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AN ELEMENTARY METHOD OF DETERMINING HAMILTONIAN PATHS IN ORIENTED GRAPHS THAT ADMIT CIRCUITS

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Abstract: *Determining Hamiltonian paths in oriented graphs that do not support circuits is a simple matter. However, to determine Hamiltonian paths in oriented graphs that admit circuits, things are more complicated. One approach in this regard is the Ford-Fulkerson method, which proves ineffective when we have only one equivalence class, and the only method that generally works is the Latin matrix method in which the concatenation operation is used. In this study, another way of determining the Hamiltonian paths is proposed, which is based on elementary operations and which uses, with certain limitations, the operations from Y. Chen's algorithm for the transition from the capacitive connected matrix to the paths matrix.*

Keywords: Latin matrix method, Y. Chen's algorithm, Ford-Fulkerson method, Hamiltonian paths

1. Introduction

The issue of determining the Hamiltonian paths remains topical given the complexity of a technological flow or of any other nature that involves determining the optimal path by which a raw material reaches a finished product. It is an important problem that immediately translates into additional costs if we deviate from this path/sequence of operations by the occurrence of redundant elements (performing an operation several times) or finding that at the end of the process we do not have the designed product.

A first problem to be solved is that of framing the given oriented graph: whether it is with or without circuits. The answer to this problem is simple: if n is the number of vertices of the graph and we have determined the final connection matrix.

In the matrix $T = \left(t_{ij}\right)_{i,j\in\overline{1,n}}$, where

$$t_{ij} = \begin{cases} 1, \text{there is a path from } x_i \text{ to } x_j \\ 0, \text{there is no path from } x_i \text{ to } x_j \end{cases},$$

we know between which vertices there is a path and is constructed with the help of the connected or Boolean matrix attached to the graph $C = \left(c_{ij}\right)_{i,j\in\overline{1,n}}$, where

$$c_{ij=} \begin{cases} 1, \text{there is arch from } x_i \text{ to } x_j \\ 0, \text{there is no arch from } x_i \text{ to } x_j \end{cases},$$

in which we know between which vertices/operations there are direct arches/connections, we can establish where

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we stand based on the sum of the matrix's elements T, $S = \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}$, as follows:

1. If $S < \frac{n \cdot (n-1)}{2}$, then the graph has no circuits or Hamiltonian path (in this case there is the problem of "correcting" the graph, in the sense of determining a minimum number of direct links that must be created between the process operations so that it provides the finished product as designed);
2. If $S = \frac{n \cdot (n-1)}{2}$, then the graph has no circuits and admits a single Hamiltonian path, i.e. a unique sequence of operations corresponding to this process leading to the desired finished product (and consequently, any deviation from this sequence will lead to the failure of the desired finished product);
3. If $S > \frac{n \cdot (n-1)}{2}$, then the graph has circuits.

Next, we will deal with this latter situation in which we have to determine the Hamiltonian paths, i.e. that succession of operations that must meet two conditions: not to perform the same operation twice (the process should not be redundant) and to go through all the operations necessary to achieve the desired finished product.

2. Presentation of the Method / Algorithm

The method is based, as mentioned before, on operations on the matrix C, similar to the operations performed in Y. Chen's algorithm to move from the connection matrix linked to the road matrix.

Specifically, the algorithm by which Hamiltonian paths are built, assuming that we have reached a line in the matrix corresponding to vertex x_i and we have already performed some of the operations that will be described below, is the following:

1. The largest element on the line corresponding to the vertex x_i that has been reached is selected, which we will call ***pivot*** (we assume that it is m and that it is on the column corresponding to the vertex x_j);
2. For all elements on the line corresponding to the vertex x_j in the matrix C which are 1 (or $\neq$ 0) except, where applicable, the one on the column corresponding to the vertex x_i (which always remains 0), the following operation is done:

*Table 1: Modification of the elements on the line corresponding to the **pivot***

Changing the line corresponding to vertex x_i	…	x_j	…	x_k	…
x_i	…	m	…	$= \begin{cases} \text{unchanged, if } x_k \text{ was previously pivot} \\ m+1, \text{ if } x_k \text{ was not previously pivot} \end{cases}$	
x_j	…	...	…	1	

3. If on the line corresponding to the vertex x_i there are several elements equal to m, then the analysis from point 2 is done, in turn, for each of them.
4. Hamiltonian paths are reached only after n – 1 steps have been taken, and on the line corresponding to such a vertex we have all the values from 0 to n – 1. The Hamiltonian path is formed in the sequence given by the vertices corresponding to the values from 0 to n – 1.

