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Negative Powers of Contractions Having a Strong AA^+ Spectrum

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The following article is the last article written by Jean Esterle who died on May 6, 2023. It was written with great affection and respect for Mohamed who wrote his thesis with Jean as director. The creation of this article was one of the things that kept Jean so aware and alive as he fought a vicious inoperable brain cancer. One of the topics that was visited often in Jean's mind and research was the story of the Wiener algebra A its subalgebra A^+ . He loved the beautiful classical set theory that came up when trying to characterize ideals or operators (and the rate of growth of the norms of their powers) whose associated zero sets or spectrum was found in a certain type of set. He brought all of this to life for Mohamed - and me - and many other people - and it is one of the legacies he will leave in this world!

By Elizabeth Strouse.

ABSTRACT. Zarrabi proved in 1993 that if the spectrum of a contraction T on a Banach space is a countable subset of the unit circle $\mathbb{T}$, and if $\lim_{n \rightarrow +\infty} \frac{\log(\|T^{-n}\|)}{\sqrt{n}} = 0$, then T is an isometry, so that $\|T^n\| = 1$ for every $n \in \mathbb{Z}$. It is also known that if $\mathcal{C}$ is the usual triadic Cantor set then every contraction T on a Banach space such that $\text{Spec}(T) \subset \mathcal{C}$ satisfying $\limsup_{n \rightarrow +\infty} \frac{\log(\|T^{-n}\|)}{n^\alpha} < +\infty$ for some $\alpha < \frac{\log(3) - \log(2)}{2\log(3) - \log(2)}$ is an isometry.

In the other direction an easy refinement of known results shows that if a closed $E \subset \mathbb{T}$ is not a "strong AA^+ -set" then for every sequence $(u_n)_{n \geq 1}$ of positive real numbers such that $\liminf_{n \rightarrow +\infty} u_n = +\infty$ there exists a contraction T on some Banach space such that $\text{Spec}(T) \subset E$, $\|T^{-n}\| = O(u_n)$ as $n \rightarrow +\infty$ and $\sup_{n \geq 1} \|T^{-n}\| = +\infty$.

We show conversely that if $E \subset \mathbb{T}$ is a strong AA^+ -set then there exists a nondecreasing unbounded sequence $(u_n)_{n \geq 1}$ such that for every contraction T on a Banach space satisfying $\text{Spec}(T) \subset E$ and $\|T^{-n}\| = O(u_n)$ as $n \rightarrow +\infty$ we have $\sup_{n > 0} \|T^{-n}\| \leq K$, where $K < +\infty$ denotes the " AA^+ -constant" of E (closed countable subsets of $\mathbb{T}$ and the triadic Cantor set are strong AA^+ -sets of constant 1).

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1. Introduction

Zarrabi showed in [15] that if the spectrum of a contraction T on a Banach space is a countable subset of the unit circle $\mathbb{T}$, and if $\lim_{n \rightarrow +\infty} \frac{\log \|T^{-n}\|}{\sqrt{n}} = 0$, then T is an isometry, and $\|T^n\| = 1$ for every $n \in \mathbb{Z}$. He also showed in [15], using singular inner functions, that if a closed set $E \subset \mathbb{T}$ of Lebesgue measure 0 is uncountable, then there exists a contraction T on some Banach space such that $\text{Spec}(T) \subset E$ and $\lim_{n \rightarrow +\infty} \frac{\log \|T^{-n}\|}{\sqrt{n}} = 0$ while $\sup_{n \geq 1} \|T^{-n}\| = +\infty$. It is easy to see that his arguments show that the compression of multiplication by z on the Hardy space H^2 to the model space associated with any singular inner function defined by a singular measure on $\mathbb{T}$ with support in E has this property.

