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Generalized functional inequalities in Banach spaces

H. Dimou[1], Y. Aribou [2], and S. Kabbaj[3]

Abstract. In this paper, we solve and investigate the generalized additive functional inequalities

$$\Big\|F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)\Big\| \le \Big\|F\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) - \frac{1}{n}\sum_{i=1}^{n} F(x_i)\Big\| \tag{0.1}$$

and

$$\Big\|F\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) - \frac{1}{n}\sum_{i=1}^{n} F(x_i)\Big\| \le \Big\|F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)\Big\|. \tag{0.2}$$

Using the direct method, we prove the Hyers-Ulam stability of the functional inequalities (0.1) in Banach spaces and (0.2) in non-Archimedian Banach spaces.

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[1]*Department of Mathematics, Faculty of Sciences Applied, Ibn Zohr University, Agadir, Morocco,*
e-mail: dimouhajira@gmail.com

[2]*Department of Mathematics and Computer Science, Technology High School of Technology, Ibn Zohr University, Laayoune, Morocco,*
e-mail:aribouyoussef3@gmail.com

[3]*Department of Mathematics, Faculty of Sciences, Ibn Tofail University, Kenitra, Morocco,*
e-mail:samkabbaj@yahoo.fr.

1. Introduction

The theory of functional equations is an ancient subject that has taken about fifty years of relevant research and a large number of mathematicians have invested in this field by publishing a large number of articles and books leaving open questions. The study of the problem of the stability of functional equations was initiated by a problem posed by Ulam [33] in a high-profile conference at the university of Wisconsin, to share a number of important unresolved problems, one of the mysterious questions and that of the stability of group homomorphisms. The relevant question asked is: Under which conditions a group homomorphism could exist being close to an approximate group homomorphism?
In other words:

Ulam's Problem 1940: *Let* $(G_1, *)$ *be a group and let* $(G_2, \diamond, d)$ *be a metric group with a metric* $d(.,.)$.
Given $\varepsilon > 0$, *does there exist a* $\delta(\varepsilon) > 0$ *such that if a mapping* $h : G_1 \to G_2$ *satisfies the inequality*

$$d\big(h(x * y), h(x) \diamond h(y)\big) < \delta \text{ for all } x, y \in G_1,$$

then there exists a homomorphism $H : G_1 \to G_2$ *with* $d\big(h(x), H(x)\big) < \varepsilon$ *for all* $x \in G_1$*?*
The concept of the stability of functional equations can be summed up by the fact that if we replace the functional equation by an inequality, the inequality will play the role of a perturbation of the equation, so the question of the stability of functional equations and that despite the perturbation caused is it that we will have the same solutions? If the answer is affirmative, then the equation is said to be stable.
In 1980, several results were proven on stability in the sense of Hyers-Ulam-Rassias on many equations and this by a large number of researchers for more details refer for example to [11, 15, 16, 27].

In 1941, Hyers [14] partially solved the Ulam's problem for the case of approximately additive functions of Cauchy under of assumptions, (the context of Banach spaces with $\delta = \epsilon$)
In fact, Hyers proved the following theorem:

Theorem 1.1 ([14], D. H. Hyers (1941))**.** *Let* $f : X \to Y$ *be a function where* X *and* Y *are Banach spaces, such that*

$$||f(x+y) - f(x) - f(y)|| \le \epsilon,$$

for all $x, y \in X$. *Then there exists a unique additive function* A *satisfying inequality*

$$||f(x) - A(x)|| \le \epsilon$$

for all $x \in X$.

Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then the function A is Linear.

