

Unscented Kalman filter with a reduced number of sigma-points and its application to the estimation of battery state of charge

João Paulo da Silva, Takashi Yoneyama

This paper presents a modified approach to implementing Sigma Point Kalman Filters (SPKFs), in which the Singular Value Decomposition (SVD) of the covariance matrix is used to generate sigma points for the unscented transform. The advantage of the proposed approach is that it requires just $(n+1)$ instead of the usual $(2n+1)$ sigma-points with only a minor reduction of precision. The proposed method for selecting the sigma-points to adjust the normal process parameters during the update phase of the filter can be intuitively visualized in the 2D case by plotting the covariance ellipsoid. The proposed reduced sigma-point filter achieved satisfactory performance in a case study involving state-of-charge (SoC) estimation for vehicular batteries when compared to the original SPKF implementation.

Keywords: sigma points, extended Kalman filter, state estimation, state of charge, SoC, batteries

1 Introduction

The Kalman filters have been used to estimate the state of charge (SoC) of batteries as in [1], as well as the extended Kalman filter [2]. Nonlinear estimation has been the focus of numerous studies over the past few decades, particularly with the introduction of the Unscented Kalman Filter (UKF) by Julier and Uhlmann in [3]. As noted by [4], the UKF has gained significant popularity due to its superior performance compared to the Extended Kalman Filter (EKF) in a variety of applications, such as estimating SoC in batteries, monitoring insulin concentration and enabling simultaneous localization and mapping (SLAM) for mobile robots, among other examples. The UKF extends the Kalman filter [5] by utilizing the Unscented Transformation (UT) to approximate the mean and the covariance matrix. The key idea behind the UKF is the intuition that it is easier to approximate a probability distribution than to directly approximate an arbitrary nonlinear transformation, such as the input-output relations of a system model. Rather than propagating only the mean and covariance, the UKF propagates sigma points, which are used to approximate the transformed state's normal distribution [6]. These principles have inspired the development of several other nonlinear filtering algorithms that also rely on sampling the system's state distribution. The term sigma point Kalman filter is adopted to reference the original UKF algorithm and its various extensions. This paper presents a modified approach to implement the SPKF

that incorporates SVD of the state covariance matrix, offering an alternative to the conventional sigma point generation method, which commonly employs Cholesky decomposition. Unlike Cholesky decomposition, which requires the covariance matrix to be positive-definite and Hermitian, SVD can be applied to any real or complex matrix, enhancing stability, especially when the covariance matrix risks degenerating due to small or zero eigenvalues. This change removes the requirement to reinitialize the filter in such circumstances, a problem frequently encountered by standard UKF implementations owing to numerical round-off errors. Moreover, the paper proposes a new sampling technique for generating sample points with zero mean and known covariance, located on the surface of an ellipsoid centered at the origin of a specified coordinate system. The effectiveness of this method is illustrated through a practical application in the estimation of the SoC of electric vehicle batteries, underscoring its advantage in improving the numerical stability.

The structure of this paper is organized as follows. In Section 3, we propose a method for obtaining sigma points using SVD, which reduces the number of sigma points from the typical $2n + 1$ to $n + 1$. This approach is demonstrated through an example of SoC estimation for batteries in Section 4, with a performance comparison to the traditional method. A brief mathematical formulation of the UKF is provided in Section 2, and conclusions are drawn in Section 5.

2 General UKF algorithm

The original Kalman Filter (KF) is used to estimate the states x_k of linear dynamic systems utilizing the measured outputs z_k

$$x_k = Ax_{k-1} + Bu_{k-1} + Cw_{k-1} \quad (1)$$

$$z_k = Hx_k + v_k \quad (2)$$

where $x_k \in R^n$ denotes the state vector, $u_k \in R^m$ the manipulated variable, $z_k \in R^p$ the measured outputs, w_k the state noise, v_k the observation noise and A, B, C, H are matrices of appropriate order. The noises w_k and v_k are assumed to be $N(0, \sigma^2)$, independent and identically distributed (i.i.d).

