_{e_j}\| \leq 2^{s_j}$. The finite dimensional sections of the domain $\mathbb{D}_{X'}^\infty$ are the polydiscs ($\mathbb{D}_{X'}^N$ are either $\mathbb{D}^N$ or $\overline{\mathbb{D}}^N$, see the beginning of Section 7).

(2) If a linear affine function

$$f(\zeta) = a + \sum_{k \geq 1} a_k \zeta_k$$

is nonvanishing on every $\mathbb{D}_{X'}^N$, $N = 1, 2, \dots$, then $|a| \geq \sum_{k \geq 1} |a_k|$.

(3) Assume $|s| := \sum_{j \geq 1} s_j < \infty$ and $f \in X_A \cap \ell^1(\mathbb{Z}_+(\infty))_A$ be a factorable function in the sense that $f = \prod_{j \geq 1} f_j$, f_j being linear affine functions, and

$$0 < \prod_{j \geq 1} \|f_j\|_{\ell^1} < \infty.$$

If $|s| \leq 1/p' = \frac{p-1}{p}$ and $f(\zeta) \neq 0$ for all $\zeta \in \mathbb{D}_{X'}$, then f is $\sigma(X, X')$ -cyclic in X_A ($\sigma(\ell^1, c_0)$ -cyclic, if $p = 1$).

Proof. Let $1 < p < \infty$ (modifications for $p = 1$ or $p = \infty$ are standard).

(1) Since $\|D_\alpha\| = \sup\{\frac{w_{\alpha+\beta}}{w_\beta} : \beta \in \mathbb{Z}_+(\infty)\}$, we have for $\alpha = e_j$:

$$\begin{aligned} \|D_{e_j}\| &= \|M_{\zeta_j}\| = \sup \left\{ \frac{w_{e_j+\beta}}{w_\beta} : \beta \in \mathbb{Z}_+(\infty) \right\} \\ &= \sup \left\{ \frac{(1 + \beta_j + 1)^{s_j}}{(1 + \beta_j)^{s_j}} : \beta \in \mathbb{Z}_+(\infty) \right\} \leq 2^{s_j}. \end{aligned}$$

The claim on the domain $\mathbb{D}_{X'}^N$ is also easy.

(2) It is clear (and is already noticed).

(3) Without loss of generality, $f_j(0) = 1$ for every $j \geq 1$ (indeed, $0 < \prod_{j \geq 1} |f_j(0)| < \infty$). Observe first, that $f_j =: \sum_{k \geq 0} a_{j,k} \zeta_k \in \text{Mult}(X_A)$ for every j :

$$\|f_j\|_{\text{Mult}} = \left\| 1 + \sum_{k \geq 1} a_{j,k} \zeta_k \right\|_{\text{Mult}} \leq 1 + \sum_{k \geq 1} |a_{j,k}| \cdot \|M_{\zeta_k}\| \leq 1 + 2^{\bar{s}} \|f_j - 1\|_{\ell^1},$$

where $\bar{s} = \max_{k \geq 1} (s_k)$. Since $\prod_j (1 + \|f_j - 1\|_{\ell^1}) < \infty$, it follows that the product f , as well as all products $\prod_{j \geq n} f_j$, are also multipliers. Hence, for the cyclicity of f , it suffices to prove that every f_j is cyclic.

Simplifying the notation, let $f_j = F(\zeta) = 1 - l(\zeta)$ and $F_r = 1 - rl(\zeta)$, $0 < r < 1$. As in Proposition 7.2(2), we show that $\|1/F_r\|_X = O(\frac{1}{1-r})$ for $r \rightarrow 1$, which implies the result. Indeed, we have pointwise $1/F_r = \sum_{k \geq 0} r^k l(\zeta)^k$ and

$$\begin{aligned} \frac{1}{F_r} &= \sum_{k \geq 0} r^k l(\zeta)^k = \sum_{k \geq 0} r^k \sum_{|\alpha|=k} a^\alpha \zeta^\alpha \frac{|\alpha|!}{\alpha!}, \\ \left\| \frac{1}{F_r} \right\|^p &= \sum_{k \geq 0} r^{kp} \left\| \sum_{|\alpha|=k} a^\alpha \zeta^\alpha \frac{|\alpha|!}{\alpha!} \right\|^p = \sum_{k \geq 0} r^{kp} \sum_{|\alpha|=k} |a^\alpha|^p \left(\frac{|\alpha|!}{\alpha!} \right)^p w_\alpha^p. \end{aligned}$$