The generalization of Hyers' theorem has been established by Aoki [3] for the additive functions, these last results have been improved by Rassias [26] he provided a remarkable generalization of Hyers' theorem for linear mappings by assuming the unbounded Cauchy difference. The approach taken by Rassias in his paper [26] above has greatly influenced the evolution of

Hyers-Ulam stability. Using a general control of the function inspired by Rassias instead of the unbounded Cauchy difference method, in 1994 Găvruta [12] has established the generalized Hyers-Ulam-Rassias stability. In the last few years, various problems concerning the stability of functional equations have been studied and developed by a large number of mathematicians.
Among the functional equations very well known in the literature we find Cauchy's one, this equation verifies

$$A(x+y) = A(x) + A(y).$$

In particular any solution of the Cauchy equation is called an additive function.
In [34] A. L. Cauchy solved this equation for the first time, to honor Cauchy the equation bears this name. The main foundations of the theory of additive functional equations are the basis for the development of the theories of the rest of the functional equations.

In 2015, Park [25] obtained the solutions of the additive functional inequality

$$\|F(x+y) - F(x) - F(y)\| \leq \left\| F\left(\frac{x+y}{2}\right) - \frac{1}{2}F(x) - \frac{1}{2}F(y) \right\|$$

and proved the Hyers-Ulam stability of this additive functional inequality in Banach spaces.

Also, Park gave the solutions of the additive functional inequality

$$\left\| F\left(\frac{x+y}{2}\right) - \frac{1}{2}F(x) - \frac{1}{2}F(y) \right\| \leq \|F(x+y) - F(x) - F(y)\|$$

and proved the Hyers-Ulam stability of this additive functional inequality in non-Archimedean Banach spaces.
The above equations are the main motivation for our work.

In 1908, K. Hensel [13] discovered a new concept of numbers called p-adics to solve problems related to algebra and number theory. More precisely his idea was to make an extension the analogies between the ring of integers $\mathbb{Z}$ and the field of rational numbers $\mathbb{Q}$ to the field of rational functions and power series in complex analysis and in 1897, he gave birth to a normed space not satisfying the Archimedean property called non-Archimedean space. This one was able to give birth to a new example on non-Archimedian numbers called p-adic numbers:
Given that p a prime number. For every nonzero rational number $a = p^k \frac{m}{n}$ such that m and n are coprime to the prime number p, define the p-adic absolute value $|a|_p = p^{-k}$. Thus $|.|$ is a non-Archimedean norm on $\mathbb{Q}$. Respecting $|.|$ the completion of $\mathbb{Q}$ is denoted by $\mathbb{Q}_p$ and is called the p-adic number field.

During the last decades, the theory of non-Archimedian spaces has attracted the attention of physicists in their research on problems related to quantum physics, p-adic strings and superstrings [19]. Even if many results in classical normalized space theory have a non-Archimedean counterpart, their different demonstrations are and require a new intuition [23, 24].

The stability of functional equations in non-Archimedean Banach spaces have been studied and developed by Moslehian and Rassias [20], Moslehian and Sadeghi [21, 22], Mirmostafaee [23] and Najati and Moradlou [24].

In this paper the main motivation for our work is the idea of Park [25], we prove the Hyers-Ulam stability of generalized additive functional inequalities and functional equations using the direct method.

This work is organized as follows: In Section 2, we introduce the definitions and terminologies that will be used throughout our paper.
In Section 3, first we determine solutions of the additive functional inequality (0.1) and second we prove the Hyers-Ulam stability of the additive functional inequality (0.1) in Banach spaces. In Section 4, first we present the solution of the additive functional inequality (0.2) and second we prove the Hyers-Ulam stability of the additive functional inequality (0.2) in non-Archimedean Banach spaces.

2. Preliminaries

In this section, we will introduce some notions and notations.

Definition 2.1 ([13]). *A non-Archimedean field is a field K equipped with a function (valuation) $|.|$ from K into $[0, \infty)$ such that $|x| = 0$ if and only if $x = 0$, $|xy| = |x||y|$, and $|x + y| \leq max\{|x|, |y|\}$ for all $x, y \in K$.*

Example 2.1 ([13]). *An example of a non-Archimedean valuation is the mapping $|.|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial.*

The p-adic number field is one of the most important examples of non-Archimedean fields ([13]), which have gained the interest of physicists for their research in some problems coming from quantum physics, p-adic strings and superstrings.
Throughout this work, we assume that the base field is a non-Archimedean field and hence call it simply a field.