Let $\mathcal{C}_{1/p} = \left\{ \exp(2i\pi(1 - 1/p) \sum_{j=1}^{+\infty} \epsilon_j p^{-j+1}), \epsilon_j \in \{0; 2\} \right\} \subset \mathbb{T}$ be the Cantor set associated with $1/p$,

where $p \geq 3$ is an integer. It was shown in [5] that if the spectrum of a contraction is contained in $\mathcal{C}_{1/p}$, and if $\log \|T^{-n}\| = O(n^\alpha)$ for some $\alpha < \frac{\log(p) - \log(2)}{2 \log(p) - \log(2)}$, then T is an isometry, and it was shown in [7] that the constant $\frac{\log(p) - \log(2)}{2 \log(p) - \log(2)}$ is sharp. Also Kellay obtained in [13] analogous results for contractions on Hilbert space with spectrum contained in a Carleson set.

Let $A(\mathbb{T})$ be the algebra of absolutely convergent Fourier series (the Wiener algebra), and let $A^+(\mathbb{T})$ be the algebra of restrictions to $\mathbb{T}$ of absolutely convergent Taylor series. If $E \subset \mathbb{T}$ is closed denote by $I(E)$ the space of functions $f \in A(\mathbb{T})$ vanishing on E , by $J_0(E)$ the space of all functions $f \in A(\mathbb{T})$ vanishing on some neighbourhood of E , and set $J(E) := \overline{J_0(E)}$, $I^+(E) := I(E) \cap A^+(\mathbb{T})$, $J^+(E) := J(E) \cap A^+(\mathbb{T})$. A closed set $E \subset \mathbb{T}$ is said to be set of synthesis if $I(E) = J(E)$, and it is said to be a AA^+ -set (resp. a strong AA^+ -set) if for every $f \in A(\mathbb{T})$ there exists $g \in A^+(\mathbb{T})$ such that $f - g \in I(E)$ (resp. $f - g \in J(E)$). Closed countable subsets of $\mathbb{T}$ and the Cantor sets $\mathcal{C}_{\frac{1}{p}}$ are examples of AA^+ -sets satisfying synthesis, and so they are strong AA^+ -sets.

The results mentioned above for closed countable subsets of $\mathbb{T}$ and for the Cantor sets $\mathcal{C}_{1/p}$ suggest to study more generally closed sets $E \subset \mathbb{T}$ for which there exists an unbounded nondecreasing sequence $(u_n)_{n \geq 1}$ of positive real numbers having the following property

$\sup_{n \geq 1} \|T^{-n}\| < +\infty$ for every contraction T on a Banach space such that $\text{Spec}(T) \subset E$ and $\|T^n\| = O(u_n)$ as $n \rightarrow +\infty$ (1).

It was shown in [5] that such sets E are necessarily strong AA^+ -sets if the sequence $(u_n)_{n \geq 1}$ is submultiplicative and satisfies $\sum_{n=1}^{+\infty} \frac{\log(u_n)}{n^2} < +\infty$. An easy argument shows that this result remains valid if we only assume that the sequence $(u_n)_{n \geq 1}$ satisfies $\liminf_{n \rightarrow +\infty} u_n = +\infty$.

The main result of the paper is that, conversely, there exists for every strong AA^+ -set E an unbounded nondecreasing sequence $(u_n)_{n \geq 1}$ of positive real numbers which satisfies (1) with respect to E .

The pairing $\langle u, f \rangle = \sum_{n=0}^{+\infty} u_n \frac{f^{(n)}(0)}{n!}$, $u \in c_0$, $f \in A^+(\mathbb{T})$, allows to identify $A^+(\mathbb{T})$ with the dual space of $c_0 := \{u = (u_n)_{n \geq 0} \in \mathbb{C}^{\mathbb{N}} \mid \lim_{n \rightarrow +\infty} u_n = 0\}$. The proof of the main result uses the fact that strong AA^+ -sets are those closed sets $E \subset \mathbb{T}$ for which $J^+(E)$ is w^* -sequentially dense in $A^+(\mathbb{T})$ with respect to the w^* -topology on $A^+(\mathbb{T})$ associated with this duality.

The results of the paper extend immediately to power bounded operators since such operators are similar to contractions.