Maximizing $\log(w_\alpha(s))$ over all $\alpha \in \mathbb{Z}_+(\infty)$ with a given (finite) support $\sigma \subset \mathbb{N}$ and $\sum \alpha_j = k$, we can restrict ourselves to the case $\#\sigma \leq k$, and then as in the proof of 7.2(2), see that $\max(w_\alpha(s))$ over all such α is

$$\begin{aligned} \max \left\{ w_\alpha(s) : \text{supp}(\alpha) \subset \sigma, \sum \alpha_j = k \right\} &= \prod_{j \in \sigma} \left(s_j \frac{\#\sigma + k}{\sum_{j \in \sigma} s_j} \right)^{s_j} \\ &\leq (2k)^{\sum_j s_j} \prod_{j \in \sigma} \left(\frac{s_j}{\sum_{j \in \sigma} s_j} \right)^{s_j} \leq (2k)^{\sum_j s_j} \leq (2k)^{|\sigma|}. \end{aligned}$$

Using $\sum_{|\alpha|=k} |a^\alpha| \left(\frac{|\alpha|!}{\alpha!} \right) \leq 1$, we obtain $\|1/F_r\|^p \leq C \sum_{k \geq 0} r^{kp} k^{|s|p}$, and so

$$\|1/F_r\|_X^p = O \left(\frac{1}{(1-r)^{1+|s|p}} \right),$$

where $1/p + |s| \leq 1$. This proves the claim. $\square$

8. Factorable functions in Dirichlet–Drury–Arveson spaces and algebras in ℓ^p -unit ball

Here we briefly consider an ℓ^p version of weighted Drury–Arveson space $\mathfrak{D}_0 = \ell_{w_\alpha}^2(\mathbb{Z}_+^N)$ (see 6.10(4) above) and its infinite variables versions, in the form $\mathfrak{D}_s^p = \ell_{w_\alpha(s)}^p(\mathbb{Z}_+(\infty))$, $s \in \mathbb{R}_+$,

$$\mathfrak{D}_s^p = \{x = (x_\alpha)_{\alpha \in \mathbb{Z}_+(\infty)} : (x_\alpha w_\alpha) \in \ell^p\}, w_\alpha = \left(\frac{\alpha!}{|\alpha|!} \right)^{1/p'} (|\alpha| + 1)^s,$$

where, as above, $\alpha! = \alpha_1! \alpha_2! \dots$, $|\alpha| = \alpha_1 + \alpha_2 + \dots$ ($\alpha \in \mathbb{Z}_+(\infty)$), $1/p + 1/p' = 1$. The Drury–Arveson spaces are the Bohr transforms $B\mathfrak{D}_0^2$ (defined in the unit balls $\mathbb{B}^N$, see 8.1 below). Since the answers to the cyclicity question are different for the algebra- and non-algebra-cases of $\mathfrak{D}_s^p$, we state preliminary results separately (Lemmas 8.1 and 8.2). The following lemma identifies the domain of the definition for the holomorphic space $B\mathfrak{D}_s^p = (\mathfrak{D}_s^p)_A$.

8.1. Lemma. *Let $X = \mathfrak{D}_s^p$ as defined above. Then, $X = \mathfrak{D}_s^p$ is an ISL, and an elementary dilation D_{e_j} , $e_j = (\delta_{j,k})_{k \geq 1}$ has the norm $\|D_{e_j}\| \leq 2^s$.*

For $sp' \leq 1$, the domain $\mathbb{D}_{X'}^\infty$ is the open unit “ $\ell^{p'}$ -ball” $\mathbb{B}_{p'}^\infty$,

$$\mathbb{D}_{X'}^\infty = \mathbb{B}_{p'}^\infty =: \left\{ \lambda = (\lambda_j)_{j \geq 1} \in \mathbb{C}^\infty : \sum_{j \geq 1} |\lambda_j|^{p'} < 1 \right\},$$

and for $sp' > 1$ it is the closed “ $\ell^{p'}$ -ball”

$$\mathbb{D}_{X'}^\infty = \overline{\mathbb{B}}_{p'}^\infty =: \left\{ \lambda = (\lambda_j)_{j \geq 1} \in \mathbb{C}^\infty : \sum_{j \geq 1} |\lambda_j|^{p'} \leq 1 \right\}$$

(for $p' = \infty$ and $s = 0$, $X = \mathfrak{D}_0^1 = \ell^1(\mathbb{Z}_+(\infty))$ and $\mathbb{D}_{X'}^\infty = \overline{\mathbb{D}}_\infty^\infty =: \{\lambda = (\lambda_j)_{j \geq 1} \in \mathbb{C}^\infty : |\lambda_j| \leq 1\}$). The finite dimensional sections of $\mathbb{D}_{X'}^\infty$ are the $\ell^{p'}$ -balls $\mathbb{B}_{p'}^N = \mathbb{D}_{X'}^\infty \cap \mathbb{C}^N$ (open or closed).