A Kalman filter can be used to provide $\hat{x}_k$, an estimate of x_k , from the previous estimate $\hat{x}_{k-1}$ together with the observed output z_k and the input u_k . The unscented Kalman filter is an extension of the traditional KF to overcome the limitation that the normality of a stochastic signal may not be preserved in non-linear dynamical models

$$x_k = f(x_{k-1}, u_{k-1}) + w_{k-1} \quad (3)$$

$$z_k = h(x_k) + v_k \quad (4)$$

where $f: R^n \times R^m \rightarrow R^n$ and $h: R^n \rightarrow R^p$ are not necessarily linear functions and, as before, the additive noises w_k and v_k are assumed to be normally distributed, independent and identically distributed (i.i.d).

In such scenarios, although the probability distribution of the estimates may not be guaranteed to be normal, the approximation using the first two moments may suffice in many applications. This can be achieved by propagating some specific sample points of the stochastic process, namely sigma-points, through the eventually nonlinear model and then fitting an appropriate second-order probability distribution to them. The obtained mapping is known as unscented transform (UT).

UKF can be implemented in a manner similar to the original Kalman filter algorithm by decomposing into two interdependent phases: prediction and update. In the prediction (or propagation) phase, the present estimates are used to calculate future estimates using the system model. In the update phase, the estimates are refined by incorporating the latest observations (measurements) by means of unscented transform.

2.1 UKF prediction step

Denoting $\hat{x}_{k-1|k-1}$ the estimate of x_{k-1} , given the observation z_{k-1} , a prediction $\hat{x}_{k|k-1}$ is obtained by using the model and the provided input variable u_{k-1} by propagation.

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_{k-1}) \quad (5)$$

However, due to the non-linearity of the model, the propagated probability distribution of the estimation error $\tilde{x}_{k|k-1} = \hat{x}_{k|k-1} - x_k$ at time k may not be normal, thus requiring the unscented transformation to approximate it by a normal distribution with the covariance matrix $P_{k|k}$.

2.2 UKF update step

The update phase improves the propagated estimate $\hat{x}_{k|k-1}$ by aggregating the information provided by the innovation process, namely $z_k - \hat{z}_k$, thus producing $\hat{x}_{k|k}$.

Introducing the notation δ as a scaling design parameter and W_m^i and W_c^i for the weights chosen for the mean and covariance matrices, the update phase comprises the following steps:

- Assemble the matrix of sigma points:

$$X_k = [\hat{x}_{k|k-1}, \hat{x}_{k|k-1} \pm \sqrt{(n + \delta)P_{k|k-1}}] \quad (6)$$

- Compute the predicted measurements:

$$Z_k^i = h(X_k^i) \quad (7)$$

- Compute the predicted measurements mean:

$$\hat{z}_k = \frac{1}{2n + \delta} \sum_{i=0}^{2n} W_m^i Z_k^i \quad (8)$$

- Obtain the measurement's covariance matrix:

$$S_k = \frac{1}{2n + \delta} \sum_{i=0}^{2n} W_c^i (Z_k^i - \hat{z}_k)(Z_k^i - \hat{z}_k)^T \quad (9)$$

- Obtain the cross-covariance matrix:

$$P_{xz} = \frac{1}{2n + \delta} \sum_{i=0}^{2n} W_c^i (X_k^i - \hat{x}_{k|k-1})(Z_k^i - \hat{z}_k) \quad (10)$$

- Compute the Kalman gain:

$$K_k = P_{xz} S_k^{-1} \quad (11)$$

- Update the state estimate:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(z_k - \hat{z}_k) \quad (12)$$

Update the error's covariance matrix:

$$P_{k|k} = P_{k|k-1} - K_k S_k K_k^T, \quad (13)$$

3 SVD sigma point Kalman filter

3.1 SVD unscented transformation

A traditional method to obtain the sigma points is to compute $\sqrt{(n + \delta)P_{k|k-1}}$ appearing in Eqn. (6) using the Cholesky decomposition [7].

