International Conference KNOWLEDGE-BASED ORGANIZATION
Vol. XXXI No 3 2025

ON THE CALCULUS OF THE NATURAL FREQUENCIES OF FUNCTIONALLY GRADED PLATES BY USING THE CONCEPTS OF MULTILAYER AND EQUIVALENT MATERIAL

Ghiță BÂRSAN*, Vasile NĂSTĂSESCU, Nicolae MORO*, Ioan VIRCA***

***"Nicolae Bălcescu" Land Forces Academy, Sibiu, Romania**
****"Ferdinand I" Military Technical Academy, Bucharest, Romania**
gbarsan@armyacademy.ro

Abstract: *Functionally graded materials represent a relatively new class of materials, which can be attributed to the broad category of composite materials. The specificity of functionally graded materials consists in the fact that in one of the directions (usually the smallest in a given body) the elastic, physical, mechanical, electrical, etc. constants vary according to a continuous and monotonic function, between two values assigned to the two materials, which are the basis for the realization of the functionally graded material. The paper presents a method for calculating the natural frequencies of functionally graded plates using two concepts of approaching the calculation of structures made of functionally graded materials, multilayer material and equivalent material. With the help of these concepts, the shapes and natural frequencies of a plate can be determined, using the theoretical foundations of plates made of homogeneous and isotropic materials.*

Keywords: functional graded material, graded plates, natural frequencies

1. Introduction

Functionally graded materials represent a relatively new class of materials, which can be attributed to the broad category of composite materials. The specificity of functionally graded materials consists in the fact that on one of the directions (usually the smallest in a given body) the elastic, physical, mechanical, electrical, etc. constants vary according to a continuous and monotonic function, between two values assigned to the two materials, which are the basis for the realization of the functionally graded material.

The respective function is called a law or material model and it describes the variation of the respective parameters (Young's modulus, Poisson's ratio, density, thermal expansion coefficient, electrical and thermal conductivity coefficient, etc.). The specialized literature already knows several material laws, which lead to the values of the material in the pure state at the level of the extreme faces [6].

So, functionally graded materials are not multilayer materials, where at the layer separation level, phenomena specific to the transition between different materials appear, and the most important aspect being the phenomenon of stress concentration.

The calculation proposed by the authors of this paper is based on two concepts, used independently or combined, namely: the concept of multilayer material and the concept of equivalent material, which can be used to solve both static and dynamic

DOI: 10.2478/kbo-2025-0074

load problems, including high-speed impacts, including in the modal analysis of structures, etc. To calculate the natural frequencies, the variation of the density of the material must be taken into account, according to the law adopted for the respective material, and an analytical calculation is necessary to determine an equivalent density. This can be done by direct analytical calculation or using the concept of multilayer material, using an average value of the values determined for each layer.

2. The Density Calculation

2.1. The Density Calculation by Direct Analytical Integration

Considering the power law of the functionally graded material [6], the density varies according to the following law (relationship):

$$\rho(z) = \rho_b + (\rho_t - \rho_b)\cdot\left(0.5 + \frac{z}{h}\right)^k$$

The mass of an element of infinitesimal volume is written:

$$dM_{placa} = \rho(z)\cdot dV = \rho(z)\cdot A\cdot dz = \pi r^2 \cdot \rho(z)\cdot \mathrm{d}z$$

By integration, the mass of the functionally graded plate is obtained:

$$M_{placa} = \pi r^2 \int_{-h/2}^{h/2} \rho(z)\mathrm{d}z = \pi r^2 \cdot I = A\cdot I$$

where it was noted with "I":

$$I = \int_{-h/2}^{h/2} \left[\rho_b + (\rho_t - \rho_b)\cdot\left(0.5 + \frac{z}{h}\right)^k\right]\cdot \mathrm{d}z$$

