

Models of planar motion of a two-wheeled experimental balancing prototype

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The subject of the study is the process of developing motion models on a plane for a two-wheeled experimental balancing prototype (TEBP) considered as a control object. The aim of the study is to develop an approach for constructing mathematical models of the translational and rotational motions of the TEBP, taking into account mechanical and electromechanical processes. A physical model of the TEBP was developed and its nonlinear mathematical description was obtained. The application of analytical linearization made it possible to derive linear approximation models in the form of differential equations with constant coefficients. Using the Laplace transform, transfer functions were obtained, enabling the construction of block diagrams in both time and frequency domains. The motion models of the TEBP are also represented in the state-space form in the controllability canonical form. The simulation results confirm the feasibility of using the developed mathematical descriptions as dynamic models of the control object. The scientific novelty lies in the development of a comprehensive approach to modeling the motions of the TEBP, taking into account the acting forces and moments.

Keywords: two-wheeled experimental balancing prototype, electric drive, control object, mathematical models, transfer function, state space

1 Introduction

In a number of classes of aircraft, the property of unstable angular motion about the center of mass is intentionally utilized. For instance, to ensure maneuverability in fighter aircraft, the center of aerodynamic forces is positioned ahead of the center of mass during design [1]. A similar approach is applied in the design of passenger and transport wide-body aircraft to achieve improved operational performance [2]. In launch vehicles, instability manifests at the initial stage of flight due to the overturning moment generated by aerodynamic forces [3]. In space flight, in the absence of a damping medium, spacecraft are subjected to a range of disturbing influences that induce unstable angular motion about the center of mass [4-6].

The simplest dynamically similar physical models of unstable angular motion are various inverted pendulum-type devices [7-9]. The use of pendulum-based systems makes it possible to transfer the study of instability in the angular motion of aircraft to simpler and more idealized physical models, enabling a deeper understanding of this phenomenon and the selection of appropriate means for its compensation [10-12].

A pendulum-type device in the form of an autonomous two-wheeled vehicle has attracted the attention of a number of researchers due to the specific challenge of stabilizing angular positions relative to the vertical axis [13-15].

Existing models of two-wheeled self-balancing vehicles account only for partial motions, which lead to instability in the angular orientation of the body. Thus, in [16-18], only the linear motion of two-wheeled [16, 18] and four-wheeled [17] pendulum platforms along the Ox axis is considered, using a mathematical model of the object in the form of state equations derived from Newton's second law. A representation of the dynamic properties of the more complex motion of a pendulum on a moving base is presented in [19, 20], where the moving base is a rotating platform with a constant radius. The mathematical model of the pendulum in these works is derived based on the balance of the pendulum's kinetic and potential energy [19] and using second-order Lagrangian equations [20].

Taking into account the characteristics of planar motion associated with the varying radius of curvature of the trajectory of a two-wheeled vehicle with a balanced body position would allow for the development of a more comprehensive nonlinear motion model for the subsequent analytical solution of problems involving the stable control of planar motion with a specified body angle, as in a dynamically similar prototype, for the study of unstable angular motions of aircraft.

The article presents the results of research on the development of dynamic, mathematical, and graphical models of the planar motion of a two-wheeled experimental balancing prototype.

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2 Dynamic models of the two-wheeled experimental balancing prototype

To study unstable angular motions relative to the center of mass, the Department of Aircraft Control Systems uses various pendulum devices. Figure 1 shows a two-wheeled experimental balancing prototype (TEBP).

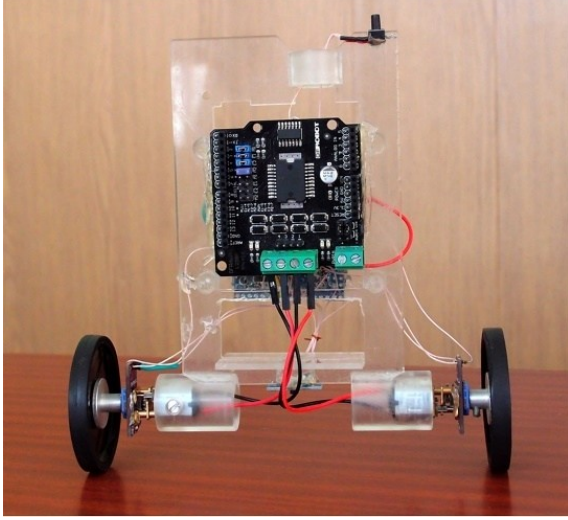


Fig. 1. Two-wheeled experimental balancing prototype

The dynamic model of the planar motion of the prototype is shown in Fig. 2.

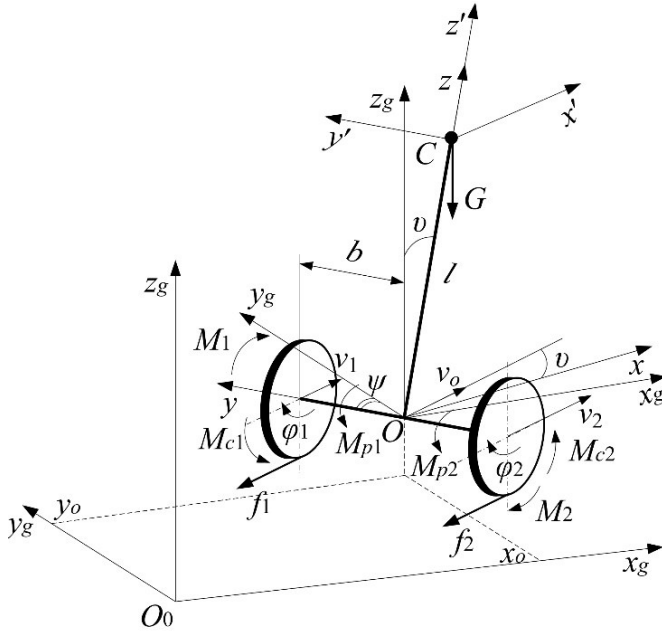


Fig. 2. Kinematic diagram of the TEBP

To describe the dynamic processes in the TEBP, the initial coordinate system $O_0x_gy_gz_g$ and the body-fixed coordinate system $Oxyz$ are used. The following forces and moments act on the TEBP: the weight force G , friction forces of the first f_1 and second wheels f_2 , driving torques of the wheels M_1 and M_2 , static friction torques in the wheel joints M_{c1} and M_{c2} , reaction torques of the electric motors with gearboxes M_{p1} and M_{p2} , and wheel friction torques M_{f1} and M_{f2} .

The dynamic model of the electric drive is shown in Fig. 3.

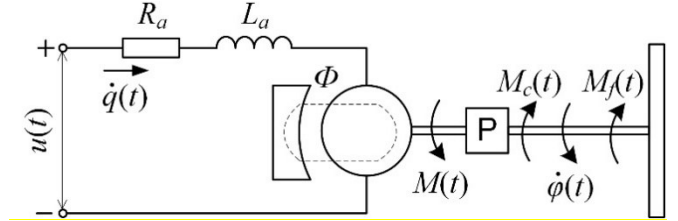


Fig. 3. Dynamic model of the electric drive

In the model, the following notations are used: $u(t)$ is the control voltage, $\dot{q}(t)$ is the armature current, R_a , L_a are the resistance and inductance of the armature winding, Φ is the flux of the permanent-magnet excitation, P is the gear ratio, $\dot{\phi}(t)$ is the angular velocity of the wheel.

The motion of the TEBP is considered under conditions of no slipping and no wheel lift-off from the surface.

3 Nonlinear mathematical model

To formalize the motion processes of the TEBP, the Lagrangian approach [21] was used in the following form:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\lambda}_i} \right) - \frac{\partial T}{\partial \lambda_i} = Q_{\lambda_i}, i = \overline{1,5}, \quad (1)$$

where T is the kinetic energy of the system, λ_i is the generalized coordinate, and Q_{λ_i} is the generalized force acting along the generalized coordinate λ_i .

The motion of the TEBP is characterized by five degrees of freedom and the following generalized coordinates: the angular position of the body $v = \lambda_1$, rotation angle of the first wheel $\varphi_1 = \lambda_2$, rotation angle of the second wheel $\varphi_2 = \lambda_3$, the electric charge of the first motor $q_1 = \lambda_4$, and the electric charge of the second motor $q_2 = \lambda_5$.

The kinetic energy of the TEBP is described by the following expression:

$$T = T_f + T_r + T_{rs} + T_e, \quad (2)$$

where T_f is kinetic energy of translational motion, T_r is the kinetic energy of rotational motion, T_{rs} is the total kinetic energy of the rotational motion of the motor rotors, gearboxes, and wheels, and T_e is the total electrical kinetic energy of motor windings.

The kinetic energy of translational motion

$$T_f = \frac{1}{2}mv_c^2, \quad (3)$$

where m is the mass of the TEBP, v_c is the absolute velocity of the center of mass of the TEBP – point C (Fig. 2).

To determine the absolute velocity, we will use the kinematic relationships between the coordinates of the center of mass C and the coordinates of the geometric center of the wheel axis O:

$$x_c = x_o + lS_vC_\psi, y_c = y_o + lS_vS_\psi, z_c = z_o + lC_v, \quad (4)$$

where x_c, y_c, z_c are the coordinates of the center of mass in the inertial (starting) coordinate system, x_o, y_o, z_o are the coordinates of the geometric center of the wheel axis in the starting coordinate system (point O), l is the distance from the wheel axis to the center of mass of the TEBP, v is the pitch angle, ψ is the yaw angle, S means sin, and C means cos.