An alternative approach for computing the sigma points is to use the singular value decomposition (SVD) is presented in [8-11]. A graphical illustration of sigma points obtained using SVD, together with scaling factor α is shown in Fig. 1. The scaling factor α determines

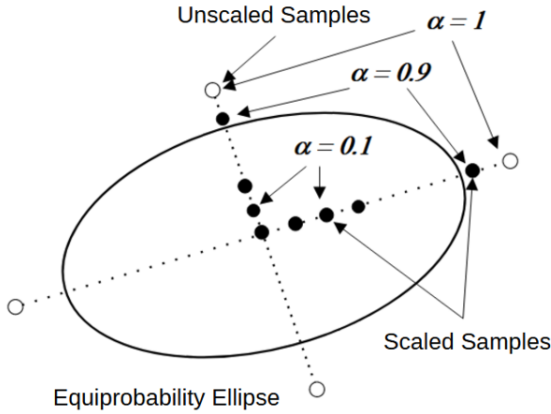


Fig. 1. Illustration of the SVD sigma points showing the effect of changes in the α scaling factor [11]

3.2 Reduced sigma points 2-dimensional case

In the proposed SVD UT, the sigma points χ_i are given by Eqn. (14), where the λ_i are the eigenvalues of the covariance matrix P (subindices dropped hereon for sake of notational simplicity) and the v_i 's are obtained as a linear combination of the associated eigenvectors u_i .

$$\chi_i = [\hat{x}(k|k), \hat{x}(k|k) \pm \sqrt{\lambda_i} v_i] \quad (14)$$

To provide a graphical representation of the process to choose the v_i 's in equation (14), we consider the case of $n = 2$.

In Fig. 3 one can visualize the procedure for selecting v_i 's given u_i 's corresponding to Eqns. (15) to (17) for the choice $\theta = \pi/3$.

$$v_1 = \sqrt{\lambda_1} u_1 \quad (15)$$

$$v_2 = -\sqrt{\lambda_1} \cos(\pi/3) u_1 + \sqrt{\lambda_2} \sin(\pi/3) u_2 \quad (16)$$

$$v_3 = -\sqrt{\lambda_1} \cos(\pi/3) u_1 - \sqrt{\lambda_2} \sin(\pi/3) u_2 \quad (17)$$

how far the sigma points are spread out from the mean. At $\alpha = 1$, these points are positioned on the equiprobability ellipse corresponding to one standard deviation along each main axis. If $\alpha < 1$, the sigma points are nearer to the mean, which can improve numerical stability yet might lower responsiveness to higher order statistics. On the other hand, setting $\alpha > 1$ (not depicted) may potentially reduce the numerical robustness [8, 9].

The sigma-points obtained by the SVD and Cholesky procedures are illustrated in Fig. 2 for the case of 2D.

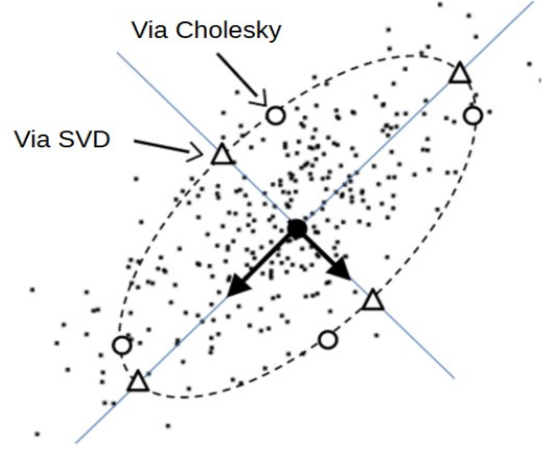


Fig. 2. Location of the sigma points obtained via Cholesky (circles) and SVD (triangle) decompositions on the equiprobability ellipse corresponding to one standard deviation σ

The selection of v_i 's must ensure the preservation of the first and second statistical moments. The verification of the property of preservation of the first statistical moment by the selected sigma points is straightforward as $v_1 + v_2 + v_3 = 0$.

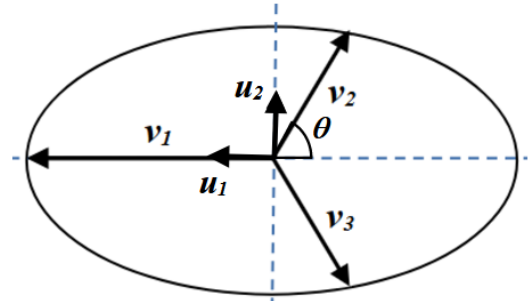


Fig. 3. Graphical representation of the process of choosing the reduced sigma points and their relation to the equiprobability ellipse of matrix P

In order to ensure the conservation of the second statistical moment, it is necessary to show that the

weighted covariance of these sigma points around the mean is equivalent to P .