More generally it is also interesting to consider sets E such that some growth conditions on the sequences $(\|T^n\|)_{n \geq 1}$ and $(\|T^{-n}\|)_{n \geq 1}$ for an operator T with spectrum contained in E imply better estimates on the growth of the sequence $(\|T^{-n}\|)_{n \geq 1}$, see [1, 2, 3, 16].

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2. Negative powers of contractions and strong AA^+ -sets

In order to use the results of [5, 6] we need to introduce some spaces of functions and hyperfunctions. A hyperfunction on $\mathbb{T}$ is by definition an analytic function ϕ on $\mathbb{C} \setminus \mathbb{T}$ such that $\lim_{|z| \rightarrow +\infty} \phi(z) = 0$, and $HF(\mathbb{T})$ will denote the space of all hyperfunctions on $\mathbb{T}$. For $\phi \in HF(\mathbb{T})$ we set $\phi^+ := \phi|_{\mathbb{D}}$ and $\phi^- := \phi|_{\mathbb{C} \setminus \mathbb{D}}$, where $\mathbb{D}$ denotes the open unit disc. The Fourier coefficients of $\phi \in HF(\mathbb{T})$ are defined by the formulae

$$\begin{cases} \phi^+(z) = \sum_{n=1}^{+\infty} \widehat{\phi}(n) z^{n-1} & |z| < 1, \\ \phi^-(z) = - \sum_{n \leq 0} \widehat{\phi}(n) z^{n-1} & |z| > 1. \end{cases}$$

We set $HF^0(\mathbb{T}) := \{\phi \in HF(\mathbb{T}) \mid \lim_{n \rightarrow -\infty} \widehat{\phi}(n) = 0\}$.

If $\phi \in HF(\mathbb{T})$, then $z \in \mathbb{T}$ is said to be ϕ -regular if there exists a neighborhood U of z and a function ψ holomorphic on U such that $\phi(z) = \psi(z)$ for $z \in U \setminus U \cap \mathbb{T}$. The set of ϕ -regular points is open, and its complement $Supp(\phi)$ in $\mathbb{T}$ is called the support of ϕ . If $E \subset \mathbb{T}$ is closed then $HF(E)$ will denote the set of all $\phi \in HF(\mathbb{T})$ such that $Supp(\phi) \subset E$.

Set

$$PM(\mathbb{T}) := \{\phi \in HF(\mathbb{T}) \mid \sup_{n \in \mathbb{Z}} |\widehat{\phi}(n)| < +\infty\},$$

$$PF(\mathbb{T}) := \{\phi \in HF(\mathbb{T}) \mid \lim_{n \rightarrow \pm\infty} \widehat{\phi}(n) = 0\}.$$

Elements of $PM(\mathbb{T})$ (resp. $PF(\mathbb{T})$) are called pseudomeasures (resp. pseudofunctions), and we set

$$PM(E) := PM(\mathbb{T}) \cap HF(E) \text{ and } PF(E) := PF(\mathbb{T}) \cap HF(E).$$

A closed set $E \subset \mathbb{T}$ is called a set of uniqueness (resp. multiplicity) when $PF(E) = \{0\}$ (resp. $PM(E) \neq \{0\}$).

We will need the following notions.

Definition 2.1. A dissymmetric (bilateral) regular weight is a submultiplicative sequence $\omega = (\omega(n))_{n \in \mathbb{Z}}$ of positive real numbers such that $\omega(n) = 1$ for $n \geq 0$ and $\sum_{n < 0} \frac{\log(\omega(n))}{n^2} < +\infty$.

Denote by $\mathcal{C}(\mathbb{T})$ the algebra of continuous functions on $\mathbb{T}$. We introduce the following classical objects.