Definition 2.2 ([20]). *Let X be a vector space over a field K with a non-Archimedean non-trivial valuation $|.|$. A function $||.|| : X \to [0, \infty)$ is said to be a non-Archimedean norm (valuation) if it satisfies the following conditions:*

(i) $||x|| = 0$ if and only if $x = 0$; for all $x \in X$;
(ii) $||kx|| = |k|||x||$ ($k \in K$, $x \in X$);
(iii) the strong triangle inequality; namely,

$$||x_1 + x_2|| \leq max\{||x_1||, ||x_2||\}, \forall x_1, x_2 \in X.$$

Then $(X, ||.||)$ is called a non-Archimedean normed space.

Definition 2.3 ([20]). *Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.*

(i) A sequence $\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is a Cauchy sequence if for a given $\epsilon > 0$ there is a positive integer N such that

$$||x_n - x_m|| \leq \epsilon,$$

for all $n, m \geq N$.

(ii) *Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called convergent if for a given $\epsilon > 0$ there are a positive integer N and an $x \in X$ such that*

$$||x_n - x|| \leq \epsilon,$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) *If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space.*

3. Additive functional inequalities in Banach spaces

In this section, we will assume that X_1 is a Banach space with norm $\|.\|$ and that X_2 is a Banach space with norm $\|.\|$.

The following lemma is an ingredient to prove the results of this section.

Lemma 3.1. *A mapping $F : X_1 \to X_2$ satisfies*

$$\Big|\Big|F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)\Big|\Big| \leq \Big|\Big|F\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) - \frac{1}{n}\sum_{i=1}^{n} F(x_i)\Big|\Big| \tag{3.1}$$

for all $x_1, \cdots, x_n \in X_1$ if and only if $F : X_1 \to X_2$ is additive.

Proof. Assume that $F : X_1 \to X_2$ satisfies (3.1).
Putting $x_i = 0$ for all $i \in [1, n]$, in (3.1), we obtain

$$||F(0)|| \leq 0.$$

Thus $F(0) = 0$.
Putting $x_2 = -x_1$ and $x_3 = x_4 = x_5 = \cdots = x_n = 0$ in (3.1), we obtain

$$||F(x_1) + F(-x_1)|| \leq \frac{1}{n}||F(x_1) + F(-x_1)|| \tag{3.2}$$

for all $x_1 \in X_1$.
Hence, $F(x_1) = -F(-x_1)$ for all $x_1 \in X_1$.
Putting $x_1 = x_2 = x_3 = \cdots = x_n = x$ in (3.1), we obtain

$$||F(nx) - nF(x)|| \leq 0$$

and thus $F(nx) = nF(x)$ for all $x \in X_1$. Then $F\Big(\frac{x}{n}\Big) = \frac{1}{n}F(x)$ for all $x \in X_1$.
It follows from (3.1) that

$$\Big|\Big|F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)\Big|\Big| \leq \frac{1}{n}\Big|\Big|F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)\Big|\Big|$$

and thus

$$F\Big(\sum_{i=1}^{n} x_i\Big) = \sum_{i=1}^{n} F(x_i)$$

for all $x_i \in X_1$.
The converse is certainly true.

Now we demonstrate the Hyers-Ulam stability of additive functional inequality (3.1) in Banach spaces.