To perform the integral *I*, the following notations and the method of integration by parts are used:

$$I = \rho_b \int_{-h/2}^{h/2} \mathrm{d}z + (\rho_t - \rho_b)\int_{-h/2}^{h/2}\left(0.5 + \frac{z}{h}\right)^k \mathrm{d}z$$

$$I = \rho_b \cdot h + (\rho_t - \rho_b)\frac{h}{k+1}$$

For the considered case study and the power coefficient, the value of the integral *I* is:

$$I = 15{,}60\left[\frac{\mathrm{kg}}{\mathrm{m}^2}\right]$$

By inserting the numerical values, the mass of the functionally graded plate is found:

$$M_{placa} = \pi r^2 \cdot I = \pi \cdot 0.05^2 \cdot 15.60 = 0.12252\ [\mathrm{kg}]$$

As the volume of the plate is:

$$V_{placa} = \pi \cdot r^2 \cdot h = \pi \cdot 0.05^2 \cdot 0.004 = 0.0000314159\ \left[\mathrm{m}^3\right]$$

the equivalent density immediately results:

$$\rho_{ech} = \rho_{ech}^a = \frac{M_{placa}}{V_{placa}} = 3899.936\left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right]$$

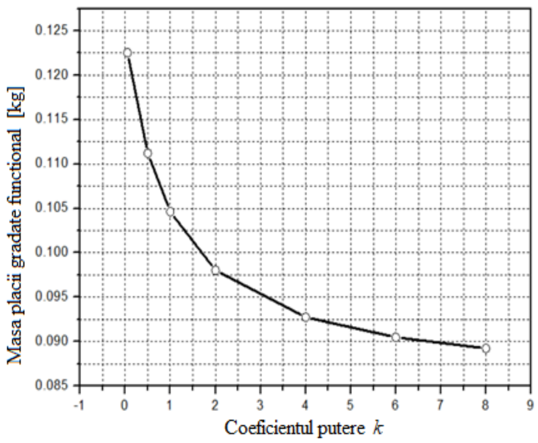

Figure 1: The variation of the mass of the functionally graded plate with the power coefficient

This density is also noted ρ_{ech}^a because it was determined purely analytically, by direct integration. An equivalent density can also be calculated based on the concept of a multilayer material, as the average of the density values of each layer, and this will be noted ρ_{ech}^m.

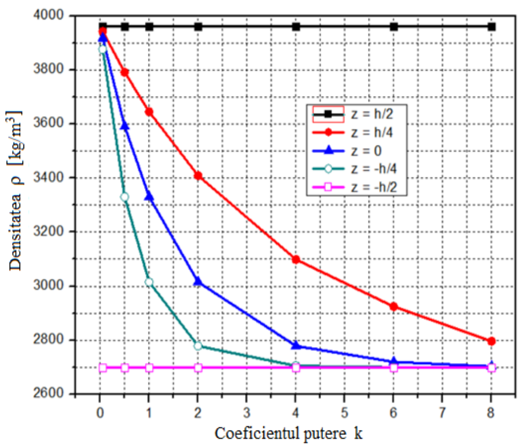

Figure 2: The density variation with power coefficient, at different elevations z

Analyzing the curves presented in Figures 1 and 2, it can be seen that both the mass and the equivalent density vary strongly with the power coefficient, in the sense that their values k decrease with increasing power coefficient.

2.2. The Density Calculation by Applying the Concept of Multilayer Material

By using the relation for integral *I*, for a value of the power coefficient *k=0,05*, the density can be calculated for each line and for each layer, using the value of the line coordinate z_{line}, respectively the average value of the coordinate of each layer $z_{layer}^{av.}$, the weighted average value of the coordinate $z_{layer}^{w.av.}$, the average value of the layer density as the average of the values $\rho(z)_{layer}^{av.}$ at the level of adjacent lines or the weighted average value of the layer density $\rho(z)_{layer}^{w.av.}$.

Analyzing the equivalent density values in table 1 and comparing the values thus calculated with the theoretical (analytical), it can be seen that all the values are extremely close. The value in column 6 is the closest (the percentage error is 0.01%), but all are below the error of $\pm 0{,}70\%$. The density are presented in table 1.