If the line corresponding to vertex x_i does not change, then the algorithm stops because it can no longer lead to a Hamiltonian path, because all values from 0

to n– 1 can no longer appear after the n – 1 steps.
The operation at point 2 assures us that no circuits can be formed and that after a number of n – 1 operations we reach the Hamiltonian paths.

3. Application of the Method

Let us suppose we have the following oriented graph, the connected matrix of which is given below:

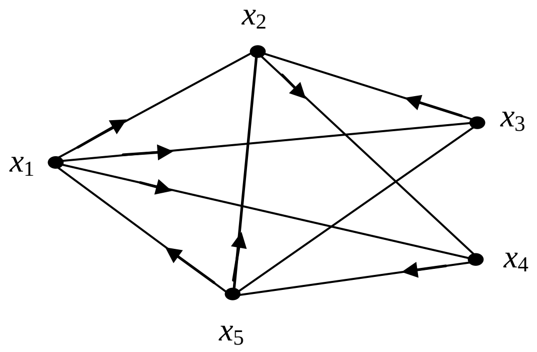

Figure 1: Oriented graph

The related or Boolean matrix corresponding to the above graph is shown in the table below:

Table 2: Connected matrix attached to the graph in figure no. 1

C	x_1	x_2	x_3	x_4	x_5
x_1	0	1	1	1	0
x_2	0	0	0	1	0
x_3	0	1	0	0	1
x_4	0	0	0	0	1
x_5	1	1	0	0	0

We proceed further to the application of the algorithm described in section 2.
We will start with the line corresponding to vertex x_1, as follows:

Table 3: Modification of the elements on the line corresponding to vertex x_1

	x_1	x_2	x_3	x_4	x_5
x_1	0	1	1	1	0

Here we have three options for 1.
We use the first option for 1.

	x_1	x_2	x_3	x_4	x_5
x_1	0	1	1	$\cancel{1}$ 2	0
x_2	0	0	0	1	0
x_1	0	1	1	2	$\cancel{0}$ 3
x_4	0	0	0	0	1
x_1	0	1	1	2	3
x_5	1	1	0	0	0

Since x_1 is ***pivot*** it results that it does not operate with the line corresponding to it, and it has already been operated with the line corresponding to the vertex x_2. Consequently, the line corresponding to vertex x_1 does not change and since only 3 < 4 steps are taken, this sequence cannot lead to a Hamiltonian path.
We continue with the second option for 1.

	x_1	x_2	x_3	x_4	x_5
x_1	0	$\cancel{1}$ 2	1	1	$\cancel{0}$ 2
x_3	0	1	0	0	1
x_1	0	2	1	1	2

Here we have two options for 2.
We use the first option for 2.

	x_1	x_2	x_3	x_4	x_5
x_1	0	2	1	$\cancel{1}$ 3	2
x_2	0	0	0	1	0
x_1	0	2	1	3	$\cancel{2}$ 4
x_4	0	0	0	0	1
x_1	0	2	1	3	4

This succession leads to the Hamiltonian path

$$x_1 \rightarrow x_3 \rightarrow x_2 \rightarrow x_4 \rightarrow x_5.$$

Further, we continue with the second option for 2:

	x_1	x_2	x_3	x_4	x_5
x_1	0	$\cancel{2}$ 3	1	1	2
x_5	1	1	0	0	0
x_1	0	3	1	$\cancel{1}$ 4	2
x_2	0	0	0	1	0
x_1	0	3	1	4	2

This succession leads to the Hamiltonian path

$$x_1 \rightarrow x_3 \rightarrow x_5 \rightarrow x_2 \rightarrow x_4.$$

Next, we will continue with the third option for 1:

	x_1	x_2	x_3	x_4	x_5
x_1	0	1	1	1	$\cancel{0}$ 2
x_4	0	0	0	0	1
x_1	0	$\cancel{1}$ 3	1	1	2
x_5	1	1	0	0	0
x_1	0	3	1	1	2
x_2	0	0	0	1	0

In this situation, too, because x_1 is ***pivot*** and therefore does not operate with its corresponding line, and it has already been operated with the line corresponding to vertex x_4, it results that the line corresponding to vertex x_1 does not change and since only $3 < 4$ steps are made, this sequence can not to lead to a Hamiltonian path.