Proof. Let $1 < p < \infty$ (modifications for $p = 1$ or $p = \infty$ are standard). Since $\|D_\alpha\| = \sup \left\{ \frac{w_{\alpha+\beta}}{w_\beta} : \beta \in \mathbb{Z}_+(\infty) \right\}$, we have for $\alpha = e_j$:

$$\begin{aligned} \|D_{e_j}\| &= \|M_{\zeta_j}\| = \sup \left\{ \frac{w_{e_j+\beta}}{w_\beta} : \beta \in \mathbb{Z}_+(\infty) \right\} = \\ &= \sup \left\{ \left(\frac{(\beta + e_j)! |\beta|!}{(|\beta| + 1)! |\beta|!} \right)^{1/p'} \left(\frac{|\beta| + 2}{|\beta| + 1} \right)^s : \beta \in \mathbb{Z}_+(\infty) \right\} \leq \\ &\leq \sup \left\{ \left(\frac{|\beta| + 2}{|\beta| + 1} \right)^s : \beta \in \mathbb{Z}_+(\infty) \right\} = 2^s. \end{aligned}$$

The domain $\mathbb{D}_{X'}^N$ consists of all $\lambda = (\lambda_j)_{j \geq 1}$ such that $(\lambda^\alpha)_\alpha \in \ell_{1/w_\alpha(s)}^{p'}(\mathbb{Z}_+(\infty))$. Summing up,

$$\begin{aligned} \sum_{\alpha} |\lambda^\alpha|^{p'} \frac{|\alpha|!}{\alpha!} (|\alpha| + 1)^{-sp'} &= \sum_{n \geq 0} (n + 1)^{-sp'} \sum_{|\alpha|=n} |\lambda^\alpha|^{p'} \frac{|\alpha|!}{\alpha!} = \\ &= \sum_{n \geq 0} (n + 1)^{-sp'} \left(\sum_{j \geq 1} |\lambda_j|^{p'} \right)^n, \end{aligned}$$

which is finite if and only if either $\sum_{j \geq 1} |\lambda_j|^{p'} < 1$ or $\sum_{j \geq 1} |\lambda_j|^{p'} \leq 1$ and $sp' > 1$. $\square$

In the next Lemma, we consider a slightly larger scale of spaces $\mathfrak{D}_{s,t}^p = \ell_{w_\alpha(s)}^p(\mathbb{Z}_+(\infty))$ introducing one more parameter as follows

$$w_\alpha = w_\alpha(s, t) := \left(\frac{\alpha!}{|\alpha|!} \right)^t (|\alpha| + 1)^s, \quad t, s \in \mathbb{R}_+.$$

Previously introduced weights are $w_\alpha = w_\alpha(s, 1/p')$.

The key point for checking the condition of Lemma 7.1 for the $\mathfrak{D}_{s,t}^p$ spaces is the use of a binomial coefficient identity, namely a “generalised Vandermond identity” for binomial coefficients (coming from the famous “Pascal’s triangle/simplex”). This point was suggested to me by A. Borichev (Aix-Marseille University) after reading a preliminary version of that paper. See also comments 8.3 below.

8.2. Lemma. $\mathfrak{D}_{s,t}^p$ is a convolution algebra (a Banach algebra after an equivalent renorming) for

$$s > 1/p', \quad t \geq 1/p' \text{ (for } p > 1; \text{ and } s \geq 0, \quad t \geq 0 \text{ for } p = 1).$$

Proof. It suffices to verify the sufficient condition of Lemma 7.1. Let $p > 1$ (modifications for $p = 1$ will be obvious), $\gamma \in \mathbb{Z}_+(\infty)$, and

$$W_\gamma^{p'} = \sum_{\alpha+\beta=\gamma} \left(\frac{w_\gamma}{w_\alpha w_\beta} \right)^{p'},$$

where $w_\alpha = w_\alpha(s, t)$. We employ the following identity for binomial (or multinomial) coefficients, sometimes called “generalised Vandermond identity” (see comments, references and a proof below): for $0 \leq k \leq |\gamma|$,

$$\sum_{\alpha+\beta=\gamma, |\alpha|=k} \frac{\gamma!}{\alpha! \beta!} = \binom{|\gamma|}{k},$$

where $\binom{l}{k}$ stands for the binomial coefficients. Using this we obtain

$$W_\gamma^{p'} = \sum_{k=0}^{|\gamma|} \sum_{\alpha+\beta=\gamma, |\alpha|=k} \left(\frac{\gamma!}{\alpha! \beta!} \right)^{tp'} c(k),$$

where

$$c(k) = \left(\frac{k!(|\gamma| - k)!}{|\gamma|!} \right)^{tp'} \left(\frac{|\gamma| + 1}{(k+1)(|\gamma| - k + 1)} \right)^{sp'},$$