Table 1: The densities at the level of the lines and the densities of the layers in the above variants

z_{line} [mm]	$\rho(z_{line})$ $\left[\frac{g}{cm^3}\right]$	$z_{layer}^{av.}$ [mm]	$\rho\left(z_{layer}^{av.}\right)$ $\left[\frac{g}{cm^3}\right]$ **a)**	$z_{layer}^{w.av.}$ [mm]	$\rho\left(z_{layer}^{w.av.}\right)$ $\left[\frac{g}{cm^3}\right]$ **b)**	$\left(\rho(z)_{layer}^{av.}\right)$ $\left[\frac{g}{cm^3}\right]$ **c)**	$\left(\rho(z)_{layer}^{w.av.}\right)$ $\left[\frac{g}{cm^3}\right]$ **d)**
1	2	3	4	5	6	7	8
–2,00	2,7000						
–1,80	3,7847	–1,90	3,7478	–1,834	3,7362	3,2424	3,2138
–1,60	3,8230	–1,70	3,8069	–1,699	3,8069	3,8039	3,8027
–1,40	3,8460	–1,50	3,8356	–1,499	3,8356	3,8345	3,8337
–1,20	3,8626	–1,30	3,8548	–1,299	3,8548	3,8543	3,8536
–1,00	3,8756	–1,10	3,8694	–1,100	3,8694	3,8691	3,8685
–0,80	3,8864	–0,90	3,8812	–0,900	3,8812	3,8810	3,8804
–0,60	3,8956	–0,70	3,8912	–0,700	3,8912	3,8910	3,8903
–0,40	3,9036	–0,50	3,8997	–0,500	3,8997	3,8996	3,8988
–0,20	3,9107	–0,30	3,9072	–0,300	3,9072	3,9071	3,9060
0,00	3,9171	–0,10	3,9140	–0,100	3,9140	3,9139	3,9107
0,20	3,9229	0,10	3,9201	0,100	3,9201	3,9200	3,9229
0,40	3,9282	0,30	3,9256	0,300	3,9256	3,9256	3,9265
0,60	3,9332	0,50	3,9307	0,500	3,9307	3,9307	3,9312
0,80	3,9377	0,70	3,9355	0,700	3,9355	3,9354	3,9358
1,00	3,9420	0,90	3,9399	0,900	3,9399	3,9399	3,9401
1,20	3,9460	1,10	3,9440	1,100	3,9440	3,9440	3,9442
1,40	3,9498	1,30	3,9479	1,300	3,9479	3,9479	3,9481
1,60	**3,9534**	**1,50**	**3,9516**	**1,500**	**3,9516**	**3,9516**	**3,9517**
1,80	3,9568	1,70	3,9551	1,700	3,9551	3,9551	3,9552
2,00	3,9600	1,90	3,9584	1,900	3,9584	3,9584	3,9585
The average density values (equivalent densities)							
Eroare [%]	**3,9068**		**3,9008**		**3,9003**	**3,8753**	**3,8736**
	0,17		***0,02***		***0,01***	***–0,63***	***–0,67***

Based on the data in table 1, the curves in figure 3 were graphically represented, which also highlights through graphic analysis the good agreement of the results according to the four methods of calculating the density of each layer.

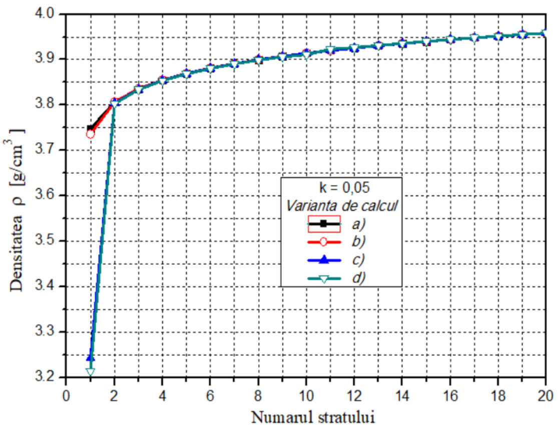

Figure 3: The density variation across the thickness of the plate, by layers

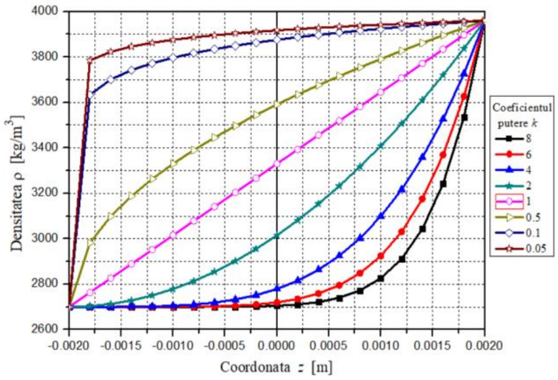

Figure 4: Density variation with power coefficient k

From the analysis of the curves presented in Figure 4, it is found that the variation mode of the density with the power coefficient follows the shape of the Young's modulus curves.