Definition 2.2. Let ω be a dissymmetric regular weight. Set

$$A_\omega(\mathbb{T}) := \{f \in \mathcal{C}(\mathbb{T}) \mid \|f\|_\omega := \sum_{n \in \mathbb{Z}} |\widehat{f}(n)| \omega(n) < +\infty\},$$

$$HF_\omega(\mathbb{T}) := \{\phi \in HF(\mathbb{T}) \mid \|\phi\|_\omega^* := \sup_{n \in \mathbb{Z}} \frac{|\widehat{\phi}(n)|}{\omega(-n)} < +\infty\}.$$

Also if $E \subset \mathbb{T}$ is closed, set $I_\omega(E) := \{f \in A_\omega(\mathbb{T}) \mid f|_E = 0\}$, denote by $J_\omega(E)$ the closure in $A_\omega(\mathbb{T})$ of the set of functions in $A_\omega(\mathbb{T})$ vanishing on some neighbourhood of E , set $HF_\omega(E) := HF_\omega(\mathbb{T}) \cap HF(E)$ and set $HF_\omega^0(E) := HF_\omega(\mathbb{T}) \cap HF^0(E)$,

Denote by $z : \zeta \rightarrow \zeta$ the identity map on $\mathbb{T}$. We have $f = \sum_{n=-\infty}^{+\infty} \widehat{f}(n)z^n$ for $f \in A_\omega(\mathbb{T})$, the series being convergent in $(A_\omega(\mathbb{T}), \|\cdot\|_\omega)$, and $\widehat{z^p f}(n) = \widehat{f}(n-p)$ for $p \in \mathbb{Z}$.

The space $HF_\omega(\mathbb{T})$ can be identified with the dual space of $A_\omega(\mathbb{T})$ by using for $f \in A_\omega(E)$, $\phi \in HF_\omega(E)$ the pairing

$$\langle f, \phi \rangle = \sum_{n \in \mathbb{Z}} \widehat{f}(n) \widehat{\phi}(-n) = \frac{1}{2\pi i} \lim_{r \rightarrow 1^-} \int_{\mathbb{T}} f(\zeta) [\phi^+(r\zeta) - \phi^-(r^{-1}\zeta)] d\zeta,$$

and we have

$$\widehat{\phi}(n) = \langle z^{-n}, \phi \rangle \quad (\phi \in HF_\omega(\mathbb{T})).$$

For $\phi \in HF(\mathbb{T})$, $p \in \mathbb{Z}$, define $z^p \phi \in HF(\mathbb{T})$ by the formula

$$\widehat{z^p \phi}(n) = \widehat{\phi}(n+p).$$

Then $z^p \phi \in HF^0(\mathbb{T})$ for $\phi \in HF^0(\mathbb{T})$. Also $z^p \phi \in HF_\omega(\mathbb{T})$ for $\phi \in HF_\omega(\mathbb{T})$, and we have

$$\langle f, z^p \phi \rangle = \langle z^p f, \phi \rangle \quad (f \in A_\omega(\mathbb{T})).$$

Since $\sum_{n < 0} \frac{\log(\omega(n))}{n^2} < +\infty$, the algebra $A_\omega(\mathbb{T})$ is a regular Banach algebra, which means that for every closed set $E \subset \mathbb{T}$ and every $\eta \in \mathbb{T} \setminus E$, there exists $f \in A_\omega(\mathbb{T})$ such that $f(\eta) = 0$ and $f(\zeta) = 1$ for every $\zeta \in E$, and we have, for every closed set $E \subset \mathbb{T}$,

$$HF_\omega(E) = J_\omega(E)^\perp := \{\phi \in HF(\mathbb{T}) \mid \langle f, \phi \rangle = 0 \forall f \in J_\omega(E)\}.$$

References for these well-known results can be found for example in [5] and [15].

Since $J_\omega(E)$ is an ideal of $A_\omega(\mathbb{T})$, this implies that $z^p \phi \in HF_\omega(E)$ (resp. $HF_\omega^0(E)$) for $\phi \in HF_\omega(E)$ (resp. $HF_\omega^0(E)$), $p \in \mathbb{Z}$ for every closed set $E \subset \mathbb{T}$.