Theorem 3.1. *Let $\phi : X_1^n \to [0,\infty)$ be a function and let $F : X_1 \to X_2$ be a mapping satisfying:*

$$\Psi(x_1, x_2, x_3, \cdots, x_n) := \sum_{j=1}^{\infty} n^j \phi\Big(\frac{x_1}{n^j}, \frac{x_2}{n^j}, \frac{x_3}{n^j}, \cdots, \frac{x_n}{n^j}\Big) < \infty \tag{3.3}$$

$$\Big|\Big|F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)\Big|\Big| \leq \Big|\Big|F\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) - \frac{1}{n}\sum_{i=1}^{n} F(x_i)\Big|\Big| + \phi\Big(x_1, x_2, x_3, \cdots, x_n\Big) \tag{3.4}$$

for all $x_1, x_2, x_3, \cdots, x_n \in X_1$.
Then there exists a unique additive mapping $H : X_1 \to X_2$ such that

$$||F(x) - H(x)|| \leq \frac{1}{n}\Psi(x, x, x, \cdots, x\Big) \text{ for all } x \in X_1. \tag{3.5}$$

Proof. Putting $x = x_1 = x_2 = x_3 = \cdots = x_n$ in (3.4), we obtain

$$||F(nx) - nF(x)|| \leq \phi\Big(x, x, \cdots, x\Big) \tag{3.6}$$

for all $x \in X_1$. Then

$$\Big|\Big|F(x) - nF\Big(\frac{x}{n}\Big)\Big|\Big| \leq \phi\Big(\frac{x}{n}, \frac{x}{n}, \cdots, \frac{x}{n}\Big)$$

for all $x \in X_1$. Hence,

$$\begin{aligned} \Big|\Big|n^l F\Big(\frac{x}{n^l}\Big) - n^m F\Big(\frac{x}{n^m}\Big)\Big|\Big| &\leq \sum_{j=l}^{m-1} \Big|\Big|n^j F\Big(\frac{x}{n^j}\Big) - n^{j+1} F\Big(\frac{x}{n^{j+1}}\Big)\Big|\Big| \\ &\leq \frac{1}{n}\sum_{j=l+1}^{m} n^j \phi\Big(\frac{x}{n^j}, \frac{x}{n^j}, \frac{x}{n^j}, \cdots, \frac{x}{n^j}\Big) \end{aligned} \tag{3.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X_1$.

It follows from (3.7) that the sequence $\Big\{n^m F\Big(\frac{x}{n^m}\Big)\Big\}$ is a Cauchy sequence for all $x \in X_1$. Since X_2 is complete, the sequence $\Big\{n^m F\Big(\frac{x}{n^m}\Big)\Big\}$ converges. We can define the function $H : X_1 \to X_2$ by

$$H(x) := \lim_{m\to\infty} n^m F\Big(\frac{x}{n^m}\Big)$$

for all $x \in X_1$.
Furthermore, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.5).
It follows from (3.3) and (3.4) that

$$\begin{aligned} \Big|\Big|H\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} H(x_i)\Big|\Big| &= \lim_{m\to\infty} n^m \Big|\Big|F\Big(\sum_{i=1}^{n} \frac{x_i}{n^m}\Big) - \sum_{i=1}^{n} F\Big(\frac{x_i}{n^m}\Big)\Big|\Big| \\ &\leq \lim_{m\to\infty} n^m \Big|\Big|F\Big(\sum_{i=1}^{n} \frac{x_i}{n^{m+1}}\Big) - \frac{1}{n}\sum_{i=1}^{n} F\Big(\frac{x_i}{n^m}\Big)\Big|\Big| \end{aligned}$$

$$
\begin{aligned}
&+ \lim_{m\to\infty} n^m \phi\Big(\frac{x_1}{n^m}, \frac{x_2}{n^m}, \frac{x_3}{n^m}, \cdots, \frac{x_n}{n^m}\Big)\\
&= \Big|\Big|H\Big(\frac{1}{n}\sum_{i=1}^n x_i\Big) - \frac{1}{n}H\Big(\sum_{i=1}^n x_i\Big)\Big|\Big|
\end{aligned}
$$

for all $x_1, x_2, \cdots, x_n \in X_1$. Thus

$$\Big|\Big|H\Big(\sum_{i=1}^n x_i\Big) - \sum_{i=1}^n H(x_i)\Big|\Big| \le \Big|\Big|H\Big(\frac{1}{n}\sum_{i=1}^n x_i\Big) - \frac{1}{n}H\Big(\sum_{i=1}^n x_i\Big)\Big|\Big|$$