and hence

$$\begin{aligned}
 W_\gamma^{p'} &\leq \sum_{k=0}^{|\gamma|} \binom{|\gamma|}{k}^{tp'} c(k) = \sum_{k=0}^{|\gamma|} \left(\frac{|\gamma|+1}{(k+1)(|\gamma|-k+1)} \right)^{sp'} \leq \\
 &\leq 2 \sum_{k=0}^{|\gamma|/2} \left(\frac{|\gamma|+1}{(k+1)(|\gamma|-k+1)} \right)^{sp'} \leq 2 \sum_{k=0}^{|\gamma|/2} \left(\frac{2}{k+1} \right)^{sp'} < \\
 &< C^{p'} =: 2 \sum_{k=0}^{\infty} \left(\frac{2}{k+1} \right)^{sp'} < \infty.
 \end{aligned}$$

So, $\sup_\gamma W_\gamma < \infty$, and by Lemma 7.1, $\mathfrak{D}_{s,t}^p$ is an algebra. $\square$

8.3. Comments. The algebra property for the Hilbert space $\mathfrak{D}_{s,t}^2(\mathbb{B}^N)$ in finite number of variables is proved for $t = 1/p' = 1/2$ and $s > 1/2$ in [47], Theorem 4.1, and then — with a very short proof — in [2], Lemma 14.2 (in both cases, using the reproducing kernel techniques and certain integration expressions for the norm coming from the Hilbert space structure).

A binomial identity used in the proof of 8.2 (“generalised Vandermond’s identity”) rose from the classical “Pascal/Tartaglia’s triangle/simplex”, and so goes back as early as to X-XIII centuries, see for this history, for example, [32] or [48, 58]. For completeness, we give a short proof.

Proof. Let $n_i \in \mathbb{Z}_+$, $m = \sum_{i=1}^N n_i$. Using the Newton binomial,

$$\begin{aligned}
 \sum_{k=0}^m \binom{m}{k} x^k &= (1+x)^m = \prod_{i=1}^N (1+x)^{n_i} = \prod_{i=1}^N \left(\sum_{k_i=0}^{n_i} \binom{n_i}{k_i} x^{k_i} \right) = \\
 &= \sum_{k=0}^m x^k \sum_{k_1+\dots+k_N=k, k_i \leq n_i} \prod_{i=1}^N \binom{n_i}{k_i}.
 \end{aligned}$$

Comparing the coefficients, we see that

$$\binom{m}{k} = \sum_{k_1+\dots+k_N=k, k_i \leq n_i} \prod_{i=1}^N \binom{n_i}{k_i}. \quad \square$$

Below, in Proposition 8.5 (an $\ell^{p'}$ -unit ball analogue of Proposition 7.4), an extra property $f \in \ell_A^1$ of a function from $X = \mathfrak{D}_s^p$ is required. The following lemma, in a sense, justifies this requirement.

8.4. Lemma. *If a linear affine function f ,*

$$f(\zeta) = a + \sum_{k \geq 1} a_k \zeta_k$$

is nonvanishing on every $\mathbb{B}_{p'}^N$, $N = 1, 2, \dots$, then $|a| \geq (\sum_{k \geq 1} |a_k|^p)^{1/p}$.

Proof. It is clear that the upper bound of the linear form

$$l(\zeta) = \sum_{k \geq 1} a_k \zeta_k$$

over the ball $\mathbb{B}_{p'}^N =: \{\zeta = (\zeta_j)_{j \geq 1} \in \mathbb{C}^N : \sum_{j \geq 1} |\zeta_j|^{p'} < 1\}$ is $R =: (\sum_{k=1}^N |a_k|^p)^{1/p}$, and the range $l(\mathbb{B}_{p'}^N)$ is the disc $\{z \in \mathbb{C} : |z| < R\}$. Since $|f(\zeta)| = |a + l(\zeta)| > 0$ on $\mathbb{B}_{p'}^N$, we have $|a| \geq (\sum_{k=1}^N |a_k|^p)^{1/p}$ for any $N \geq 1$, and so the claimed inequality. $\square$

8.5. Proposition. (1) Let $X = \mathfrak{D}_s^p$ be as above, and let $f \in X_A \cap \ell^1(\mathbb{Z}_+(\infty))_A$ be a factorable function in the sense that $f = \prod_{j \geq 1} f_j$, f_j being linear affine functions, and

$$0 < \prod_{j \geq 1} \|f_j\|_{\ell^1} < \infty.$$

Then if $s \leq 1/p'$ and $f(\zeta) \neq 0$ for all $\zeta \in \mathbb{B}_{p'}^\infty$, then f is $\sigma(X, X')$ -cyclic in X_A .