Given the assumptions made regarding the motion characteristics of the TEBP – namely, the absence of motion about the roll axis – we can write

$$v_o = \frac{r_w}{2}(\dot{\phi}_1 + \dot{\phi}_2), v_o^2 = \dot{x}_o^2 + \dot{y}_o^2. \quad (5)$$

After differentiating Eqns. (4) and taking Eqn. (5) into account, the expression for the square of the center-of-mass velocity takes the following form:

$$\begin{aligned} v_c^2 &= \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2 = \\ &= v_o^2 + l^2\dot{v}^2 + l^2S_v^2\dot{\psi}^2 + 2lC_vv_o\dot{v} = \\ &= \frac{r_w^2}{4}(\dot{\phi}_1 + \dot{\phi}_2)^2 + l^2\dot{v}^2 + \frac{l^2r_w^2}{4b^2}S_v^2(\dot{\phi}_2 - \dot{\phi}_1)^2 + \\ &\quad + lr_wC_v(\dot{\phi}_1 + \dot{\phi}_2)\dot{v}. \end{aligned} \quad (6)$$

Let us substitute expression (6) for the square of velocity v_c^2 into Eqn. (3) and write down the expression for the kinetic energy of translational motion

$$T_f = \frac{1}{2}m \left(\frac{r_w^2}{4}(\dot{\phi}_1 + \dot{\phi}_2)^2 + l^2\dot{v}^2 + \frac{l^2r_w^2}{4b^2}S_v^2(\dot{\phi}_2 - \dot{\phi}_1)^2 + lr_wC_v(\dot{\phi}_1 + \dot{\phi}_2)\dot{v} \right). \quad (7)$$

Kinetic energy of rotational motion

$$T_r = \frac{1}{2}I_v\omega_v^2, \quad (8)$$

where I_v is the moment of inertia of the TEBP relative to the instantaneous axis of rotation v (Fig. 4), ω_v is the angular velocity of the TEBP relative to the instantaneous axis of rotation v .

The instantaneous axis of rotation passes through point O of the wheel axis in the plane Oyz_g .

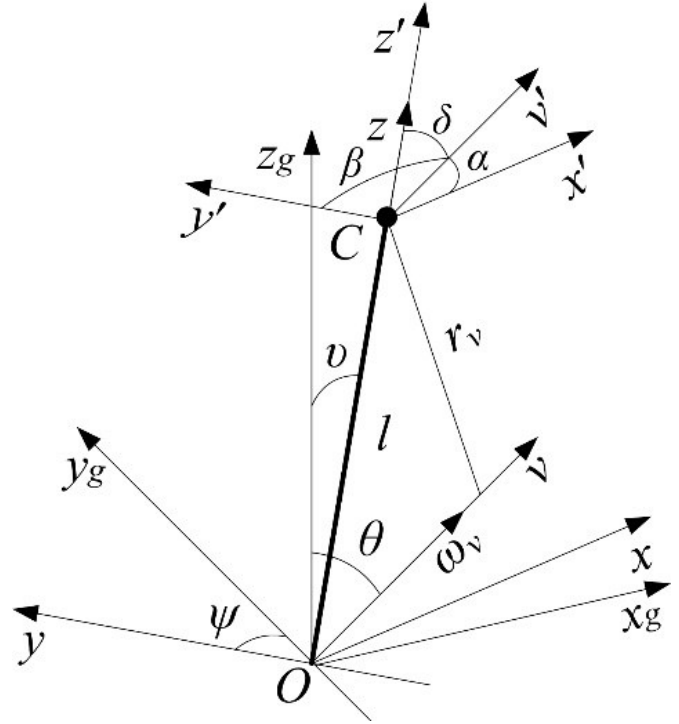


Fig. 4. Determination of the absolute angular velocity of rotation TEBP

To find I_v we use Steiner's theorem [21]:

$$I_v = I_{v'} + mr_v^2, \quad (9)$$

where $I_{v'}$ is the moment of inertia of the TEBP with respect to axis Cv' that passes through the center of mass and is parallel to the instantaneous axis of rotation Ov , r_v is the distance between axes Ov and the instantaneous axis Cv' .

Represent the moment of inertia $I_{v'}$:

$$I_{v'} = I_{x'}C_\alpha^2 + I_{y'}C_\beta^2 + I_{z'}C_\delta^2, \quad (10)$$

where $I_{x'}, I_{y'}, I_{z'}$ are the central moments of inertia of the TEBP, α, β, δ are the angles between axis Cv' and axes Cx', Cy' and Cz' respectively.

Expressions for the direction cosines in Eqn. (10) according to Fig. 2 take the form:

$$C_\alpha = -S_v C_\theta, C_\beta = -S_\theta, C_\delta = C_v C_\theta, C_\theta = \frac{\omega_{vz}}{\omega_v},$$

$$S_\theta = \frac{\sqrt{\omega_{vx}^2 + \omega_{vy}^2}}{\omega_v}. \quad (11)$$

The square of the distance between axes Ov and the instantaneous axis Cv' is

$$r_v^2 = (1 - C_v^2 C_\theta^2) l^2. \quad (12)$$

The components of the instantaneous angular velocity ω_{vx} , ω_{vy} , ω_{vz} in the starting coordinate system are determined by using the direction-cosine matrix for the transformation from the body-fixed (linked) coordinate system to the starting system at $\gamma = 0$:

$$\begin{aligned} \begin{bmatrix} \omega_{vx} \\ \omega_{vy} \\ \omega_{vz} \end{bmatrix} &= M_{Oxyz}^{Oxgygzg} \begin{bmatrix} \dot{\psi} \\ \dot{\psi} \\ \dot{v} \end{bmatrix} = \\ &= \begin{bmatrix} C_\psi C_v & S_\psi C_v - C_\psi S_v & S_\psi \\ S_v & C_v & 0 \\ -S_\psi C_v & S_\psi S_v & C_\psi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \dot{\psi} \\ \dot{v} \end{bmatrix} = \\ &= \begin{bmatrix} C_\psi C_v & S_{\psi-v} & S_\psi \\ S_v & C_v & 0 \\ -S_\psi C_v & S_\psi S_v & C_\psi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \dot{\psi} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} S_{\psi-v} \dot{\psi} + S_\psi \dot{v} \\ C_v \dot{\psi} \\ S_\psi S_v \dot{\psi} + C_\psi \dot{v} \end{bmatrix}, \end{aligned} \quad (13)$$

where $M_{Oxyz}^{Oxgygzg}$ is the direction cosine matrix, $[0 \ \dot{\psi} \ \dot{v}]^T$ is the vector of projections ω_v onto the axes of the body-fixed coordinate system $Oxyz$. Let us perform the transformation and rewrite Eqn. (9) taking into account relations (10-13). As a result, we obtain:

$$T_r = \frac{1}{2} \left((I_{x'} - I_{y'} + (I_{z'} - I_{x'} - ml^2) C_v^2) \omega_{vz}^2 + (I_{y'} + ml^2) \omega_v^2 \right), \quad (14)$$

where $\omega_{vz}^2 = S_\psi^2 S_v^2 \dot{\psi}^2 + S_{2\psi} S_v \dot{v} \dot{\psi} + C_\psi^2 \dot{v}^2$,

$$\omega_v^2 = (S_{\psi-v}^2 - C_\psi^2 S_v^2 + 1) \dot{\psi}^2 + (2S_{\psi-v} S_\psi + S_{2\psi} S_v) \dot{v} \dot{\psi} + \dot{v}^2.$$

The total kinetic energy of the rotational motion of the rotors of the electric motors with gearboxes and wheels is described by the following expression:

$$\begin{aligned} T_{rs} &= T_{rs1} + T_{rs2} = \\ &= \frac{1}{2} I_{rw} \omega_1^2 + \frac{1}{2} I_{rw} \omega_2^2 = \frac{1}{2} I_{rw} \dot{\phi}_1^2 + \frac{1}{2} I_{rw} \dot{\phi}_2^2 \end{aligned} \quad (15)$$

where $I_{rw} = (i_{red}^2 + 1) I_a + \frac{1}{2} m_w r_w^2$ is the moment of inertia of the motor armature with the gearbox and wheel

reduced to the wheel axis, i_{red} is the gear ratio of the gearbox, I_a is the moment of inertia of the motor armature, m_w is the mass of the wheel, and r_w is the wheel radius.

The total electrical kinetic energy of the motor windings

$$T_e = T_{e1} + T_{e2} = \frac{1}{2} L_a i_1^2 + \frac{1}{2} L_a i_2^2 = \frac{1}{2} L_a \dot{q}_1^2 + \frac{1}{2} L_a \dot{q}_2^2, \quad (16)$$

where L_a is the inductance of the motor armature winding.