for all $x_1, x_2, \cdots, x_n \in X_1$. The mapping $H : X_1 \to X_2$ is additive, By Lemma 3.1.
Now, let $T : X_1 \to X_2$ be another additive mapping fulfilling (3.5), then we obtain

$$
\begin{aligned}
\Big|\Big|H(x) - T(x)\Big|\Big| &= n^m\Big|\Big|H\Big(\frac{x}{n^m}\Big) - T\Big(\frac{x}{n^m}\Big)\Big|\Big|\\
&\le n^m\Big(\Big|\Big|H\Big(\frac{x}{n^m}\Big) - F\Big(\frac{x}{n^m}\Big)\Big|\Big| + \Big|\Big|T\Big(\frac{x}{n^m}\Big) - F\Big(\frac{x}{n^m}\Big)\Big|\Big|\Big)\\
&\le n^m\Big(\frac{1}{n}\Psi\Big(\frac{x}{n^m}, \frac{x}{n^m}, \frac{x}{n^m}, \cdots, \frac{x}{n^m}\Big) + \frac{1}{n}\Psi\Big(\frac{x}{n^m}, \frac{x}{n^m}, \frac{x}{n^m}, \cdots, \frac{x}{n^m}\Big)\Big)\\
&\le n^m \times \frac{2}{n}\Psi\Big(\frac{x}{n^m}, \frac{x}{n^m}, \frac{x}{n^m}, \cdots, \frac{x}{n^m}\Big)\\
&\le n^m\Psi\Big(\frac{x}{n^m}, \frac{x}{n^m}, \frac{x}{n^m}, \cdots, \frac{x}{n^m}\Big)
\end{aligned}
$$

which tends to zero as $m \to \infty$ for all $x \in X_1$.
Thus we can conclude that $H(x) = T(x)$ for all $x \in X_1$. This demonstrates the uniqueness of H. Thus the mapping $H : X_1 \to X_2$ is a unique additive mapping fulfilling (3.5).

Corollary 3.1. *Let $k > 1$ and Θ be nonnegative real numbers, and let $F : X_1 \to X_2$ be a mapping satisfying:*

$$\Big|\Big|F\Big(\sum_{i=1}^n x_i\Big) - \sum_{i=1}^n F(x_i)\Big|\Big| \le \Big|\Big|F\Big(\frac{1}{n}\sum_{i=1}^n x_i\Big) - \frac{1}{n}\sum_{i=1}^n F\Big(x_i\Big)\Big|\Big| + \Theta\Big(\sum_{i=1}^n ||x_i||^k\Big) \tag{3.8}$$

for all $x_1, \cdots, x_n \in X_1$. Then there exists a unique additive mapping $H : X_1 \to X_2$ satisfying

$$||F(x) - H(x)|| \le \frac{n\Theta}{n^k - n}||x||^k$$

for all $x \in X_1$.

Theorem 3.2. *Let $\phi : X_1^n \to [0, \infty)$ be a function and let $F : X_1 \to X_2$ be a mapping satisfying (3.4) and*

$$\Psi(x_1, x_2, \cdots, x_n) := \sum_{j=0}^n \frac{1}{n^j}\phi(n^j x_1, n^j x_2, \cdots, n^j x_n) < \infty$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique additive mapping $H : X_1 \to X_2$ such that

$$||F(x) - H(x)|| \le \frac{1}{n}\Psi(x, x, \cdots, x) \tag{3.9}$$

for all $x \in X_1$.