We continue with the line corresponding to vertex x_2, as follows:

Table 4: Modification of the elements on the line corresponding to vertex x_2

	x_1	x_2	x_3	x_4	x_5
x_2	0	0	0	1	$\cancel{0}$ 2
x_4	0	0	0	0	1
x_2	$\cancel{0}$ 3	0	0	1	2
x_5	1	1	0	0	0
x_2	3	0	$\cancel{0}$ 4	1	2
x_1	0	1	1	1	0
x_2	3	0	4	1	2

This succession leads to the Hamiltonian path

$$x_2 \rightarrow x_4 \rightarrow x_5 \rightarrow x_1 \rightarrow x_3.$$

We continue with the line corresponding to vertex x_3, as follows:

Table 5. Modification of the elements on the line corresponding to vertex x_3

	x_1	x_2	x_3	x_4	x_5
x_3	0	1	0	0	1

Here we have two options for 1.

We use the first option for 1.

	x_1	x_2	x_3	x_4	x_5
x_3	0	1	0	$\cancel{0}$ 2	1
x_2	0	0	0	1	0
x_3	0	1	0	2	$\cancel{1}$ 3
x_4	0	0	0	0	1
x_3	$\cancel{0}$ 4	1	0	2	3
x_5	1	1	0	0	0
x_3	4	1	0	2	3

This succession leads to the Hamiltonian path

$$x_3 \rightarrow x_2 \rightarrow x_4 \rightarrow x_5 \rightarrow x_1.$$

Next we continue with the second option for 1:

	x_1	x_2	x_3	x_4	x_5
x_3	$\cancel{0}$ 2	$\cancel{1}$ 2	0	0	1
x_5	1	1	0	0	0
x_3	2	2	0	0	1

Here we have two options for 2.

We use the first option for 2.

	x_1	x_2	x_3	x_4	x_5
x_3	2	$\cancel{2}$ 3	0	$\cancel{0}$ 3	1
x_1	0	1	1	1	0
x_3	2	3	0	3	1

Here we have two options for 3.

We use the first option for 3.

	x_1	x_2	x_3	x_4	x_5
x_3	2	3	0	$\cancel{3}$ 4	1
x_2	0	0	0	1	0
x_3	2	3	0	4	1

This succession leads to the Hamiltonian path

$$x_3 \rightarrow x_5 \rightarrow x_1 \rightarrow x_2 \rightarrow x_4.$$

Next we continue with the second option for 3:

	x_1	x_2	x_3	x_4	x_5
x_3	2	3	0	3	1
x_4	0	0	0	0	1

Since the line corresponding to vertex x_2 has already been operated, it results that the line corresponding to vertex x_3 does not change and since only $3 < 4$ are taken, this

succession cannot lead to a Hamiltonian path.
We further continue with the second option for 2:

	x_1	x_2	x_3	x_4	x_5
x_3	2	2	0	~~0~~ 3	1
x_2	0	0	0	1	0
x_3	2	2	0	3	1
x_4	0	0	0	0	1

Since the line corresponding to vertex x_5 has already been operated, it results that the line corresponding to vertex x_3 no longer changes and since only $3 < 4$ steps are covered, this sequence cannot lead to a Hamiltonian path.
We continue with the line corresponding to vertex x_4, as follows:

Table 6: Modification of the elements on the line corresponding to the vertex x_4

	x_1	x_2	x_3	x_4	x_5
x_4	~~0~~ 2	~~0~~ 2	0	0	1
x_5	1	1	0	0	0
x_4	2	2	0	0	1

Here we have two options for 2.
We use the first option for 2.

	x_1	x_2	x_3	x_4	x_5
x_4	2	~~2~~ 3	~~0~~ 3	0	1
x_1	0	1	1	1	0
x_4	2	3	3	0	1

Here we have two options for 3.
We use the first option for 3.