If w is an unbounded dissymmetric regular weight, then there exists a singular inner function U such that the ideal generated by U is not dense in $A_\omega(\mathbb{T})$ [4], which makes the closed ideal structure of $A_\omega(\mathbb{T})$ very complicated. In fact there is no such weight for which a classification of closed ideals of $A_\omega(\mathbb{T})$ is known..

In the other direction there are submultiplicative unbounded dissymmetric weights ω satisfying $\sum_{n < 0} \frac{\log(\omega(n))}{n^2} = +\infty$ for which all closed ideals of $A_\omega(\mathbb{T})$ are generated by singular inner functions, see [10].

3. Main results

As mentioned in the introduction, a closed set $E \subset \mathbb{T}$ is said to be a strong AA^+ -set if for every $f \in A(\mathbb{T})$ there exists $g \in A^+(\mathbb{T})$ such that $f - g$ satisfies synthesis with respect to E .

The following results were obtained in [5].

Theorem 3.1. *Let E be a closed subset of $\mathbb{T}$, and let ω be an unbounded dissymmetric regular weight. Then the following are equivalent:*

- (i) $HF_\omega(E) = PM(E)$,
- (ii) For every Banach space X and for every contraction T on X such that $Sp(T) \subset E$ and $\|T^{-n}\| = O(\omega(-n))$ as $n \rightarrow +\infty$ we have $\sup_{n \geq 1} \|T^{-n}\| < +\infty$,
- (iii) $\sup_{n \geq 1} \|\pi_\omega(z)^{-n}\| < +\infty$, where $\pi_\omega = A_\omega(\mathbb{T}) \rightarrow A_\omega(\mathbb{T})/J_\omega(E)$ denotes the quotient map.
- (iv) E is a strong AA^+ -set, and $HF_\omega^0(E) = \{0\}$,
- (v) E is a strong AA^+ -set, and $J_\omega(E) = J(E) \cap A_\omega(\mathbb{T})$,
- (vi) E is a strong AA^+ -set, and $J^+(E) = J_\omega(E) \cap A^+(\mathbb{T})$.

Let $E \subset \mathbb{T}$ be a strong AA^+ -set, and let $\pi : A^+(\mathbb{T}) \rightarrow A^+(\mathbb{T})/J^+(E)$ be the quotient map. Then $\pi(z)$ is invertible in $A^+(\mathbb{T})/J^+(E)$, $\sup_{n \geq 1} \|\pi(z)^{-n}\| < +\infty$ and the constant $K := \sup_{n \geq 1} \|T^{-n}\|$ is called the AA^+ constant of E . It is showed more precisely in [5] that if a dissymmetric weight ω satisfies the conditions of the theorem then $\sup_{n \geq 1} \|T^{-n}\| \leq K$ for every contraction T such that $Spec(T) \subset E$ and $\|T^{-n}\| = O(\omega(-n))$ as $n \rightarrow +\infty$. We also have $\|\phi\|_\omega^* \leq K \limsup_{n \rightarrow +\infty} |\hat{\phi}(n)|$ for every $\phi \in HF_\omega(E)$.

We have the following trivial observation.

Proposition 3.2. *Let $(u_n)_{n \geq 1}$ be a sequence of positive real numbers such that $\liminf_{n \rightarrow +\infty} u_n = +\infty$. Then there exists a dissymmetric unbounded regular weight ω such that $\sup_{n \geq 1} \frac{\omega(-n)}{u_n} < +\infty$.*

Proof: Set $v_n = \min(u_n, e^{\sqrt{n}})$, so that $\sum_{n=1}^{+\infty} \frac{\log(v_n)}{n^2} < +\infty$. We now use a standard argument (see for example Lemma 5.1 of [4]). With the convention $n_0 = 1$, construct by induction a strictly increasing sequence $(n_p)_{p \geq 0}$ of positive integers such that $n_{p+1} - n_p > n_p - n_{p-1}$ and $\log(v_n) \geq p+1$ for $n \geq n_p$, $p \geq 1$. Let θ be the continuous function on $[0, +\infty)$ which is affine on $[n_p, n_{p+1}]$ and satisfies $\theta(n_p) = p$ for $p \geq 0$. Then θ is concave, so $\theta(m+n) - \theta(m) \leq \theta(n) - \theta(0) = \theta(n)$ for $m, n \geq 0$, and θ is subadditive. So the weight ω defined by the formulae $\omega(n) = 1$ for $n \geq 0$ and $\omega(n) = e^{\theta(-n)}$ for $n < 0$ satisfies the required conditions. $\square$