Proof. It results from (3.6) that

$$\Big|\Big|F(x)-\frac{1}{n}F(nx)\Big|\Big|\leq\frac{1}{n}\phi(x,x,\cdots,x)$$

for all $x\in X_1$.
Thus

$$\begin{aligned}\Big|\Big|\frac{1}{n^l}F(n^lx)-\frac{1}{n^m}F(n^mx)\Big|\Big| &\leq \sum_{j=l}^{m-1}\Big|\Big|\frac{1}{n^j}F(n^jx)-\frac{1}{n^{j+1}}F(n^{j+1}x)\Big|\Big| \\ &\leq \sum_{j=l}^{m-1}\frac{1}{n^{j+1}}\phi(n^jx,n^jx,\cdots,n^jx) \qquad (3.10)\end{aligned}$$

for all nonnegative integers m and l with $m>l$ and all $x\in X_1$. It follows (3.10) that the sequence $\left\{\frac{1}{n^m}F\Big(n^mx\Big)\right\}$ is a Cauchy sequence for all $x\in X_1$. Since X_2 is complete, the sequence $\left\{\frac{1}{n^m}F\Big(n^mx\Big)\right\}$ converges. So one can define the mapping $H:X_1\to X_2$ by

$$H(x):=\lim_{m\to\infty}\frac{1}{n^m}F\Big(n^mx\Big)$$

for all $x\in X_1$.
Furthermore, putting $l=0$ and passing the limit $m\to\infty$ in (3.10), we obtain mapping satisfying (3.9).
We use the similar manner to the proof of Theorem 3.1 for the rest of the proof.

Corollary 3.2. *Let $k<1$ and Θ be positive real numbers and let $F:X_1\to X_2$ be a mapping satisfying (3.8). Then there exists a unique additive mapping $H:X_1\to X_2$ such that*

$$||F(x)-H(x)||\leq\frac{n\Theta}{n-n^k}||x||^k$$

for all $x\in X_1$.

4. Generalized additive functional inequalities in non-Archimedian Banach spaces

In this section, consider that X_1 is a non-Archimedean normed space and that X_2 is a non-Archimedean Banach space. Assume that $|n|\neq 1$.

Lemma 4.1. *A mapping $F:X_1\to X_2$ satisfies $F(0)=0$ and*

$$\Big|\Big|F\Big(\frac{1}{n}\sum_{i=1}^{n}x_i\Big)-\frac{1}{n}\sum_{i=1}^{n}F(x_i)\Big|\Big|\leq\Big|\Big|F\Big(\sum_{i=1}^{n}x_i\Big)-\sum_{i=1}^{n}F(x_i)\Big|\Big| \qquad (4.1)$$

for all $x_1,x_2,\cdots,x_n\in X_1$ if and only if F is additive.

Proof. Putting $x_n=x_{n-1}=\cdots=x_2=0$ in (4.1), we obtain

$$||F\Big(\frac{x_1}{n}\Big)-\frac{1}{n}F(x_1)||\leq 0$$

and thus $F\Big(\frac{x_1}{n}\Big) = \frac{1}{n}F(x_1)$ and putting $x_1 = x$, we obtain $F(\frac{x}{n}) = \frac{1}{n}F(x)$ for all $x \in X_1$.
Thus

$$\begin{aligned}\frac{1}{|n|}\Big|\Big|F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)\Big|\Big| &= \Big|\Big|\frac{1}{n}\Big(F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)\Big)\Big|\Big| \\ &= ||F\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) - \frac{1}{n}\sum_{i=1}^{n} F(x_i)|| \\ &\leq ||F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)||\end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in X_1$.
Since $|n| < 1$,

$$F\Big(\sum_{i=1}^{n} x_i\Big) = \sum_{i=1}^{n} F(x_i)$$

for all $x_1, x_2, \cdots, x_n \in X_1$.

On the other hand the converse is obviously true.

Now we prove the Hyers-Ulam stability of the additive functional inequality (4.1) in non-Archimedean Banach spaces.