After substituting the resulting expressions (7), (14), (15), and (16) for the partial movements, we obtain the following expression for the kinetic energy of the TEBP:

$$\begin{aligned} T &= \frac{1}{2} m (r_w^2 (\dot{\phi}_1 + \dot{\phi}_2)^2 / 4 + l^2 \dot{v}^2 + l^2 S_v^2 \dot{\psi}^2 + \\ &\quad + l r_w C_v (\dot{\phi}_1 + \dot{\phi}_2) \dot{v}) + \\ &\quad + \frac{1}{2} ((I_{x'} - I_{y'} + (I_{z'} - I_{x'} - ml^2) C_v^2) \omega_{vz}^2 + \\ &\quad + (I_{y'} + ml^2) \omega_v^2) + \frac{1}{2} I_{rw} (\dot{\phi}_1^2 + \dot{\phi}_2^2) + \frac{1}{2} L_a (\dot{q}_1^2 + \dot{q}_2^2). \end{aligned} \quad (17)$$

According to the degrees of freedom, the following generalized forces act on the TEBP:

$$Q_v = mgl S_v - \frac{1}{2} m l r_w C_v (\ddot{\phi}_1 + \ddot{\phi}_2) - M_{p1} - M_{p2} - M_{c1} - M_{c2}, \quad (18)$$

$$Q_{\phi_1} = M_1 - f_1 r_w - M_{c1}, \quad (19)$$

$$Q_{\phi_2} = M_2 - f_2 r_w - M_{c2}, \quad (20)$$

$$Q_{q_1} = u_1 - R_1 \dot{q}_1 - e_1, \quad (21)$$

$$Q_{q_2} = u_2 - R_2 \dot{q}_2 - e_2, \quad (22)$$

where $M_i = i_{red} c_m \Phi \dot{q}_i = i_{red} k_m \dot{q}_i$, $e_i = c_e \Phi \dot{\phi}_{ai} = i_{red} k_e \dot{\phi}_i$, $M_{pi} = M_i$, $\dot{\phi}_{ai}$ is the angular velocity of the motor armature rotation, k_m and k_e are conversion coefficients of the electric motor.

After transforming the kinetic energy expression (17) in accordance with the second-order Lagrangian Eqn. (1) and applying the corresponding generalized forces (18-22) for each generalized coordinate, a system of five nonlinear differential equations is obtained, see (1-5) in Appendix.

The resulting equations must be supplemented with kinematic relations that relate the coordinates of the geometric center of the wheel axis x_o , y_o , and the steering angle ψ to the steering angles of the first and second wheels:

$$\begin{aligned}\psi &= \frac{r_w}{2b}(\varphi_2 - \varphi_1), & x_o &= \frac{r_w}{2}(\varphi_1 + \varphi_2)C_{\frac{\psi}{2}}, \\ y_o &= \frac{r_w}{2}(\varphi_1 + \varphi_2)S_{\frac{\psi}{2}}.\end{aligned}\quad (23)$$

Equations (1-5) in Appendix and Eqn. (23) together describe the nonlinear motion processes of the TEBP. The nonlinear relationships between the variables characterizing the sample's motion significantly complicate the analysis and understanding of motion processes over a wide range of their variations. To simplify the study of the sample's functional properties, the range of variables can be significantly reduced, and the analysis can be conducted in a small range using first-order approximation equations.

4 Linear mathematical model

To obtain mathematical models of the TEBP as a controlled object, which are necessary for solving the problem of automatic control system synthesis, we will perform linearization by expanding the nonlinear dependencies in equations (1-5) in Appendix and Eqn. (23) into a Taylor series under the following initial conditions:

$$\begin{aligned}\dot{v}_0 &= 0, \dot{\psi}_0 = 0, \dot{\varphi}_{10} = 0, \dot{\varphi}_{20} = 0, \dot{q}_{10} = 0, \\ \dot{q}_{20} &= 0, M_{10} = 0, M_{20} = 0, v_0 = 0, \psi|_{t=0} = \psi_0, \\ \varphi_1|_{t=0} &= \varphi_{10}, \varphi_2|_{t=0} = \varphi_{20}, q_{10} = 0, q_{20} = 0.\end{aligned}\quad (24)$$

Linearization yields linear differential equations in small deviations from the initial conditions of the following form:

$$a_{11}\Delta\ddot{v} + a_{12}\Delta v = b_{11}(\Delta\ddot{\varphi}_1 + \Delta\ddot{\varphi}_2) + b_{12}(\Delta\dot{q}_1 + \Delta\dot{q}_2), \quad (25)$$

$$a_{21}\Delta\ddot{\varphi}_1 + a_{22}\Delta\ddot{v} + a_{23}\Delta\ddot{\varphi}_2 = b_{21}\Delta\dot{q}_1 + b_{22}\Delta f_1, \quad (26)$$

$$a_{31}\Delta\ddot{\varphi}_1 + a_{32}\Delta\ddot{v} + a_{33}\Delta\ddot{\varphi}_2 = b_{31}\Delta\dot{q}_2 + b_{32}\Delta f_2, \quad (27)$$

$$a_{41}\Delta\ddot{q}_1 + a_{42}\Delta\dot{q}_1 = b_{41}\Delta\dot{\varphi}_1 + b_{42}\Delta u_1, \quad (28)$$

$$a_{51}\Delta\ddot{q}_2 + a_{52}\Delta\dot{q}_2 = b_{41}\Delta\dot{\varphi}_2 + b_{42}\Delta u_2, \quad (29)$$

$$\Delta x_o = d_{11}(\Delta\varphi_1 + \Delta\varphi_2), \quad (30)$$

$$\Delta y_o = d_{21}(\Delta\varphi_1 + \Delta\varphi_2), \quad (31)$$

where the coefficient expressions are local derivatives of the Taylor series, see (6) in Appendix.

The equations obtained through linearization reflect only the linear relationships between variables in the vicinity of the initial conditions. This circumstance allows us to apply the principle of superposition and reduce the problem of analyzing the complex motion

of the TEBP to the analysis of simpler local motions caused by individual control and disturbance influences in both the frequency and time domains using simpler mathematical models.

Let us consider, using specific examples, the features of constructing such mathematical models.

5 Mathematical models in the frequency domain

Using the principle of superposition, let us consider the equations of motion for the TEBP using the first electric motor, based on equations (25), (26), (28), (30), and (31). Then

$$a_{11}\Delta\ddot{v} + a_{12}\Delta v = b_{11}\Delta\ddot{\varphi}_1 + b_{12}\Delta\dot{q}_1, \quad (32)$$

$$a_{21}\Delta\ddot{\varphi}_1 + a_{22}\Delta\ddot{v} = b_{21}\Delta\dot{q}_1 + b_{22}\Delta f_1, \quad (33)$$

$$a_{41}\Delta\ddot{q}_1 + a_{42}\Delta\dot{q}_1 = b_{41}\Delta\dot{\varphi}_1 + b_{42}\Delta u_1, \quad (34)$$

$$\Delta x_o = d_{11}\Delta\varphi_1, \quad (35)$$

$$\Delta y_o = d_{21}\Delta\varphi_1. \quad (36)$$

Let us take Eqn. (33) and substitute it into Eqns. (32) and (34). After the appropriate transformations, we obtain the following equations:

$$\Delta\ddot{v} = a'_{11}\Delta v + a'_{12}\Delta\ddot{\varphi}_1 + b'_{11}\Delta f_1, \quad (37)$$

$$\begin{aligned}\Delta\ddot{\varphi}_1 &= a'_{21}\Delta\ddot{\varphi}_1 + a'_{22}\Delta\ddot{v} + a'_{23}\Delta\ddot{v} + a'_{24}\Delta\ddot{v} + \\ &+ b'_{21}\Delta\dot{q}_1 + b'_{22}\Delta f_1 + b'_{23}\Delta u_1,\end{aligned}\quad (38)$$

where
$$a'_{11} = -\frac{a_{12}b_{21}}{a_{11}b_{21} - b_{12}a_{22}},$$

$$a'_{12} = -\frac{a_{12}b_{21}}{a_{11}b_{21} - b_{12}a_{22}}, b'_{11} = -\frac{b_{12}b_{22}}{a_{11}b_{21} - b_{12}a_{22}},$$

$$a'_{21} = -\frac{a_{42}}{a_{41}}, a'_{22} = \frac{b_{21}b_{41}}{a_{21}}, a'_{23} = \frac{a_{22}}{a_{21}},$$

$$a'_{24} = \frac{a_{42}a_{22}}{a_{41}a_{21}}, b'_{21} = -\frac{b_{22}}{a_{21}}, b'_{22} = -\frac{a_{42}b_{22}}{a_{41}a_{21}},$$

$$b'_{23} = \frac{b_{21}b_{42}}{a_{21}}.$$

Using the obtained Eqns. (37) and (38), we form the transfer functions

$$\begin{aligned}W_{11}(s) &= \frac{\Delta Y_1(s)}{\Delta \Phi_1(s)} = \frac{a'_{12}s^2}{s^2 - a'_{11}}, W_{12}(s) = \frac{\Delta Y_1(s)}{\Delta F_1(s)} = \\ &= \frac{b'_{11}}{s^2 - a'_{11}}, W_{13}(s) = \frac{\Delta \Phi_1(s)}{\Delta U_1(s)} = \frac{b'_{23}}{s^3 - a'_{21}s^2 - a'_{22}s}, \\ W_{14}(s) &= \frac{\Delta \Phi_1(s)}{\Delta Y_1(s)} = \frac{a'_{22}s^3 + a'_{21}s^2}{s^3 - a'_{21}s^2 - a'_{22}s}, \\ W_{15}(s) &= \frac{\Delta \Phi_1(s)}{\Delta F_1(s)} = \frac{b'_{21}s + b'_{22}}{s^3 - a'_{21}s^2 - a'_{22}s}.\end{aligned}\quad (39)$$