(2) Let $X = \mathfrak{D}_{s,t}^p$ be as above, and either $p > 1$ and $s > 1/p'$, $t \geq 1/p'$, or $p = 1$ and $s \geq 0$, $t \geq 0$. Then, $X_A = B\mathfrak{D}_{s,t}^p$ is a function algebra on its maximal ideal space $\mathbb{D}_X^\infty$ (for $t = 1/p'$, $\mathbb{D}_X^\infty = \overline{\mathbb{B}}_{p'}^\infty$), and so a function $f \in X_A$ is cyclic if and only if $f(\zeta) \neq 0$, $\forall \zeta \in \mathbb{D}_X^\infty$ (for $t = 1/p'$, $\forall \zeta \in \overline{\mathbb{B}}_{p'}^\infty$).

Proof. (1) Let $1 < p < \infty$ (modifications for $p = 1$ or $p = \infty$ are standard). Here, the computations similar to these of 7.2, 7.4 are applied. Without loss of generality, $f_j(0) = 1$ for every $j \geq 1$. Also, all f_j and f are multipliers of X_A : $\|f_j\|_{Mult} \leq 1 + \|l_j\|_{Mult} \leq 1 + \sum_{j \geq 1} |a_j| \cdot \|D_{e_j}\| \leq 2^s \|f_j\|_1$ (see Lemma 8.1). So, to prove point (1) it suffices to show that every f_j is cyclic.

Simplifying the notation, let $f_j = F(\zeta) = 1 - l(\zeta)$ and $F_r = 1 - rl(\zeta)$, $0 < r < 1$. As in Proposition 7.4(3), we show that $\|1/F_r\|_X = O(\frac{1}{1-r})$ for $r \rightarrow 1$, which implies the result. Indeed, we have pointwise $1/F_r = \sum_{k \geq 0} r^k l(\zeta)^k$ and

$$\begin{aligned} \frac{1}{F_r} &= \sum_{k \geq 0} r^k l(\zeta)^k = \sum_{k \geq 0} r^k \sum_{|\alpha|=k} a^\alpha \zeta^\alpha \frac{|\alpha|!}{\alpha!}, \\ \left\| \frac{1}{F_r} \right\|^p &= \sum_{k \geq 0} r^{kp} \left\| \sum_{|\alpha|=k} a^\alpha \zeta^\alpha \frac{|\alpha|!}{\alpha!} \right\|^p = \sum_{k \geq 0} r^{kp} \sum_{|\alpha|=k} |a^\alpha|^p \left(\frac{|\alpha|!}{\alpha!} \right)^p w_\alpha^p. \end{aligned}$$

Since $w_\alpha^p = \left(\frac{\alpha!}{|\alpha|!}\right)^{p/p'} (|\alpha| + 1)^{sp}$, we have

$$\begin{aligned} \left\| \frac{1}{F_r} \right\|^p &= \sum_{k \geq 0} r^{kp} k^{sp} \sum_{|\alpha|=k} |a^\alpha|^p \frac{|\alpha|!}{\alpha!} = \\ &= \sum_{k \geq 0} r^{kp} k^{sp} \left(\sum_{k \geq 1} |a_k|^p \right)^k \leq \sum_{k \geq 0} r^{kp} k^{sp} = O\left(\frac{1}{(1-r)^{1+sp}}\right), \end{aligned}$$

and so $\|1/F_r\|_X = O\left(\frac{1}{(1-r)^{s+1/p}}\right)$ as $r \rightarrow 1$. This proves the claim.

(2) It is immediate from Lemmas 8.1 and 8.2. $\square$

Remark. One can show that $\overline{\mathbb{B}}_{1/t} \subset \mathbb{D}_{X'}^\infty \subset \overline{\mathbb{B}}_{p'}$ for any $t \geq 1/p'$.

9. Brief notes on the history of the periodic Dilation Completeness Problem (DCP).

Initially, the Dilation Completeness Problem consists in a search of functions $f \in L_{\text{loc}}^p[0, \infty)$ such that

$$E_f =: \text{span}_{L^p(0,1)} (f(nx)|(0,1) : n = 1, 2, 3, \dots) = L^p(0,1).$$

The famous Riemann Hypothesis (RH) on zeros of the ζ -function is closely related to the DCP for $p = 2$ and $f(x) = \frac{1}{x} - \left[\frac{1}{x}\right]$, $x > 0$: (RH) is equivalent to the inclusion $1 \in E_f$, and/or to the equality $\text{span}_{L^2(0,1)}(f(sx)|(0,1) : s \geq 1) = L^2(0,1)$ (see [46, 4], or Chapter 6 of [45]).