Theorem 4.1. *Let $\phi : X_1^n \to [0, \infty)$ be a function and let $F : X_1 \to X_2$ be a mapping with $F(0) = 0$ satisfying*

$$\Psi(x_1, x_2, \cdots, x_n) := \sum_{j=0}^{\infty} |n|^j \phi\Big(\frac{x_1}{n^j}, \frac{x_2}{n^j}, \cdots, \frac{x_n}{n^j}\Big) < \infty \tag{4.2}$$

$$\Big|\Big|F\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) - \frac{1}{n}\sum_{i=1}^{n} F(x_i)\Big|\Big| \leq \Big|\Big|F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F\Big(x_i\Big)\Big|\Big| + \phi\Big(x_1, x_2, \cdots, x_n\Big) \tag{4.3}$$

for all $x_1, x_2, \cdots, x_n \in X_1$. Then there exists a unique additive mapping $A : X_1 \to X_2$ such that

$$||F(x) - A(x)|| \leq |n|\Psi(x, 0, 0, \cdots, 0) \tag{4.4}$$

for all $x \in X_1$.

Proof. Putting $x_2 = x_3 = x_4 = \cdots = x_n = 0$ and $x_1 = x$ in (4.3), we obtain

$$\Big|\Big|F\Big(\frac{x}{n}\Big) - \frac{1}{n}F(x)\Big|\Big| \leq \phi(x, 0, 0, \cdots 0) \tag{4.5}$$

for all $x \in X_1$.
Then

$$\Big|\Big|F(x) - nF\Big(\frac{x}{n}\Big)\Big|\Big| \leq |n|\phi(x, 0, 0, \cdots, 0)$$

for all $x \in X_1$.
Thus

$$\begin{aligned}\Big|\Big|n^l F\Big(\frac{x}{n^m}\Big) - n^l F\Big(\frac{x}{n^m}\Big)\Big|\Big| &\leq max\Big\{\Big|\Big|n^l F\Big(\frac{x}{n^l}\Big) - n^{l+1} F\Big(\frac{x}{n^{l+1}}\Big)\Big|\Big|, \cdots, \Big|\Big|n^{m-1} F\Big(\frac{x}{n^{m-1}}\Big) - n^m F\Big(\frac{x}{n^m}\Big)\Big|\Big|\Big\} \\ &\leq max\Big\{|n|^l\Big|\Big|F\Big(\frac{x}{n^l}\Big) - nF\Big(\frac{x}{n^{l+1}}\Big)\Big|\Big|, \cdots, |n|^{m-1}\Big|\Big|F\Big(\frac{x}{n^{m-1}}\Big) - nF\Big(\frac{x}{n^m}\Big)\Big|\Big|\Big\}\end{aligned}$$

$$\leq \sum_{j=l}^{\infty} |n|^{j+1}\phi\Big(\frac{x}{n^l},0,0,\cdots,0\Big) \tag{4.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X_1$. It follows from (4.6) that the sequence $\Big\{n^k F\Big(\frac{x}{n^k}\Big)\Big\}$ is Cauchy for all $x \in X_1$. Since X_2 is a non-Archimedean Banach space, the sequence $\Big\{n^k F\Big(\frac{x}{n^k}\Big)\Big\}$ converges. So we can define the mapping $A : X_1 \to X_2$ by

$$A(x) := \lim_{k\to\infty} n^k F\Big(\frac{x}{n^k}\Big)$$

for all $x \in X_1$.
Futhermore, putting $l = 0$ and passing the limit $m \to \infty$ in (4.6), we obtain (4.4).

Now, let $T : X_1 \to X_2$ be another additive mapping fulfilliyng (4.4).
Thus we have

$$\begin{aligned} \Big|\Big|A(x) - T(x)\Big|\Big| &= \Big|\Big|n^q A\Big(\frac{x}{n^q}\Big) - n^q T\Big(\frac{x}{n^q}\Big)\Big|\Big| \\ &\leq max\Big\{\Big|\Big|n^q A\Big(\frac{x}{n^q}\Big) - n^q F\Big(\frac{x}{n^q}\Big)\Big|\Big|, \Big|\Big|n^q T\Big(\frac{x}{n^q}\Big) - n^q F\Big(\frac{x}{n^q}\Big)\Big|\Big|\Big\} \\ &\leq \frac{1}{|n|}|n|^q \Psi\Big(\frac{x}{n^q},0,0,\cdots,0\Big), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X_1$. Then we can conclude that $A(x) = T(x)$ for all $x \in X_1$. This proves the uniqueness of A.