The images of the yaw angle $\Delta\Phi_1(s)$ and the pitch angle $\Delta\gamma_1(s)$ will be determined by the following operator equations:

$$\begin{aligned}\Delta\Phi_1(s) &= W_{13}(s)\Delta U_1(s) + W_{14}(s)\Delta\gamma_1(s) + \\ &\quad + W_{15}(s)\Delta F_1(s), \\ \Delta\gamma_1(s) &= W_{11}(s)\Delta\Phi_1(s) + W_{12}(s)\Delta F_1(s). \quad (40)\end{aligned}$$

By performing analogous transformations with the equations of motion of the TEBP driven by the second electric motor, we obtain five transfer functions:

$$\begin{aligned}W_{21}(s) &= \frac{\Delta\gamma_2(s)}{\Delta\Phi_2(s)} = \frac{a''_{12}s^2}{s^2 - a''_{11}}, W_{22}(s) = \frac{\Delta\gamma_2(s)}{\Delta F_2(s)} = \\ &= \frac{b''_{11}}{s^2 - a''_{11}}, W_{23}(s) = \frac{\Delta\Phi_2(s)}{\Delta U_2(s)} = \frac{b''_{23}}{s^3 - a''_{21}s^2 - a''_{22}s}, \\ W_{24}(s) &= \frac{\Delta\Phi_2(s)}{\Delta\gamma_2(s)} = \frac{a''_{22}s^3 + a''_{21}s^2}{s^3 - a''_{21}s^2 - a''_{22}s}, \\ W_{25}(s) &= \frac{\Delta\Phi_2(s)}{\Delta F_2(s)} = \frac{b''_{21}s + b''_{22}}{s^3 - a''_{21}s^2 - a''_{22}s}. \quad (41)\end{aligned}$$

Using these transfer functions, the output variables are described by the following differential equations:

$$\begin{aligned}\Delta\Phi_2(s) &= W_{23}(s)\Delta U_2(s) + W_{24}(s)\Delta\gamma_2(s) + \\ &\quad + W_{25}(s)\Delta F_2(s), \\ \Delta\gamma_2(s) &= W_{21}(s)\Delta\Phi_2(s) + W_{22}(s)\Delta F_2(s). \quad (42)\end{aligned}$$

The transfer functions $W_X(s)$ and $W_Y(s)$ represent the relationship between the Laplace transforms of the coordinates $\Delta X_o(s)$ and $\Delta Y_o(s)$ and the sum of the transforms $\Delta\Phi_1(s)$ and $\Delta\Phi_2(s)$, respectively, in accordance with equations (30) and (31):

$$\begin{aligned}W_X(s) &= \frac{\Delta X_o(s)}{\Delta\Phi_1(s) + \Delta\Phi_2(s)} = d_{11}, \\ W_Y(s) &= \frac{\Delta Y_o(s)}{\Delta\Phi_1(s) + \Delta\Phi_2(s)} = d_{21}, \\ \Delta X_o(s) &= W_X(s)(\Delta\Phi_1(s) + \Delta\Phi_2(s)), \\ \Delta Y_o(s) &= W_Y(s)(\Delta\Phi_1(s) + \Delta\Phi_2(s)). \quad (43)\end{aligned}$$

The transfer functions obtained make it possible to construct a graphical model of the transformation processes in the TEBP in the form of a block diagram (Fig. 5).

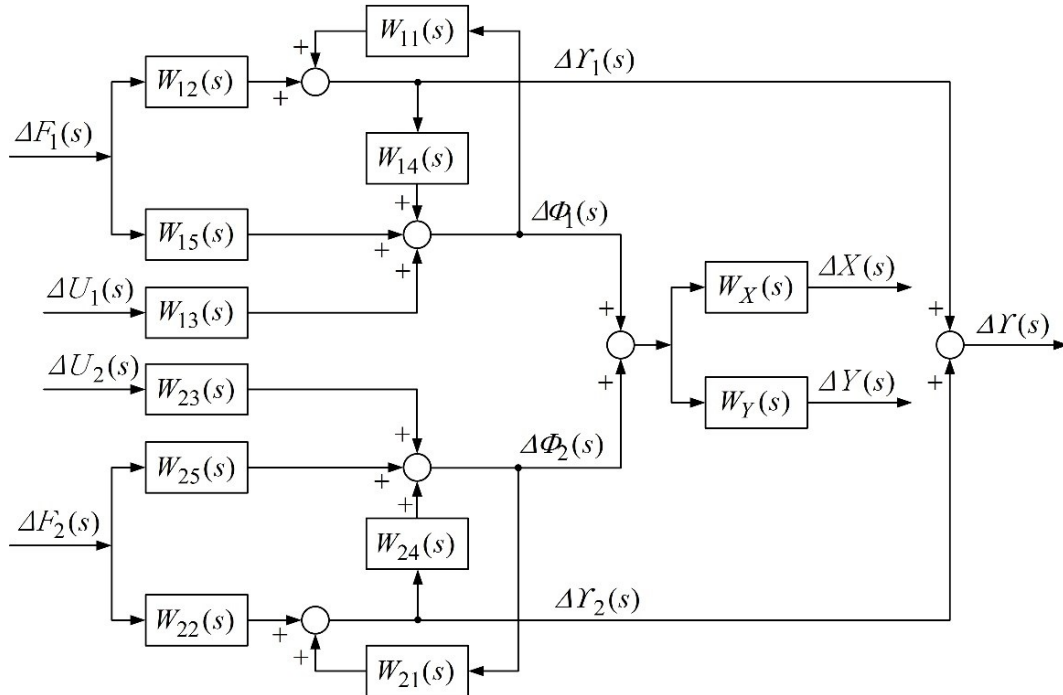


Fig. 5. Block diagram of the TEBP

The block diagram allows us to form transfer functions that reflect the functional relationships between the control and disturbance actions and the output signals accessible to measurement. Thus, the transfer function that describes the relationship between $\Delta\Phi_1(s)$ and $\Delta U_1(s)$ is determined by the following expression:

$$W_{U1}(s) = \frac{\Delta\Phi_1(s)}{\Delta U_1(s)} = \frac{W_{13}(s)}{1 - W_{14}(s)W_{11}(s)} = \frac{b_{33}(s^2 - a'_{11})}{s^5 + a_{55}s^4 + a_{54}s^3 + a_{53}s^2 + a_{52}s}, \quad (44)$$

where $a_{52} = \frac{a'_{11}a'_{22}}{1 - a'_{12}a'_{22}}, a_{53} = \frac{a'_{11}a'_{21}}{1 - a'_{12}a'_{22}},$
 $a_{54} = -\frac{a'_{11} + a'_{22}}{1 - a'_{12}a'_{22}}, a_{55} = -\frac{(1 + a'_{12})a'_{21}}{1 - a'_{12}a'_{22}},$
 $b_{33} = \frac{b'_{23}}{1 - a'_{12}a'_{22}}.$

The influence of the control action on the TEBP body deflection angle is described by a transfer function of the form:

$$W_{Y1}(s) = \frac{\Delta Y_1(s)}{\Delta U_1(s)} = W_{11}(s)W_{U1}(s) = \frac{a'_{12}b_{33}s^2}{s^5 + a_{55}s^4 + a_{54}s^3 + a_{53}s^2 + a_{52}s}. \quad (45)$$

The influence of the disturbance action on the wheel angular position in operational form is described by the following expression:

$$W_{F1}(s) = \frac{\Delta\Phi_1(s)}{\Delta F_1(s)} = \frac{W_{12}(s)W_{14}(s) + W_{15}(s)}{1 - W_{14}(s)W_{11}(s)} = \frac{b'_{11}a'_{21}s^3 + b'_{11}a'_{21}s^2 + b'_{21}s + b'_{22}}{s^5 + a_{55}s^4 + a_{54}s^3 + a_{53}s^2 + a_{52}s}. \quad (46)$$

Similar mathematical models in the form of transfer functions can be formed using equivalent transformations of the block diagram (Fig. 5). There are sixteen such transfer functions, and they allow us to evaluate both the necessary conditions for operability–stability and the sufficient conditions – the quality of TEBP functioning.

6 Mathematical models in the time domain

The state-space method makes it possible to describe the motion processes of the TEBP in the time domain in a vector-matrix form [22]. The structure of the vector-matrix equations depends on the choice of the basis used to form the state variables. As an example, we present the time-domain description in the form of a controllable canonical form.