2.3. The Calculation of The Natural Frequencies Of The Functionally Graded Plate

Using the concept of an equivalent plate, made of a homogeneous and isotropic material, with fictitious properties, but determined as presented above, with the same geometry (shape and mass distribution) and with the same support conditions, the natural frequencies can be calculated analytically as in the case of homogeneous and isotropic plates.

Therefore, using the relation:

$$f_i = \frac{\alpha_i}{R^2}\sqrt{\frac{D}{\rho h}}$$

for the calculation of the natural pulsation, denoted or , has for the "*i*" vibration mode the expression:

$$p_i \equiv \omega_i = \frac{\alpha_i}{R^2}\sqrt{\frac{D_{ech}}{\rho_{ech}\cdot h}}$$

where α_i is a constant that depends on the number of nodal circles *(s)* and the number of nodal diameters *(n)*, according to the values determined and presented in the literature.

Some values α_i of this parameter are presented in Table 2.

Table 2: The parameter values α_i for the first eigenfrequencies

s	$n=0$	$n=1$	$n=2$
0	10,21	21,22	34,84
1	39,78		
2	88,90		

Table 3 presents the first three natural frequencies of the functionally graded plate from the case study, compared to cases where the same plate would be made of only ceramic material or only aluminium.

Table 3: The values of frequencies for the first three eigenfrequencies

Plate type Natural frequencies	Ceramic material	**Functional graded material**	Aluminum
First frequency [Hz]	7709,30	**7571,83**	4007,17
Second frequency [Hz]	16018,37	**15732,73**	8326,09
Third frequency [Hz]	26299,72	**25830,74**	13670,17

Table 4 presents the first three natural frequencies of the functionally graded plate, defined in the case study, determined analytically.

Table 4: The first three natural frequencies of the functionally graded plate

	α_i	Number of frequency	D^a_{ech}	E^a_{ech}	ρ^a_{ech}	ω	**f**
			Nm	Pa	kg/m^3	rad/s	Hz
k = 0,05	10,217	1	2115,8	3,610E+11	3899,94	47575,21	**7571,83**
	21,22	2	2115,8	3,610E+11	3899,94	98851,65	**15732,73**
	34,84	3	2115,8	3,610E+11	3899,94	162299,32	**25830,74**
	α_i	Number of frequency	D^a_{ech}	E^a_{ech}	ρ^a_{ech}	ω	**f**
			Nm	Pa	kg/m^3	rad/s	Hz
k = 0,50	10,217	1	1449,26	2,473E+11	3539,99	41327,99	6577,56
	21,22	2	1449,26	2,473E+11	3539,99	85871,20	13666,84
	34,84	3	1449,26	2,473E+11	3539,99	140987,39	22438,86
	α_i	Number of frequency	D^a_{ech}	E^a_{ech}	ρ^a_{ech}	ω	**f**
			Nm	Pa	kg/m^3	rad/s	Hz
k = 1,00	10,217	1	1111,12	1,896E+11	3330,00	37310,45	5938,15
	21,22	2	1111,12	1,896E+11	3330,00	77523,57	12338,27
	34,84	3	1111,12	1,896E+11	3330,00	127281,86	20257,55
	α_i	Number of frequency	D^a_{ech}	E^a_{ech}	ρ^a_{ech}	ω	**f**
			Nm	Pa	kg/m^3	rad/s	Hz
k = 2,00	10,217	1	865,58	1,477E+11	3120,00	34021,11	5414,63
	21,22	2	865,58	1,477E+11	3120,00	70688,98	11250,51
	34,84	3	865,58	1,477E+11	3120,00	116060,50	18471,62
	α_i	Number of frequency	D^a_{ech}	E^a_{ech}	ρ^a_{ech}	ω	**f**
			Nm	Pa	kg/m^3	rad/s	Hz
k = 4,00	10,217	1	751,98	1,283E+11	2952,00	32599,96	5188,45
	21,22	2	751,98	1,283E+11	2952,00	67736,12	10780,55
	34,84	3	751,98	1,283E+11	2952,00	111212,37	17700,01
	α_i	Number of frequency	D^a_{ech}	E^a_{ech}	ρ^a_{ech}	ω	**f**
			Nm	Pa	kg/m^3	rad/s	Hz
k = 6,00	10,217	1	713,71	1,218E+11	2880,00	32154,13	5117,49
	21,22	2	713,71	1,218E+11	2880,00	66809,78	10633,12
	34,84	3	713,71	1,218E+11	2880,00	109691,45	17457,95
	α_i	Number of frequency	D^a_{ech}	E^a_{ech}	ρ^a_{ech}	ω	**f**
			Nm	Pa	kg/m^3	rad/s	Hz
k = 8,00	10,217	1	686,24	1,171E+11	2839,99	31750,53	5053,26
	21,22	2	686,24	1,171E+11	2839,99	65971,18	10499,65
	34,84	3	686,24	1,171E+11	2839,99	108314,60	17238,82