	x_1	x_2	x_3	x_4	x_5
x_4	2	3	3	0	1
x_2	0	0	0	1	0

Because the line corresponding to the pivot is not operated, it results that the line corresponding to the x_4 vertex does not change and since only $3 < 4$ steps are taken, this sequence cannot lead to a Hamiltonian path.
Next we continue with the second option for 3:

	x_1	x_2	x_3	x_4	x_5
x_4	2	~~3~~ 4	3	0	1
x_3	0	1	0	0	1
x_4	2	4	3	0	1

This succession leads to the Hamiltonian path

$$x_4 \rightarrow x_5 \rightarrow x_1 \rightarrow x_3 \rightarrow x_2.$$

We continue with the second option for 2:

	x_1	x_2	x_3	x_4	x_5
x_4	2	2	0	0	1
x_3	0	1	0	0	1

Since the lines corresponding to vertices x_2 and x_5 have already been operated with, it results that the line corresponding to vertex x_4 does not change and since only $2 < 4$ steps are covered, this sequence cannot lead to a Hamiltonian path.
We continue with the line corresponding to the last vertex, x_5, as follows:

Table 7: Modification of the elements on the line corresponding to the vertex x_5

	x_1	x_2	x_3	x_4	x_5
x_5	1	1	0	0	0

Here we have two options for 1.
We use the first option for 1.

	x_1	x_2	x_3	x_4	x_5
x_5	1	~~1~~ 2	~~0~~ 2	~~0~~ 2	0
x_1	0	1	1	1	0
x_5	1	2	2	2	0

Here we have three options for 2.
We use the first option for 2.

	x_1	x_2	x_3	x_4	x_5
x_5	1	2	2	~~2~~ 3	0
x_2	0	0	0	1	0
x_5	1	2	2	3	0
x_4	0	0	0	0	1

In this situation, too, because x_5 is ***pivot*** and therefore does not operate with its corresponding line, it results that the line corresponding to vertex x_5 does not change and since only $3 < 4$ steps are covered, this succession cannot lead to a Hamiltonian path.

We further continue with the second option for 2:

	x_1	x_2	x_3	x_4	x_5
x_5	1	$\cancel{2}$ 3	2	2	0
x_3	0	1	0	0	1
x_5	1	3	2	$\cancel{2}$ 4	0
x_2	0	0	0	1	0
x_5	1	3	2	4	0

This succession leads to the Hamiltonian path

$x_5 \rightarrow x_1 \rightarrow x_3 \rightarrow x_2 \rightarrow x_4$.

We further continue with the third option for 2:

	x_1	x_2	x_3	x_4	x_5
x_5	1	2	2	2	0
x_4	0	0	0	0	1

In this situation, too, because x_5 is ***pivot*** and therefore does not operate with its corresponding line, it results that the line corresponding to vertex x_5 does not change and only $3 < 4$ steps are covered, this sequence cannot lead to a Hamiltonian path.

We further continue with the second option for 1:

	x_1	x_2	x_3	x_4	x_5
x_5	1	1	0	$\cancel{0}$ 2	0
x_2	0	0	0	1	0
x_5	1	1	0	2	0
x_4	0	0	0	0	1

Here also, because x_5 is ***pivot*** and therefore does not operate with its corresponding line, it results that the line corresponding to vertex x_5 does not change and only $2 < 4$ steps are covered, this sequence cannot lead to a Hamiltonian path.

In this way, the algorithm is completed and we can conclude that we determined the 6 Hamiltonian paths of the graph.

4. Discussions and Conclusions

The presented method is comparable to the Latin matrix method in terms of complexity of calculations. If the Latin matrix method involves the operation with matrices in which the elements are vectors and the concatenation operation is applied, the method or the presented algorithm involves only elementary operations, according to some very simple rules, but with many variants which should be verified.

This approach will continue with the study of the following problems:

- the possibility to determine when such a graph admits Hamiltonian paths;
- the possibility of simplifying the presented algorithm, simultaneously with the determination of all Hamiltonian paths (if it can be seen more simply when a certain succession does not lead to a Hamiltonian path);
- the development of a computer application to provide Hamiltonian path for an oriented graph.

Also, another problem that can be studied by means of this method or algorithm is that of the circuits in an oriented graph, in correspondence with the way in which this can be done with the method of Latin matrices.

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