We represent the transfer function $W_{U1}(s)$ (44) as the product of transfer functions

$$W_{U1}(s) = W'_{U1}(s)W''_{U1}(s), \quad (47)$$

by introducing a new image $\Delta Z(s)$. Then

$$W'_{U1}(s) = \frac{\Delta Z(s)}{\Delta U_1(s)}, W''_{U1}(s) = \frac{\Delta\Phi_1(s)}{\Delta Z(s)}. \quad (48)$$

In detailed form

$$W'_{U1}(s) = \frac{\Delta Z(s)}{\Delta U_1(s)} = \frac{1}{s^5 + a_{55}s^4 + a_{54}s^3 + a_{53}s^2 + a_{52}s},$$

$$W''_{U1}(s) = \frac{\Delta\Phi_1(s)}{\Delta Z(s)} = b_{33}(s^2 - a'_{11}). \quad (49)$$

If we pass to differential equations and introduce five state variables: $\Delta z(t) = \Delta x_1(t)$, $\Delta \dot{x}_1(t) = \Delta x_2(t)$, $\Delta \dot{x}_2(t) = \Delta x_3(t)$, $\Delta \dot{x}_3(t) = \Delta x_4(t)$, $\Delta \dot{x}_4(t) = \Delta x_5(t)$, then the influence of $\Delta u_1(t)$ on the variable $\Delta\varphi_1(t)$ can be expressed in the following vector–matrix form:

$$\begin{bmatrix} \Delta \dot{x}_1(t) \\ \Delta \dot{x}_2(t) \\ \Delta \dot{x}_3(t) \\ \Delta \dot{x}_4(t) \\ \Delta \dot{x}_5(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ \Delta x_3(t) \\ \Delta x_4(t) \\ \Delta x_5(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Delta u_1(t), \quad (50)$$

$$\Delta\varphi_1(t) = [c_{11} \quad 0 \quad c_{13} \quad 0 \quad 0] \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ \Delta x_3(t) \\ \Delta x_4(t) \\ \Delta x_5(t) \end{bmatrix} + \varphi_{10}, \quad (51)$$

where $c_{11} = -b_{33}a'_{11}$, $c_{13} = b_{33}$.

By performing similar transformations on the transfer function $W_{Y1}(s)$ (45), we obtain a description in a five-dimensional state space:

$$\begin{bmatrix} \Delta \dot{x}_1(t) \\ \Delta \dot{x}_2(t) \\ \Delta \dot{x}_3(t) \\ \Delta \dot{x}_4(t) \\ \Delta \dot{x}_5(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ \Delta x_3(t) \\ \Delta x_4(t) \\ \Delta x_5(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Delta u_1(t), \quad (52)$$

$$\Delta v_1(t) = \begin{bmatrix} 0 & 0 & c_{23} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ \Delta x_3(t) \\ \Delta x_4(t) \\ \Delta x_5(t) \end{bmatrix}, \quad (53) \quad \begin{bmatrix} \Delta \varphi_1(t) \\ \Delta v_1(t) \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & c_{13} & 0 & 0 \\ 0 & 0 & c_{23} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ \Delta x_3(t) \\ \Delta x_4(t) \\ \Delta x_5(t) \end{bmatrix} + \begin{bmatrix} \varphi_{10} \\ 0 \end{bmatrix}. \quad (55)$$

where $c_{23} = a'_{12}b_{33}$.

By combining Eqns. (50-53), we obtain a state-space representation that describes the effect of the control input $\Delta u_1(t)$ on variables $\Delta \varphi_1(t)$ and $\Delta v_1(t)$:

$$\begin{bmatrix} \Delta \dot{x}_1(t) \\ \Delta \dot{x}_2(t) \\ \Delta \dot{x}_3(t) \\ \Delta \dot{x}_4(t) \\ \Delta \dot{x}_5(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ \Delta x_3(t) \\ \Delta x_4(t) \\ \Delta x_5(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Delta u_1(t), \quad (54)$$

The resulting state-space description is presented as a graphical model in the form of a block (structural) diagram (Fig. 6).

The influence of the disturbance input $\Delta f_1(t)$ on the angular position of the wheel $\Delta \varphi_1(t)$ according to Eqn. (46), is added to the obtained block diagram, resulting in the following graphical model (Fig. 7).

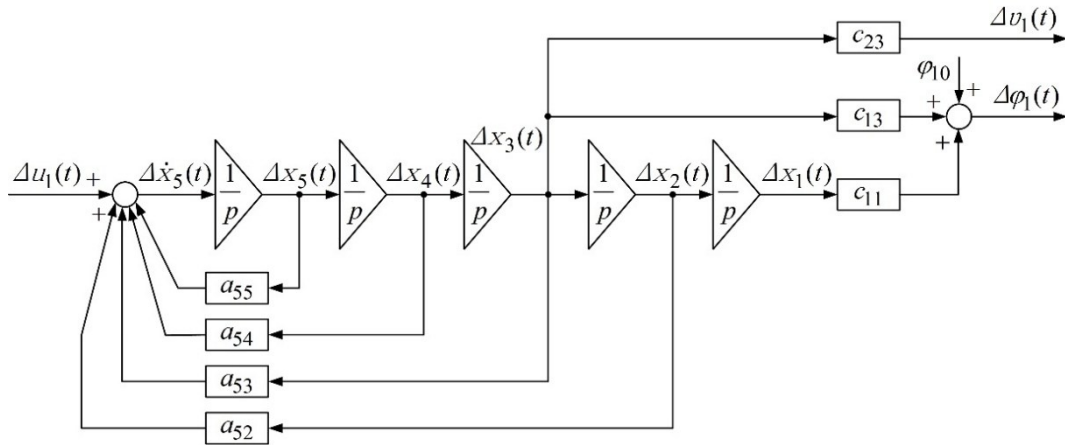


Fig. 6. Structural diagram of the TEBP

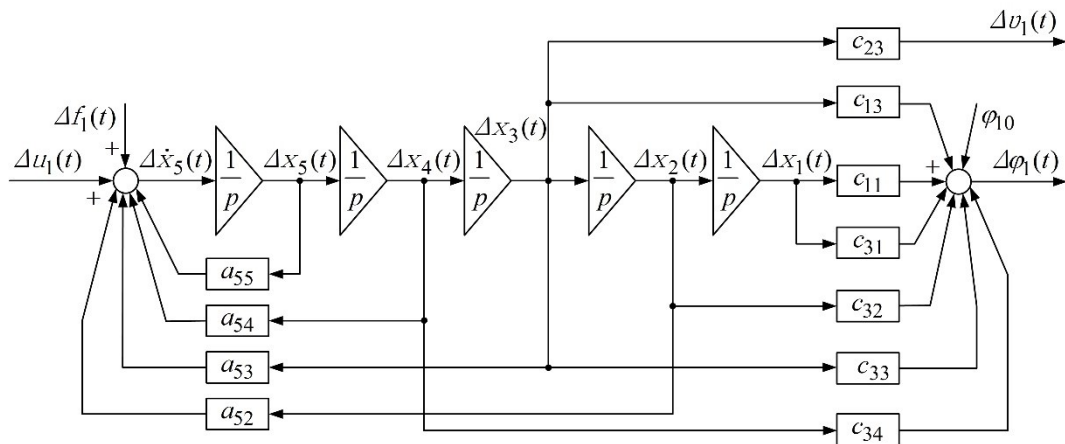


Fig. 7. Resulting structural diagram of the TEBP

The influence of the disturbance input $\Delta f_1(t)$ on the angular position of the wheel $\Delta \varphi_1(t)$ according to Eqn. (46) is added to the obtained block diagram, resulting in the following graphical model (Fig. 7).

In the presented block diagram $c_{31} = b'_{22}$, $c_{32} = b'_{21}$, $c_{33} = b'_{11}$, and $c_{34} = b'_{11}a'_{21}$.

In a similar manner, state-space mathematical models can be formed for other input and output variables and in terms of other bases of state variables, which makes it possible to obtain the desired structure of the matrices in the vector-matrix description of the TEBP motion. Mathematical models in the state space make it possible to assess the functional properties of the TEBP, such as stability, controllability and observability, identifiability, diagnosability, and other properties required for the synthesis of an autonomous motion control system in the plane. With an adaptive approach to TEBP control in the face of a range of destabilizing factors, mathematical models in state space allow for a more comprehensive representation of operating conditions and the synthesis of effective algorithms for rational control.

7 Results of the numerical study

The further use of linearized mathematical models of TEBP for solving the problems of synthesizing stabilization algorithms is associated with determining the conditions of controllability and observability of the control object. The mathematical model of TEBP in the frequency domain, in the form of transfer functions (44-46), reflects the properties of the controllable and observable parts of the object based on the output signals $\Delta \varphi_1(t)$ and $\Delta v_1(t)$ does not require additional analysis. In the case of forming state-based stabilization algorithms using Eqns. (54), (55), satisfying the conditions of controllability and observability is mandatory.

The study of these conditions can be carried out for a specific automatic control object (Fig. 1) with the following parameters: mass of the TEBP $m = 0.52$ kg, mass of the wheel $m_w = 0.0454$ kg, the central moments of inertia of the TEBP $I_y = 3.88 \times 10^{-3}$ kg m², $I_z = 1.2 \times 10^{-3}$ kg m², the moment of inertia of the wheel $I_w = 9.2 \times 10^{-5}$ kg m², wheel radius $r_w = 0.045$ m, the distance from the wheel axis to the center of mass of the TEBP $l = 0.116$ m, the distance between the wheel plane and the vertical plane of symmetry of the TEBP $b = 0.06$ m, the resistance of the electric motor winding $R_a = 2$ Ω , the inductance of the electric motor winding $L_a = 0.005$ H, motor conversion coefficients $k_e = 2.2 \times 10^{-4}$ V s, $k_m = 3.5 \times 10^{-3}$ kg m²/A, and gear ratio of the reducer $i_{red} = 380$.