The values of the natural frequencies presented in table 4 and the curves in figures 5 and 6 show a decrease with increasing power coefficient *k*.

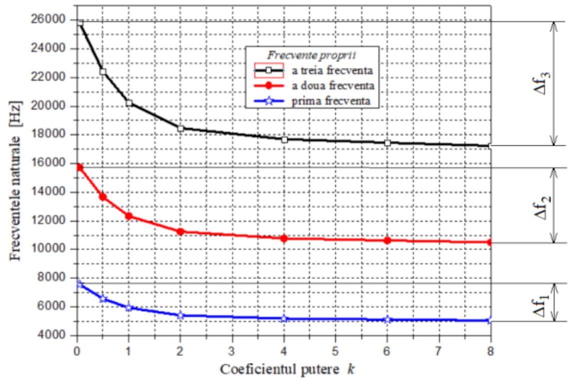

Figure 5: The variation of the first three natural frequencies with the power coefficient k

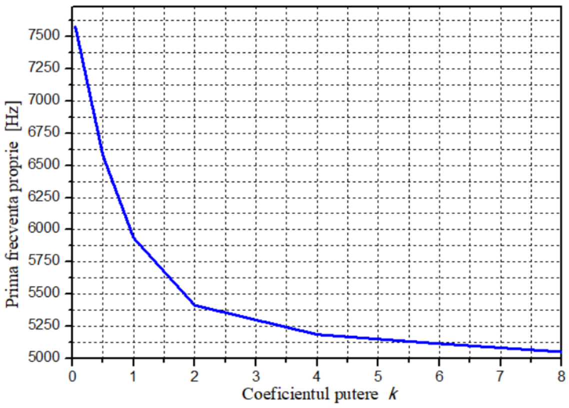

Figure 6: The variation of the first natural frequency with the power coefficient k

From figure 6, it is found that the frequency decrease rate with the power coefficient is the lowest for the first frequency, and the decrease rate increases with the frequency number $\Delta f_3 > \Delta f_2 > \Delta f_1$.

3. Conclusions

The use of structural elements from this category of composite materials (functionally graded materials) is expanding, and the need to calculate them with the most recent and efficient calculation tools is also becoming a necessity. If there is already a rich and developing literature on functionally graded materials, often essentially known, about the calculation of functionally graded plates and especially about the use of newer methods such as meshfree or meshless, the situation is not similar.

The purpose of this paper is to complete the existing literature through models, methods and methodologies available to those interested, without proposing a new, dedicated software, but using an existing, high-performance and accessible software.

The quantitative examples and arguments should be illustrated by using ELG method for the calculation of functionally graded plates which are based on the use of five calculation models, which cover practically all modelling options [9].

The calculation errors compared to the analytical solution, using the Galerkin free element method, regardless of the concept used, the 2D or 3D model and the type of finite element (solid or shell) are below 0.50 % for the calculation of the frequency of the first vibration mode and below 2.50 % for the calculation of the frequency of the third vibration mode [8].