Let $E \subset \mathbb{T}$ be a closed set which is not a strong AA^+ -set. It follows from this observation for every sequence $(u_n)_{n \geq 1}$ of positive real numbers such that $\liminf_{n \rightarrow +\infty} u_n = +\infty$ there exists a Banach space X and a contraction $T : X \rightarrow X$ such that $Spec(T) \subset E$, $\|T^{-n}\| = O(u_n)$ and $\sup_{n \geq 1} \|T^{-n}\| = +\infty$. There exists also a nonzero hyperfunction $\phi \in HF^0(E)$ such that $\hat{\phi}(n) = O(u_n)$ as $n \rightarrow +\infty$.

We now turn to the main result of the paper, which shows that, conversely, there exists for every closed AA^+ -set $E \subset \mathbb{T}$ a sequence $(u_n)_{n \geq 1}$ of positive reals satisfying $\lim_{n \rightarrow +\infty} u_n = +\infty$ such that $\sup_{n \geq 1} \|T^{-n}\| \leq K$ for every contraction T on a Banach space such that $Spec(T) \subset E$ and $\|T^{-n}\| = O(u_n)$ as $n \rightarrow +\infty$, where $K < +\infty$ denotes the strong AA^+ constant of E .

Theorem 3.3. *For every closed strong AA^+ -set $E \subset \mathbb{T}$ there exists an unbounded dissymmetric weight satisfying the equivalent conditions of theorem 3.1 with respect to E .*

Proof: Denote by $\|\cdot\|_1$ the usual norm of $A(\mathbb{T})$. Since $A(\mathbb{T})$ is an ℓ^1 -space we can identify $A(\mathbb{T})$ with the dual of $PF(\mathbb{T})$, which induces a w^* -topology on $A(\mathbb{T})$. It follows to a characterization of strong AA^+ -sets given in [5], which is analogous to a characterization of AA^+ -sets given by Kahane and Katznelson in [11], that $J^+(E)$ is w^* -dense in $A(\mathbb{T})$. Let $(f_p)_{p \geq 1}$ be a sequence of elements of $J^+(E)$ such that $\lim_{p \rightarrow +\infty} f_p = 1$ with respect to the w^* -topology, and let $(g_p)_{p \geq 1}$ be a sequence of elements of $A(\mathbb{T})$ such that $\text{Supp}(g_p) \cap E = \emptyset$ and $\|f_p - g_p\|_1 \leq 2^{-p}$ for $p \geq 1$.

Set $a_n := \sup_{n \geq 1} |\widehat{g}_p(-n)|$ pour $n \geq 1$. Since $\sum_{n \leq -1} |\widehat{g}_p(n)| \leq \|g_p - f_p\|_1 \leq 2^{-p}$, we have

$$\sum_{n=1}^{+\infty} a_n \leq \sum_{n \leq -1, p \geq 1} |\widehat{g}_p(n)| = \sum_{p=1}^{+\infty} \sum_{n \leq -1} |\widehat{g}_p(n)| \leq 1.$$