In accordance with the physical parameters of the TEBP, the coefficients in the state equations (54), (55) take the following values:

$$\begin{aligned} a_{52} &= -133.4, a_{53} = -904.3, a_{54} = -0.11, \\ a_{55} &= 7.6, c_{11} = 1600, c_{13} = -25.6, \\ c_{23} &= 12.4, \end{aligned} \quad (56)$$

and the state matrix A , the input matrix b and the output matrix c are respectively equal to

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -133.4 & -904.3 & -0.11 & 7.6 \end{bmatrix},$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1600 & 0 & -25.6 & 0 & 0 \\ 0 & 0 & 12.4 & 0 & 0 \end{bmatrix}.$$

The ability to control all state variables of the vector $\Delta x(t)$ and to observe all its changes is evaluated using the Kalman criteria [23]. Thus, the controllability of the TEBP will be assessed using the controllability criterion

$$\text{rank } R = \text{rank}[b, Ab, A^2b, A^3b, A^4b],$$

where rank – denotes the rank of the controllability matrix. By performing the operations required by the criterion on matrices A and b , we obtain the controllability matrix of dimension $\dim R = 5 \times 5$:

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 1 & 8 & 58 \\ 0 & 1 & 8 & 58 & -467 \\ 1 & 8 & 58 & -467 & -10562 \end{bmatrix}.$$

There are no columns in the matrix that are identical in structure. Therefore, $\text{rank } R = 5$ and the object is completely controllable.

The ability to observe changes in the state vector $\Delta x(t)$ based on the available measurements is evaluated using the observability criterion

$$\text{rank } Q = \text{rank}[c^T (cA)^T (cA^2)^T (cA^3)^T].$$

By performing the corresponding transformations with matrices A and c , we obtain the observability matrix of dimension $\dim Q = 10 \times 5$:

$$Q = \begin{bmatrix} 1600 & 0 & -30 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 1600 & 0 & -30 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 1600 & 0 & -30 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 3420 & 23150 & 1600 & -190 \\ 0 & -1650 & -11210 & -1,4 & 90 \\ 0 & 25950 & 179360 & 23170 & 120 \\ 0 & -12570 & -86880 & -11220 & 710 \end{bmatrix}$$

The observability matrix contains five linearly independent columns, which indicates that the control object is fully observable. Therefore, the system is completely observable $rank\ Q = 5$.

Controllability and observability are structural criteria that reflect the properties of an automatic control system. These properties can be qualitatively assessed using the block diagram (Fig. 7) by analyzing the relationships between the control input $\Delta u_1(t)$ and the components of the state vector $\Delta x(t)$, as well as the relationships between the state vector $\Delta x(t)$ and the error $\Delta u_{v1}(t)$. Controllability and observability also follow from the fact that the transfer function (53) reflects only the controllable and observable properties of the controlled object, which includes the electric drive of the first wheel and the TEBP housing.

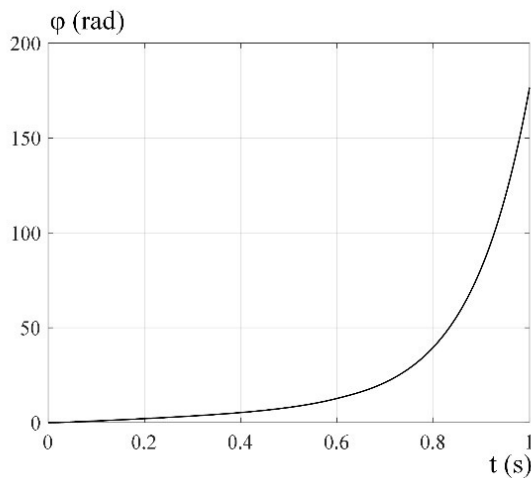
A characteristic feature of the operation of the TEBP as an inverted pendulum is the form of dynamic processes in the vicinity of two vertical body positions: the upper unstable position $v_0 = 0$ and the lower stable position ($v_0 = \pm\pi$ rad). To describe the dynamic processes about the lower body position, the values of the state-space equation coefficients (54) and (55) are as follows:

$$\begin{aligned} a_{52} &= -625.8, a_{53} = -4244, a_{54} = -17.8, \\ a_{55} &= -55, c_{11} = 7486, c_{13} = 86.5, \\ c_{23} &= 10.5. \end{aligned} \quad (57)$$

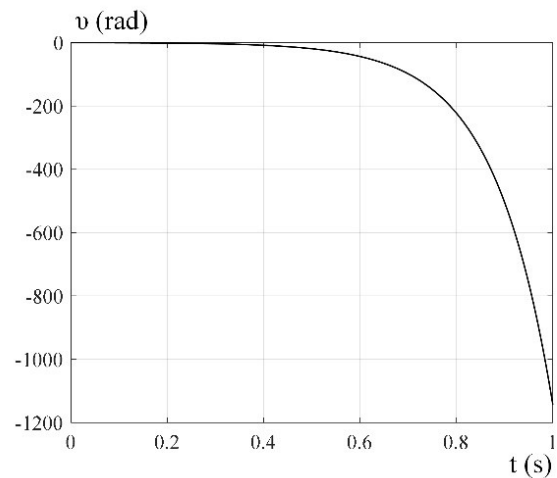
Using the simulation scheme constructed based on the block diagram in Fig. 7, as well as the coefficient values (56), (57) for the two initial body positions, impulse transient responses of the TEBP were obtained (Fig. 8) with respect to the control input applied to the first electric motor. The responses illustrate the time variation of the angular position of the first wheel $\Delta\varphi_1(t)$ and the angular position of the body in the vertical plane $\Delta v(t)$. The parameters of the impulse variation of the control voltage $\Delta u_1(t)$ are: amplitude $\Delta u_1 = 1$ V, pulse duration $\tau_1 = 0.8$ s.

The plots in Fig. 8 confirm that the linearized mathematical model of the TEBP correctly captures the characteristic features of the dynamic processes in the upper (Figs. 8 a, b) and lower (Figs. 8 c, d) body positions. The processes in the lower position (Fig. 8 c, d) are stable and exhibit oscillations with a frequency of $\omega = 3\pi\text{ s}^{-1}$, which indicates the presence of complex conjugate roots in the characteristic Eqn. (53) of the TEBP. The angular motion of the body in the impulse transient process is spatial, with components relative to the Oy axis due to the pitch angle of the body in the vertical plane v and relative to the Oz_g axis due to the yaw angle ψ of the body in the horizontal plane Ox_gz_g (Fig. 2), with an angular velocity ω_v .

Thus, there is a mutual influence of the inertial properties of the body and the first electric motor on the spatial angular motion of the TEBP, which is reflected in the transfer function $W_{11}(s)$ of the conservative link (47). A similar effect is observed when a control input is applied to the second electric motor.



(a)



(b)

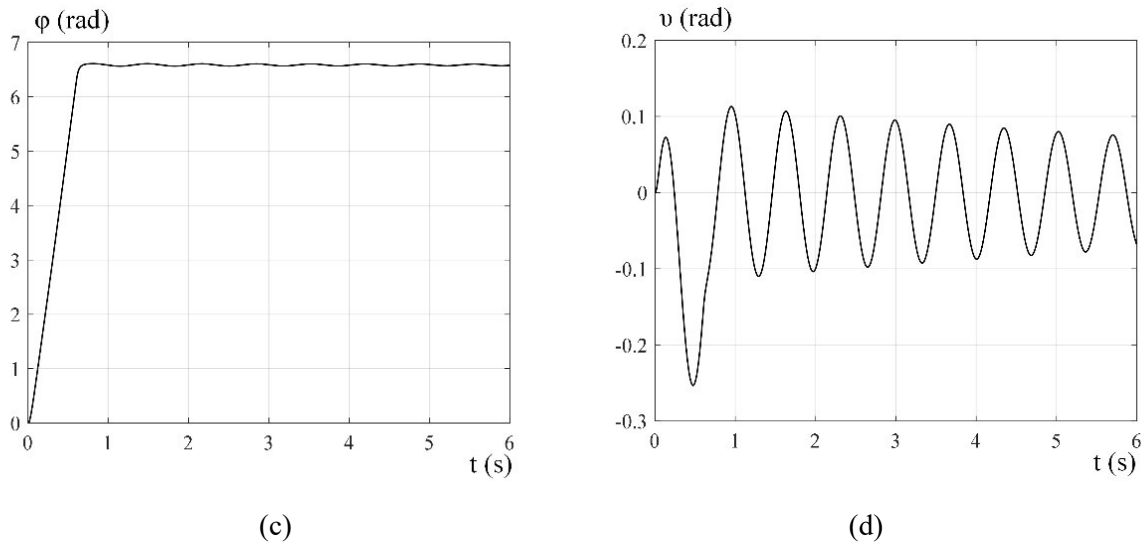


Fig. 8. Impulse transient responses of the TEBP for the angle of rotation of the first wheel φ and the angular position of the housing v : (a), (b) in the upper position of the housing $v_0 = 0$, and (c), (d) in the lower position of the housing $v_0 = \pi$ rad

8 Conclusion

As a result of the investigation of the translational and angular motions of the two-wheeled experimental balancing prototype on a plane, dynamic models of mechanical and electromechanical processes were developed. These models enabled the formulation, using the Lagrangian approach, of a nonlinear mathematical model in the form of seven nonlinear differential equations with constant coefficients, describing the motion in the “large”. To analyze the motion in the “small”, a linear approximation model was obtained using the method of analytical linearization in the form of differential equations with constant coefficients. Based on these equations, mathematical models of local motions in the frequency domain were derived in the form of ten transfer functions, and a block diagram of the transformation processes in the two-wheeled experimental balancing prototype was constructed. Using the block diagram, the possibility of constructing state-space mathematical models was demonstrated both in vector-matrix form and in graphical form as structural (block) diagrams.