According to A. Wintner [60] and A. Beurling [6], the following partial case of the DCP (2-periodic DCP for $p = 2$) is also related to some number theoretic questions (in Diophantine analysis): to determine *odd 2-periodic functions* $f \in L_{\text{odd}}^2(-1, 1)$ on $\mathbb{R}$ such that $E_f = L^2(0, 1)$. Since the functions $e_k = \frac{\sin(\pi k x)}{\sqrt{2}}$, $k = 1, 2, \dots$ form an orthonormal basis in $L_{\text{odd}}^2(-1, 1)$ and the dilations $f \mapsto f(nx)$ act as $e_k \mapsto e_{nk}$ ($n, k = 1, 2, \dots$), one can unitarily change the basis (e_k) for (z^k) ($k = 1, 2, \dots$) on the Hardy space $H_0^2(\mathbb{D})$ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$,

$$H_0^2(\mathbb{D}) = \left\{ f = \sum_{k \geq 1} \hat{f}(k) z^k : \|f\|^2 =: \sum_{k \geq 1} |\hat{f}(k)|^2 < \infty \right\},$$

and get the following equivalent form of the periodic DCP: to describe $f \in H_0^2(\mathbb{D})$ such that the dilations $T_n f$

$$T_n f = f(z^n)$$

generate the whole space,

$$E_f =: \text{span}_{H_0^2}(T_n f : n = 1, 2, \dots) = H_0^2;$$

such a function f is called “dilation cyclic”. As we saw above, the problems of cyclic (polynomially cyclic, or weakly invertible) functions in one or several complex variables are often partial cases of the DCP in sequence spaces on $\mathbb{N}$. These partial cases are treated in a lot of papers and books; quoting just a few, [54, 55, 40, 61, 14] Ch. 9. The most facts and references known on the periodic DCP are gathered in [19, 20, 33, 41, 43, 44, 45, 11, 12] (and other Dan–Guo arXiv papers), [13], as well as in references of Section 6.10 above. B. Mityagin pointed out (see for example [33]) to an applied interest of the dilation cyclicity of binomials $f = z(\lambda - z)^N$ ($\lambda \in \mathbb{C}$, $N = 1, 2, \dots$) related to the Cohn–Lehmer–Schur algorithm (localization of zeros of polynomials). This special choice of f was already partially treated in [45] and [42].

References

- [1] N. I. Ahiezer, *Lectures on the theory of approximation*, OGIZ, Moscow-Leningrad, 1947 (Russian), 323 pp (MR0025598).
- [2] A. Aleman, M. Hartz, J. McCarthy, and S. Richter, *Free outer functions in complete Pick spaces*, arXiv:2203.08179v1, 15 Mar 2022.
- [3] N. Arcozzi, R. Rochberg, E. Sawyer, *Two variations on the Drury-Arveson space*, in “Hilbert spaces of analytic functions” CRM Proc. Lecture Notes, 51, 41–58 (AMS, Providence RI), 2010.
- [4] L. Báez-Duarte, *A strengthening of the Nyman–Beurling criterion for the Riemann hypothesis*, Atti Acad. Naz. Lincei 14 (2003), 5–11.
- [5] L. Bergqvist, *A note on cyclic polynomials in polydiscs*, Anal. Math. Phys. (2018), 8: 197–211.
- [6] A. Beurling, *On the completeness of $\{\psi(nt)\}$ on $L^2(0, 1)$* , a seminar talk 1945, *The collected Works of Arne Beurling, Vol. 2: Harmonic Analysis*, Contemp. Mathematicians, Birkhäuser, Boston, 1989, 378–380.
- [7] A. Beurling, *A critical topology in harmonic analysis on semigroups*, Acta Math. 112 (1964), 215–228.
- [8] H. Bohr, *Über die Bedeutung der Potenzreihen unendlich vieler Variablen in der Theorie der Dirichletschen Reihen $\sum \frac{a_n}{n^s}$* , Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl., 1913, A9, 441–488.
- [9] A. Borichev, O. El-Fallah, A. Hanine, *Cyclicity in weighted Bergman type spaces*, J. Math. Pures Appl., (9)102:6(2014), 1041–1061.
- [10] A. Borichev, H. Hedenmalm, A. Volberg, *Large Bergman spaces: invertibility, cyclicity, and subspaces of arbitrary index*, J. Funct. Anal. 207:1 (2004), 111–160.
- [11] H. Dan and K. Guo, *The periodic dilation completeness problem: cyclic vectors in the Hardy space over the infinite-dimensional polydisc*, J. London Math. Soc., (2) 00 (2020), 1–34.