For the calculation of transverse displacements, the Galerkin free element models and the presented methodologies (simplified structures, concepts used, fineness of the discretization grid, etc.) led to fully acceptable error values [7].

All results are obtained for the *2 mm* size of the largest side of the Galerkin free elements [6].

The presented analysis does not take into account the influence of Poisson's ratio in the calculation of functionally graded plates (materials). In the specialized literature it is specified that neglecting the variation of Poisson's ratio is all the more acceptable as the power coefficient has a lower value. This aspect leads to an acceptable analytical calculation, but it is not the closest to reality and depending on the requirements regarding the accuracy of the calculations, it can be decided to take into account or neglect the variation of Poisson's ratio.

The use of the concepts of multilayer material and equivalent material allows taking into account the variation of Poisson's ratio by numerical methods, and integrating its variation in a specialized analysis software program.

Obviously, the influence of Poisson's ratio can be studied for any functionally graded material (different basic materials) and for any material law.

Thus, we conducted research under "limit" conditions of some parameters, so that those interested would have the most accessible models.

References

[1] Anju, M., Subha, K., A Review on Functionally Graded Plates, International Research Journal of Engineering and Technology (IRJET), **05**, 04 (Apr-2018), eISSN: 2395-0056.

[2] Birsan, M., Sadowski, T., Marsavina, L., Linul, E., Pietras, D., Mechanical Behavior of Sandwich Composite Beams Made of Foams and Functionally Graded Materials, International Journal of Solids and Structures, **50**, 519-530 (2013), www.elsevier.com/locate/ijsolstr, http://dx.doi.org/10.1016/j.ijsolstr.2012.10.011.

[3] Hadji, Lazreg, Bernard, Fabrice, Bending and Free Vibration Analysis of Functionally Graded Beams on Elastic Foundations with Analytical Validation, Advances in Materials Research, **9**, 1, 63-98 (2020), DOI: https://doi.org./10.12989/amr.2020.9.1.063.

[4] Islam M. El Galy, Bassiouny I. Saleh, Mahmoud H. Ahmed, Functionally Graded Materials Classifications and Development Trends from Industrial Point of View, SN Applied Sciences ISSN 2523-3963, 1, 11, SN Appl. Sci. (2019) 1:1-23, DOI 10.1007/s42452-019-1413-4.

[5] Marzavan, S., Năstăsescu, V., Free Vibration Analysis of a Functionally Graded Plate by Finite Element Method, Ain Shams Engineering Journal, 102024 (2022), https://doi.org/10.1016/j.asej.2022.102024.

[6] V. Năstăsescu, G. Bârsan, O.A. Mocian, Calculul unor elemente structurale din materiale gradate funcţionale, Editura Academiei Române, Bucuresti, 2025, ISBN:9789732739501, 268 pages.

[7] Năstăsescu, V., Research and Developments in Engineering Research, Volume 8, Chapter 1: Study about Calculus of Functionally Graded Plates by using Multylayer Plate Concept, Print ISBN 978-81-19761-42-5, eBook ISBN 978-81-19761-53-1, DOI: 10.9734/bpi/rader/v8, Print ISBN: 978-81-19217-45-8, eBook ISBN: 978-81-19217-47-2, DOI: 0.9734/bpi/rader/v3/19159D.

[8] Năstăsescu, V., Bârsan, Gh., Marzavan, S., On the Calculus of Functionally Graded Plates, în „The 28th International Conference KBO 2022, The Knowledge-Based Organization”, Sibiu, 16-18 June 2022, Conference Proceedings, Vol. 28, No. 3, Applied Technical Sciences and Advanced Military Technologies, pp. 71-85.

[9] Năstăsescu, V., Bârsan, Gh., Metoda elementelor libere Galerkin în analiza structurilor, Bucureşti, Editura Academiei Române, 2018.

[10] Năstăsescu, V., Bârsan, Gh., Metoda particulelor libere în analiza numerică a mediilor continue, Bucureşti, Editura AGIR, 2015.

[11] Toudehdehdehghan, A., Lim, J. W., Foo, K. E., and others, A Brief Review of Functionally Graded Materials, în MATEC Web of Conferences, Vol. 131, 03010 (2017), UTP-UMP SES 2017.