The feasibility and appropriateness of using the obtained linearized mathematical models of the TEBP are related to the satisfaction of controllability and observability conditions. For this control object, these conditions are confirmed by the fact that the ranks of the controllability matrix $\text{rank}(R)$ and the observability matrix $\text{rank}(Q)$, obtained from the state-space equations, are equal to the order of the characteristic equation $n = 5$. In addition, controllability and observability also follow from the fact that the state-space equations are derived from the transfer function $W_{U1}(s)$, which reflects only the controllable and observable properties

of the control object comprising the electric drive of the first wheel and the body of the TEBP.

Inverted pendulums on a moving base of various designs exhibit characteristic features regarding the stability of dynamic processes in the vicinity of two pendulum positions – upper and lower. Instability in the upper vertical position is caused by the destabilizing effect of the gravitational moment $mgL\sin v$, which promotes deviation of the pendulum from the vertical equilibrium position $v = 0$, even in the absence of external disturbances. Conversely, in the lower position $v = \pm\pi$ rad, the action of the gravitational moment leads to the stabilization of the free motion of the pendulum. The impulse transient response plots obtained using the linearized mathematical model of the TEBP correctly reflect the characteristic features of the dynamic processes in the vicinity of both the upper and lower body positions. The processes corresponding to the lower position are stable and accompanied by oscillations with a frequency of $\omega = 3\pi \text{ s}^{-1}$, which indicates the presence of complex conjugate roots in the characteristic equation of the TEBP. The angular motion of the body in the impulse transient process is spatial, with components relative to the Oy axis corresponding to the pitch angle v of the body in the vertical plane and relative to the Oz_g axis corresponding to the yaw angle ψ of the body in the horizontal Ox_gz_g plane, with an angular velocity ω_v . Thus, there is a mutual influence between the inertial properties of the body and the first electric motor on the spatial angular motion of the TEBP, which is associated with the structural features of the system and is reflected in the transfer function $W_{11}(s)$ of the conservative link.

The obtained models of planar motion of the two-wheeled experimental balancing prototype (TEBP) provide opportunities for further research aimed at the

development of autonomous control algorithms in both the frequency and time domains using classical and modern adaptive synthesis methods, as well as for conducting experimental studies on the implementation of complex planar motions. Control of individual motion components can be achieved through the synthesis of compensating elements within the stabilization loop for vertical position and the positioning loop for planar motion coordinates of the TEBP. The choice of synthesis method depends on multiple factors (including the intended application of the two-wheeled pendulum platform, operating conditions, control objectives for pendulum and translational motion, and requirements for motion parameter measurement) and requires the use of an appropriate form of the mathematical model. For example, compensating elements of series and cascade types can be synthesized using the logarithmic amplitude-frequency characteristics method based on the transfer functions of the TEBP. In this case, ensuring control performance requires only a minimal set of motion parameter sensors.

For stable and high-quality control of the state of the two-wheeled experimental balancing prototype (TEBP), modal control methods based on state-space equations can be employed. The transformation from a transfer function to a state-space representation is not unique, as it depends on the choice of state variables. Accordingly, to ensure high-quality state control, the conditions of controllability, observability, and measurability of all state variables must be satisfied, which is not always feasible in practice. In this work, a mathematical model in the time domain is derived in the controllable canonical form, ensuring the fulfillment of controllability and observability conditions. However, the state variables themselves do not possess direct physical interpretation. Therefore, estimation of the state variables – required for the synthesis of feedback compensators – necessitates the use of a full-order Luenberger observer.

Thus, the obtained mathematical models of the TEBP in different representations provide a more comprehensive description of its dynamic properties and enable the selection of an appropriate motion control system synthesis method in accordance with the intended application of the two-wheeled pendulum platform.

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Appendix

Nonlinear differential equations of motion of the two-wheeled experimental balancing prototype (TEBP)