Since the series $\sum_{n=1}^{+\infty} a_n$ is convergent, a standard elementary argument shows that there exists an unbounded nondecreasing sequence $(u_n)_{n \geq 1}$ of positive real numbers such that $\sum_{n=1}^{+\infty} a_n u_n < +\infty$, and so there exists an unbounded dissymmetric regular weight ω such that $\sum_{n=1}^{+\infty} a_n \omega(-n) < +\infty$. It follows from the definition of the sequence $(a_n)_{n \geq 1}$ that $\sum_{n=1}^{+\infty} |\widehat{g}_p(-n)| \omega(-n) < +\infty$, so that $g_p \in A_\omega(\mathbb{T})$ for $p \geq 1$. For $N \geq 1$ we have

$$\|f_p - g_p\|_\omega \leq \|f_p - g_p\|_1 + 2^{-p} \sum_{n=1}^N \omega(-n) + \sum_{n=N+1}^{+\infty} a_n \omega(-n),$$

and so $\lim_{p \rightarrow +\infty} \|f_p - g_p\|_\omega = 0$, and $\lim_{p \rightarrow +\infty} \langle f_p - g_p, \phi \rangle = 0$ for every $\phi \in HF_\omega(\mathbb{T})$.

Let $\phi \in HF_\omega^0(E)$. Since the sequence $(f_p)_{p \geq 1}$ converges to 1 with respect to the w^* -topology on $A^+(\mathbb{T})$, and since $g_p \in J_\omega(E)$ for $p \geq 1$, we have

$$\langle 1, \phi \rangle = \lim_{p \rightarrow +\infty} \langle f_p, \phi \rangle = \lim_{p \rightarrow +\infty} (\langle f_p - g_p, \phi \rangle + \langle g_p, \phi \rangle) = 0.$$

Since $z^{-n}\phi \in HF_\omega^0(E)$, we obtain, for $n \in \mathbb{Z}$,

$$\widehat{\phi}(n) = \langle z^{-n}, \phi \rangle = \langle 1, z^{-n}\phi \rangle = 0,$$

and so $\phi = 0$. Hence $HF_\omega^0(E) = \{0\}$, which shows that ω satisfies the equivalent conditions of Theorem 3.1 with respect to E . $\square$

The algebra $A^+(\mathbb{T})$ provides a functional calculus for contractions on general Banach spaces, and Theorem 3.1 is a byproduct of the research on closed ideals of $A^+(\mathbb{T})$ pursued at University of Bordeaux in the early nineties [5, 8, 9].

The situation is different if we restrict attention to contractions on Hilbert space, because such contractions are polynomially bounded and the disc algebra $\mathcal{A}(\mathbb{D})$ provides a functional calculus for them. For example set, for $\zeta \in (0, 1/2)$

$$\mathcal{C}_\zeta := \left\{ \exp(2i\pi(1 - \zeta) \sum_{j=1}^{+\infty} \epsilon_j \zeta^{j-1}, \epsilon_j \in \{0, 2\}) \right\}.$$

If $p \geq 3$, then $E_{1/p}$ is a strong AA^+ -set of AA^+ constant 1, and it is shown in [5] that every contraction T such that $\text{Spec}(T) \subset \mathcal{C}_{1/p}$ and $\log \|T^{-n}\| = O(n^\alpha)$ for some $\alpha < \frac{\log(p)-\log(2)}{2\log(p)-\log(2)}$ is an isometry. In the other direction $\mathcal{C}_\zeta$ is a set of multiplicity if ζ is not a Pisot number, so it is not a strong AA^+ -set and there exists for every sequence $(u_n)_{n \geq 1}$ such that $\liminf_{n \rightarrow +\infty} u_n = +\infty$ a contraction T on some Banach space of spectrum contained in $\mathcal{C}_\zeta$ satisfying $\|T^{-n}\| = O(u_n)$ and $\sup_{n \geq 1} \|T^{-n}\| = +\infty$.

Nevertheless it is shown in [5] that if $\zeta \in (0, 1/2)$ then every contraction T on a Hilbert space such that $\text{Spec}(T) \subset \mathcal{C}_\zeta$ and $\log \|T^{-n}\| = O(n^\alpha)$ for some $\alpha < \frac{\log(\zeta)-\log(2)}{2\log(\zeta)-\log(2)}$ is unitary. Properties analogous to those described in Theorem 3.1 and Theorem 3.3 for Hilbert space contractions will be discussed elsewhere.

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