- [12] H. Dan, K. Guo, The Beurling–Wintner problem for step functions, *arXiv:205.09779v3*, 2021, 53pp.
- [13] H. Dan, K. Guo, J. Ni, *Invariant subspaces of the Bergman spaces in infinitely many variables*, *arXiv:2103.04145v2*, 2021.
- [14] O. El-Fallah, K. Kellay, J. Mashregghi, Th. Ransford, *A primer on the Dirichlet spaces*, Cambridge Univ. Press, Cambridge, 2014.
- [15] O. El-Fallah, K. Kellay, Th. Ransford, *Cyclicity in the Dirichlet spaces*, *Ark. Mat.* 44:1, 2006, 61–86.
- [16] O. El-Fallah, N. Nikolski, and M. Zarrabi, *Resolvent estimates in Beurling–Sobolev algebras*, *Algebra i Analiz*, 6 (1998), 1–80 (Russian); English transl.: *St. Petersburg Math. J.*, 6 (1999), 1–69.
- [17] Q. Fang, *Multipliers of Drury–Arveson space: a survey*, pp. 99–116 in: *Oper. Theory Adv. Appl.*, vol. 272, Birkhäuser, 2019.
- [18] Ph. Hartman, *Multiplicative sequences and Töplerian (L^2) -bases*, *Duke Math. J.*, 14 (1947), 755–767.
- [19] H. Hedenmalm, P. Lindquist, and K. Seip, *A Hilbert space of Dirichlet series and sytems of dilated functions in $L^2(0, 1)$* , *Duke Math. J.*, 86 (1997), 1–37.
- [20] H. Hedenmalm, P. Lindquist, and K. Seip, *Addendum to “A Hilbert space of Dirichlet series and sytems of dilated functions in $L^2(0, 1)$ ”*, *Duke Math. J.*, 99 (1999), 175–178.
- [21] D. Hilbert, *Wesen und Ziele einer Analysis der unendlich vielen unabhängigen Variablen*, *Rend. Cir. Mat. Palermo* 27 (1909).
- [22] K. Kellay, F. LeManach, M. Zarrabi, *Cyclicity in Dirichlet type spaces*, *Contemp. Math.*, v.743, 181–193, Amer. Math. Soc., Providence, RI, 2020.
- [23] E. Kerlin, A. Lambert, *Strictly cyclic shifts on ℓ_p* , *Acta Sci. Math. (Szeged)*, 35 (1973), 87–94.
- [24] G. Knese, L. Kosinski, T. Ransford, A. Sola, *Cyclic polynomials in anisotropic Dirichlet spaces*, *J. Anal. Math.*, 138:1 (2019), 23–47.
- [25] L. Kosinski, D. Vavitsas, *Cyclic polynomials in Dirichlet-type spaces in the unit ball of $\mathbb{C}^2$* , *arXiv:2212.12013v1*, 19pp, 2022.
- [26] V. Ya. Kozlov, *On the completeness of systems of functions $\{\varphi(nx)\}$ in the space $L^2(0, 2\pi)$* (Russian), *Doklady Akad. Nauk SSSR*, 61 (1948), 977–980.
- [27] V. Ya. Kozlov, *On the completeness of the system of functions $\{\varphi(nx)\}$ in the space of odd functions from $L^2(0, 2\pi)$* (Russian), *Doklady Akad. Nauk SSSR*, 62 (1948), 13–16.
- [28] V. Ya. Kozlov, *On the completeness of a system of functions of type $\{\varphi(nx)\}$ in the space L^2* (Russian), *Doklady Akad. Nauk SSSR*, 73 (1950), 441–444.
- [29] F. Le Manach, *Sur l’approximation et la complétude des translatés dans les espaces de fonctions*, Thèse de Doctorat, 108 pp. Univ. Bordeaux, 2018 (HAL <https://Imf.alea.ovh/Recherche/These/memoire.pdf>).
- [30] F. Le Manach, *Cyclicity in weighted ℓ^p spaces*, Preprint, <https://HAL.archives-ouvertes.fr/hal-01483933v1>.
- [31] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, vol. I & II, Springer-Verlag, Berlin etc., 1979.
- [32] R. Merris, *Combinatorics*, J.Wiley, 2003.
- [33] B. Mityagin, *Systems of dilated functions: completeness, minimality, basisness*, *Funct. Anal. Appl.*, 51:3 (2017), 236–239; see also *arXiv:1612.06791v* (20 Dec 2016).