It follows from (4.2) and (4.3) that

$$\begin{aligned} \left\|A\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) - \frac{1}{n}\sum_{i=1}^{n} A(x_i)\right\| &= \lim_{m\to\infty}\left\|n^m\Big(F\left(\frac{\sum_{i=1}^{n} x_i}{n^{m+1}}\right) - \frac{1}{n}\sum_{i=1}^{n} F\Big(\frac{x_i}{n^m}\Big)\Big)\right\| \\ &\leq \lim_{m\to\infty}\left\|n^m\Big(F\left(\frac{\sum_{i=1}^{n} x_i}{n^m}\right) - \sum_{i=1}^{n} F\Big(\frac{x_i}{n^m}\Big)\Big)\right\| \\ &+ \lim_{m\to\infty}|n|^m\phi\Big(\frac{x_1}{n^m},\frac{x_2}{n^m},\cdots,\frac{x_n}{n^m}\Big) \\ &= \Big|\Big|A\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} A(x_i)\Big|\Big| \end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in X_1$.

$$\left\|A\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) - \frac{1}{n}\sum_{i=1}^{n} A(x_i)\right\| \leq \Big|\Big|A\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} A(x_i)\Big|\Big|$$

for all $x_1, x_2, \cdots, x_n \in X_1$.
The mapping $A : X_1 \to X_2$ is additive by Lemma 4.1.

Corollary 4.1. *Let $k < 1$ and Θ be nonnegative real numbers, and let $F : X_1 \to X_2$ be a mapping with $F(0) = 0$ satisfying :*

$$\Big|\Big|F\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) - \frac{1}{n}\sum_{i=1}^{n} F(x_i)\Big|\Big| \le \Big|\Big|F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)\Big|\Big| + \Theta\Big(\sum_{i=1}^{n} ||x_i||^k\Big) \tag{4.7}$$

for all $x_1, x_2, \cdots, x_n \in X_1$. Then there exists a unique additive mapping $A : X_1 \to X_2$ such that

$$||F(x) - A(x)|| \le \frac{|n|^{k+1}\Theta}{|n|^k - |n|}||x||^k$$

for all $x_1 \in X_1$.

Theorem 4.2. *Let $\phi : X_1^n \to [0, \infty)$ be a function and let $F : X_1 \to X_2$ be a mapping fulfilliyng $F(0) = 0$, (4.3) and*

$$\Psi(x_1, x_2, \cdots, x_n) := \sum_{j=1}^{\infty} \frac{1}{|n|^j}\phi(n^j x_1, n^j x_2, \cdots, n^j x_n) < \infty$$

for all $x_1, x_2, \cdots, x_n \in X_1$.
Then there exists a unique additive mapping $A : X_1 \to X_2$ such that

$$||F(x) - A(x)|| \le |n|\Psi(x, 0, 0, \cdots, 0)$$

for all $x \in X_1$.

Proof. It follows from (4.5) that

$$\Big|\Big|F(x) - \frac{1}{n}F(nx)\Big|\Big| \le \phi(nx, 0, 0, \cdots, 0)$$

for all $x \in X_1$.
We use the similar manner to the proof of Theorem 4.1 for the rest of the proof.

Corollary 4.2. *Let $k > 1$ and Θ be nonnegative real numbers, and let $F : X_1 \to X_2$ be a mapping fulfilliyng $F(0) = 0$ and (4.7). Then there exists a unique additive mapping $A : X_1 \to X_2$ such that*

$$||F(x) - A(x)|| \le \frac{|n|^{r+1}\Theta}{|n| - |n|^k}||x||^k$$

for all $x \in X_1$.

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