$$\begin{aligned}
 & \frac{1}{2} m (2l^2 \ddot{v} - lr_w S_v (\dot{\varphi}_1 + \dot{\varphi}_2) \dot{v} + lr_w C_v (\ddot{\varphi}_1 + \ddot{\varphi}_2)) - \frac{1}{2} (I_{z'} - I_{x'} - ml^2) \times \\
 & \times \left(\frac{r_w}{2b} S_{2\psi} S_v S_{2v} (\dot{\varphi}_2 - \dot{\varphi}_1) \dot{v} + 2C_{\psi}^2 S_{2v} \dot{v}^2 \right) + \frac{1}{2} (I_{x'} - I_{y'} + (I_{z'} - I_{x'} - ml^2) C_v^2) \times \\
 & \times \left(\frac{r_w^2}{2b^2} C_{2\psi} S_v (\dot{\varphi}_2 - \dot{\varphi}_1)^2 - \frac{r_w}{2b} S_{2\psi} (2 - C_v) (\dot{\varphi}_2 - \dot{\varphi}_1) \dot{v} + \frac{r_w}{2b} S_{2\psi} S_v (\ddot{\varphi}_2 - \ddot{\varphi}_1) + 2C_{\psi}^2 \ddot{v} \right) + \frac{1}{2} (I_{y'} + ml^2) \times \\
 & \times \left(\frac{r_w}{2b} (2S_{\psi-v} S_{\psi} + S_{2\psi} S_v) (\ddot{\varphi}_2 - \ddot{\varphi}_1) + 2\ddot{v} \right) + \frac{r_w^2}{4b^2} (I_{y'} + ml^2) (C_{\psi-v} S_{\psi} + S_{\psi-v} C_{\psi} + C_{2\psi} S_v) (\dot{\varphi}_2 - \dot{\varphi}_1)^2 + \\
 & + \frac{r_w}{4b} (I_{y'} + ml^2) (S_{2\psi} C_v - 2C_{\psi-v} S_{\psi}) (\dot{\varphi}_2 - \dot{\varphi}_1) \dot{v} - \frac{1}{2} m \left(\frac{l^2 r_w^2}{4b^2} S_{2v} (\dot{\varphi}_2 - \dot{\varphi}_1)^2 - lr_w S_v (\dot{\varphi}_1 + \dot{\varphi}_2) \dot{v} \right) + \frac{1}{2} \times \\
 & \times (I_{z'} - I_{x'} - ml^2) S_{2v} \omega_{vz}^2 - \frac{1}{2} (I_{x'} - I_{y'} + (I_{z'} - I_{x'} - ml^2) C_v^2) \times \\
 & \times \left(\frac{r_w^2}{4b^2} S_{\psi}^2 S_{2v} (\dot{\varphi}_2 - \dot{\varphi}_1)^2 + \frac{r_w}{2b} S_{2\psi} C_v \dot{v} (\dot{\varphi}_2 - \dot{\varphi}_1) \right) + \frac{r_w^2}{8b^2} (I_{y'} + ml^2) (S_{2(\psi-v)} + C_{\psi}^2 S_{2v}) (\dot{\varphi}_2 - \dot{\varphi}_1)^2 - \\
 & - \frac{r_w}{4b} (I_{y'} + ml^2) (S_{2\psi} C_v - 2C_{\psi-v} S_{\psi}) \dot{v} (\dot{\varphi}_2 - \dot{\varphi}_1) = mgl S_v - \frac{1}{2} mlr_w C_v (\ddot{\varphi}_1 + \ddot{\varphi}_2) - i_{red} k_m \dot{q}_1 - \\
 & - i_{red} k_m \dot{q}_2 - M_{c1} - M_{c2} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{2} m \left(\frac{r_w^2}{2} (\ddot{\varphi}_1 + \ddot{\varphi}_2) - \frac{l^2 r_w^2}{2b^2} S_v^2 (\ddot{\varphi}_2 - \ddot{\varphi}_1) - \frac{l^2 r_w^2}{2b^2} S_{2v} (\ddot{\varphi}_2 - \dot{\varphi}_1) \dot{v} - lr_w S_v \dot{v}^2 + lr_w C_v \ddot{v} \right) + \frac{r_w}{4b} (I_{z'} - I_{x'} - ml^2) \times \\
 & \times \left(\frac{r_w}{b} S_{\psi}^2 S_v^2 S_{2v} (\dot{\varphi}_2 - \dot{\varphi}_1) \dot{v} + S_{2\psi} S_v S_{2v} \dot{v}^2 \right) - \frac{r_w}{4b} (I_{x'} - I_{y'} + (I_{z'} - I_{x'} - ml^2) C_v^2) \times \\
 & \times \left(\frac{r_w^2}{2b^2} S_{2\psi} S_v^2 (\dot{\varphi}_2 - \dot{\varphi}_1)^2 + \frac{r_w}{b} S_{\psi}^2 S_{2v} (\dot{\varphi}_2 - \dot{\varphi}_1) \dot{v} + \frac{r_w}{b} S_{\psi}^2 S_v^2 (\ddot{\varphi}_2 - \ddot{\varphi}_1) + \frac{r_w}{b} C_{2\psi} S_v (\dot{\varphi}_2 - \dot{\varphi}_1) \dot{v} + S_{2\psi} C_v \dot{v}^2 + \right. \\
 & \left. + S_{2\psi} C_v \ddot{v} \right) - \frac{r_w^2}{4b^2} (I_{y'} + ml^2) (S_{\psi-v}^2 + C_{\psi}^2 S_v^2 + 1) (\ddot{\varphi}_2 - \ddot{\varphi}_1) - \frac{r_w^2}{4b^2} (I_{y'} + ml^2) \times \\
 & \times \left(S_{2(\psi-v)} \left(\frac{r_w}{2b} (\dot{\varphi}_2 - \dot{\varphi}_1) - \dot{v} \right) + \frac{r_w}{2b} S_{2\psi} S_v^2 (\dot{\varphi}_2 - \dot{\varphi}_1) - C_{\psi}^2 S_{2v} \dot{v} \right) (\dot{\varphi}_2 - \dot{\varphi}_1) - \frac{r_w}{4b} (I_{y'} + ml^2) \times \\
 & \times \left(2S_{\psi-v} S_{\psi} + S_{2\psi} S_v \right) \ddot{v} - \frac{r_w}{4b} (I_{y'} + ml^2) \left(2C_{\psi-v} S_{\psi} \left(\frac{r_w}{2b} (\dot{\varphi}_2 - \dot{\varphi}_1) - \dot{v} \right) \dot{v} + \frac{r_w}{b} S_{\psi-v} C_{\psi} (\dot{\varphi}_2 - \dot{\varphi}_1) \dot{v} + \right. \\
 & \left. + \frac{r_w}{b} C_{2\psi} S_v (\dot{\varphi}_2 - \dot{\varphi}_1) \dot{v} + S_{2\psi} C_v \dot{v}^2 \right) + I_{rw} \ddot{\varphi}_1 + \frac{1}{2} (I_{x'} - I_{y'} + (I_{z'} - I_{x'} - ml^2) C_v^2) \times \\
 & \times \left(\frac{r_w^3}{2b^3} S_{2\psi} S_v^2 (\dot{\varphi}_2 - \dot{\varphi}_1)^2 + \frac{r_w^2}{2b^2} C_{2\psi} S_v (\dot{\varphi}_2 - \dot{\varphi}_1) \dot{v} + \frac{r_w}{2b} S_{2\psi} \dot{v}^2 \right) + \frac{r_w^3}{16b^3} (I_{y'} + ml^2) (S_{2(\psi-v)} + S_{2\psi} S_v^2) \times \\
 & \times (\dot{\varphi}_2 - \dot{\varphi}_1)^2 + \frac{r_w^2}{4b^2} (I_{y'} + ml^2) (C_{\psi-v} S_{\psi} + S_{\psi-v} C_{\psi} + 2C_{2\psi} S_v) (\dot{\varphi}_2 - \dot{\varphi}_1) \dot{v} = i_{red} k_m \dot{q}_1 - f_1 r_w - M_{c1} \quad (2)
 \end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}m \left(\frac{r_w^2}{2}(\ddot{\phi}_1 + \ddot{\phi}_2) + \frac{l^2 r_w^2}{2b^2} S_v^2(\ddot{\phi}_2 - \ddot{\phi}_1) + \frac{l^2 r_w^2}{2b^2} S_{2v}(\dot{\phi}_2 - \dot{\phi}_1)\dot{v} - l r_w S_v \dot{v}^2 + l r_w C_v \ddot{v} \right) - \frac{r_w}{4b} (I_{z'} - I_{x'} - m l^2) \times \\
& \quad \times \left(\frac{r_w}{b} S_{\psi}^2 S_{2v}(\dot{\phi}_2 - \dot{\phi}_1)\dot{v} + S_{2\psi} S_v S_{2v} \dot{v}^2 \right) + \frac{r_w}{4b} (I_{x'} - I_{y'} + (I_{z'} - I_{x'} - m l^2) C_v^2) \times \\
& \quad \times \left(\frac{r_w^2}{2b^2} S_{2\psi} S_v^2 (\dot{\phi}_2 - \dot{\phi}_1)^2 + \frac{r_w}{b} S_{\psi}^2 S_{2v}(\dot{\phi}_2 - \dot{\phi}_1)\dot{v} + \frac{r_w}{b} S_{\psi}^2 S_v^2 (\ddot{\phi}_2 - \ddot{\phi}_1) + \frac{r_w}{b} C_{2\psi} S_v (\dot{\phi}_2 - \dot{\phi}_1)\dot{v} + S_{2\psi} C_v \dot{v}^2 + \right. \\
& \quad \left. + S_{2\psi} C_v \ddot{v} \right) + \frac{r_w^2}{4b^2} (I_{y'} + m l^2) (S_{\psi-v}^2 - C_{\psi}^2 S_v^2 + 1) (\ddot{\phi}_2 - \ddot{\phi}_1) + \frac{r_w^2}{4b^2} (I_{y'} + m l^2) \times \\
& \quad \times \left(S_{2(\psi-v)} \left(\frac{r_w}{2b} (\dot{\phi}_2 - \dot{\phi}_1) - \dot{v} \right) + \frac{r_w}{2b} S_{2\psi} S_v^2 (\dot{\phi}_2 - \dot{\phi}_1) - C_{\psi}^2 S_{2v} \dot{v} \right) (\dot{\phi}_2 - \dot{\phi}_1) + \frac{r_w}{4b} (I_{y'} + m l^2) \times \\
& \quad \times \left(2 S_{\psi-v} S_{\psi} + S_{2\psi} S_v \right) \dot{v} + \frac{r_w}{4b} (I_{y'} + m l^2) \left(2 C_{\psi-v} S_{\psi} \left(\frac{r_w}{2b} (\dot{\phi}_2 - \dot{\phi}_1) - \dot{v} \right) \dot{v} + \frac{r_w}{b} S_{\psi-v} C_{\psi} (\dot{\phi}_2 - \dot{\phi}_1) \dot{v} + \right. \\
& \quad \left. + \frac{r_w}{b} C_{2\psi} S_v (\dot{\phi}_2 - \dot{\phi}_1) \dot{v} + S_{2\psi} C_v \dot{v}^2 \right) + I_{rw} \ddot{\phi}_2 + \frac{1}{2} (I_{x'} - I_{y'} + (I_{z'} - I_{x'} - m l^2) C_v^2) \times \\
& \quad \times \left(\frac{r_w^3}{2b^3} S_{2\psi} S_v^2 (\dot{\phi}_2 - \dot{\phi}_1)^2 + \frac{r_w^2}{2b^2} C_{2\psi} S_v (\dot{\phi}_2 - \dot{\phi}_1) \dot{v} - \frac{r_w}{2b} S_{2\psi} \dot{v}^2 \right) - \frac{r_w^3}{16b^3} (I_{y'} + m l^2) (S_{2(\psi-v)} + S_{2\psi} S_v^2) \times \\
& \quad \times (\dot{\phi}_2 - \dot{\phi}_1)^2 - \frac{r_w^2}{4b^2} (I_{y'} + m l^2) (C_{\psi-v} S_{\psi} + S_{\psi-v} C_{\psi} + 2 C_{2\psi} S_v) (\dot{\phi}_2 - \dot{\phi}_1) \dot{v} = i_{red} k_m \dot{q}_2 - f_2 r_w - M_{c2} \quad (3)
\end{aligned}$$

$$L_a \ddot{q}_1 = u_1 - R_1 \dot{q}_1 - i_{red} k_e \dot{\phi}_1 \quad (4)$$

$$L_a \ddot{q}_2 = u_2 - R_2 \dot{q}_2 - i_{red} k_e \dot{\phi}_2 \quad (5)$$

Expressions for the coefficients of the linearized differential equations of motion of the TEBP

$$\begin{aligned}
a_{11} &= 2m l^2 + I_{y'} + (I_{z'} - I_{x'} - m l^2) C_{\psi_0}^2, \quad a_{12} = -m g l C_v, \quad b_{11} = -m l r_w C_v - \frac{r_w}{4b} (I_{y'} + m l^2) S_{\psi_0}^2 \\
b_{12} &= i_{red} k_m, \quad a_{21} = \frac{1}{4} m r_w^2 + \frac{r_w^2}{4b^2} (I_{y'} + m l^2) (S_{\psi_0}^2 + 1) + I_{rw}, \quad a_{22} = \frac{1}{2} m l r_w - \frac{r_w}{2b} (I_{y'} + m l^2) S_{\psi_0}^2 \\
a_{23} &= \frac{1}{4} m r_w^2 - \frac{r_w^2}{4b^2} (I_{y'} + m l^2) (S_{\psi_0}^2 + 1), \quad b_{21} = i_{red} k_m, \quad b_{22} = -r_w \\
a_{31} &= \frac{1}{4} m r_w^2 - \frac{r_w^2}{4b^2} (I_{y'} + m l^2) (S_{\psi_0}^2 + 1) + I_{rw}, \quad a_{32} = \frac{1}{2} m l r_w + \frac{r_w}{2b} (I_{y'} + m l^2) S_{\psi_0}^2 \\
a_{33} &= \frac{1}{4} m r_w^2 - \frac{r_w^2}{4b^2} (I_{y'} + m l^2) (S_{\psi_0}^2 + 1), \quad b_{31} = i_{red} k_m, \quad b_{32} = -r_w, \quad a_{41} = L_a \\
a_{42} &= R_a, \quad b_{41} = -i_{red} k_e, \quad b_{42} = 1, \quad c_{11} = \frac{r_w}{2} C_{\psi_0}, \quad c_{21} = \frac{r_w}{2} S_{\psi_0} \quad (6)
\end{aligned}$$

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