- [34] *Multiplicative function*, https://en.wikipedia.org/wiki/Multiplicative_function.
- [35] J. Neuwirth, J. Ginsberg, and D. Newman, *Approxiamtion by $f(kx)$* , J. Funct. Anal., 5 (1970), 194–203.
- [36] N. Nikolski, *Basicity and unicellularity of weighted shift operators*, Izvestia Akad. Nauk SSSR, Ser. Mat. 32 (1968), 1123–1137 (Russian); Engl. transl.: Math. URSS Izv. 2 (1968), 1077–1090.
- [37] N. Nikolski, *Spectral synthesis for a shift operator and zeroes in certain classes of analytic functions smooth up to the boundary*, Dokl. Akad. Nauk SSSR, 190 (1970), 780–783 (Russian); Engl. transl.: Soviet Math. Dokl. 11 (1970), 206–209.
- [38] N. Nikolski, *Selected problems of weighted approximation and spectral analysis*, Trudy Math. Inst. Steklova, vol. 120, Moscow (Russian); English transl.: Proc. Steklov Math. Inst., No 120 (1974), AMS, Providence, 1976.
- [39] N. Nikolski, *Two problems on the spectral synthesis*, Zapiski Nauchn. Seminarov LOMI, 81, 139–141; reprinted in: J. Soviet Math. 26 (1984), 2185–2186; reprinted in Lect. Notes Math., Springer-Verlag, vol. 1043 (1984), 378–381.
- [40] N. Nikolski, *Treatise on the shift operator*, Springer-Verlag, Berlin etc., 1986 (Russian original: *Lekzii ob opereatore sdviga*, “Nauka”, Moskva, 1980)
- [41] N. Nikolski, *In a shadow of the RH: Cyclic vectors of Hardy spaces on the Hilbert multidisc*, Ann. Inst. Fourier, 62:5 (2012), 1601–1626.
- [42] N. Nikolski, *Binomials whose dilations generate $H^2(\mathbb{D})$* , Algebra and Analysis (St. Petersburg), 29:6 (2017), 159–177.
- [43] N. Nikolski, *Addendum and Corrigendum to the paper “In a shadow of the RH: cyclic vectors of Hardy spaces on the Hilbert multidisc*, Ann. Inst. Fourier, 62:5 (2012), 1601–1626”, Ann. Inst. Fourier (Grenoble), 68:2 (2018), 563–567.
- [44] N. Nikolski, *The current state of the dilation completeness problem*, A PPT talk in the MSU analysis seminar 2018, and in the KCL (London) analysis seminar 2017.
- [45] N. Nikolski, *Hardy Spaces*, Cambridge Univ. Press, 2019.
- [46] B. Nyman, *On the one-dimensional translation group and semi-group in certain function spaces*, Thesis, Uppsala Univ., 1950.
- [47] J. Ortega, J. Fabrega, *Multipliers in Hardy–Sobolev spaces*, Integral Eq. Operator Theory, 55(4) (2006), 535–560.
- [48] https://en.wikipedia.org/wiki/Pascal%27s_triangle
- [49] G. Polya and G. Szegő, *Problems and theorems in Analysis*, Vol. I & II, Springer, 1976 (for 3rd English edition); for original German edition “*Aufgaben und Lehrsätze aus der Analysis*”, Band I & II, Springer, Berlin, 1925).
- [50] H. Queffélec, *Espaces de séries de Dirichlet et leurs opérateurs de composition*, Annales math. Blaise Pascal, 22 N°S2 (2015), 267–314.
- [51] S. Richter, C. Sundberg, *Cyclic vectors in the Drury-Arveson space*, ESI Workshop on “Operator related function theory”, 2012 (web.math.utk.edu/~richter/talk/), <https://web.math.utk.edu/~richter/talk/Richter-Cyclic-Drury-Arveson2012.pdf>.
- [52] S. Richter, J. Sunkes, *Hankel operators, invariant subspaces, and cyclic vectors in the Drury-Arveson space*, Proc. Amer. Math. Soc., 114:6 (2016), 2575–2586.
- [53] N. P. Romanov, *Hilbert space and the theory of numbers*, Izvestia Acad. Nauk SSSR, 10 (1946), 3–34 (Russian); MR0016390.
- [54] W. Rudin, *Function theory in polydiscs*, W. A. Benjamin, Inc, N.Y.-Amsterdam, 1969.

- [55] W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$* , Springer-Verlag, Berlin-N.Y., 1980.
- [56] O. Shalit, *Operator theory and function theory in Drury-Arveson space and its quotients*, in “Operator Theory, Adv. and Appl.” (ed.: D.Alpay), pp. 1–50, Springer, Basel, 2014.
- [57] A. Shields, *Weighted shift operators and analytic function theory*, pp. 49–128 in: *Topics in operator theory*, ed. C. Pearcy, Math. Surveys No 13, Amer. Math. Soc., 1974, Providence.
- [58] https://en.wikipedia.org/wiki/Vandermonde%27s_identity
- [59] J. Wermer, *On a class of normed rings*, Ark. Mat. 2:6 (1954), 537–551.
- [60] A. Wintner, *Diophantine approximation and Hilbert’s space*, Amer. J. Math., 66 (1944), 564–578.
- [61] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Springer, N.Y., 2005.