Noting that

$$\begin{aligned}
 \sum_{i=1}^3 v_i v_i^T &= \lambda_1 u_1 u_1^T \\
 &+ (-\sqrt{\lambda_1} \cos(\theta) u_1 + \sqrt{\lambda_2} \sin(\theta) u_2) \times (-\sqrt{\lambda_1} \cos(\theta) u_1 + \sqrt{\lambda_2} \sin(\theta) u_2)^T \\
 &+ (-\sqrt{\lambda_1} \cos(\theta) u_1 - \sqrt{\lambda_2} \sin(\theta) u_2) \times (-\sqrt{\lambda_1} \cos(\theta) u_1 - \sqrt{\lambda_2} \sin(\theta) u_2)^T \\
 &= \lambda_1 (1 + 2\cos^2(\theta)) u_1 u_1^T + \lambda_2 2\sin^2(\theta) u_2 u_2^T \\
 &= \frac{3}{2} (\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T) = \frac{3}{2} \text{cov}(P)
 \end{aligned} \tag{18}$$

and choosing $W^i = W^j, i \neq j$ with $i \neq 0$ and $j \neq 0$, one confirms that the structure of the 2nd moment is preserved.

$$P_{corrected} = W^i \sum_{i=1}^{2n} \lambda_i v_i v_i^T = 2 W^i P \tag{19}$$

Basically, by positioning the sigma points on the equiprobability ellipsoid as proposed, one requires only $n + 1$ sigma points instead of the usual $2n + 1$ (where n represents the dimension of the matrix P).

The generalization for $n > 2$ follows the same principle for $n + 1$ vectors v_i 's pointing in spacially symmetrical directions and with modules equals to the distance from the center of the ellipsoid to the ellipsoid proper. For instance, for $n = 3$ the directions would be given by the vertices of a tetrahedron.

4 Applying the SVD-SPKF to battery's state of charge estimation

A key factor in the functionality of electric vehicles is the energy stored within the battery, which is essential for its conversion to mechanical movement. In this context, the state of charge (*SoC*) is a parameter of paramount importance [12].

SoC is characterized as the current charge level of a battery, expressed as a percentage of its full capacity as expressed as

$$SoC = \frac{Q_{res}}{Q_{rated}} \times 100\% \tag{20}$$

where Q_{rated} is the total capacity (the rated capacity under the same specific condition) and Q_{res} is the residual capacity (the battery's remaining capacity under a given testing condition). *SoC* is a dimensionless indicator commonly expressed as a percentage.

4.1 The battery model

The equivalent electric circuit models (EECM) use analog electrical circuit elements to characterize the behavioral or phenomenological approximation of how the voltage of a battery cell responds to different input current stimuli [13, 14]. An equivalent circuit model of a lithium-ion battery is of form [15].

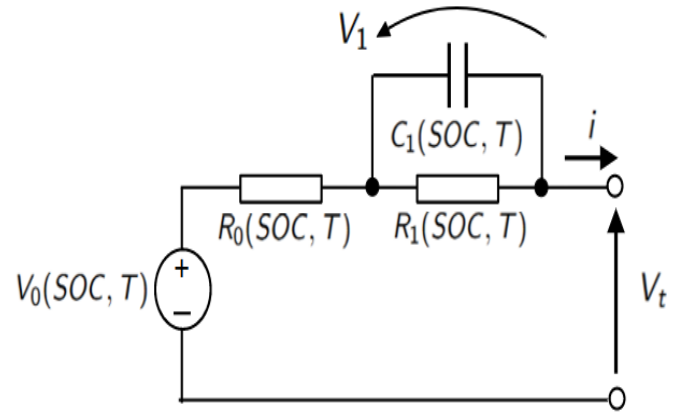


Fig. 4. Electric circuit model of a battery, para-metrized in terms of the state of charge – *SoC* and temperature – *T*. (Adapted from MATLAB®)

$$\frac{d SoC}{dt} = -\frac{i}{C_Q} \tag{21}$$

$$\frac{dV_1}{dt} = \frac{i}{C_1(SoC, T)} - \frac{V_1}{R_1(SoC, T)C_1(SoC, T)} \tag{22}$$

$$V_t = V_0(SoC, T) - iR_0 - V_1 \tag{23}$$

where the parameters are

SoC	state of charge
i	output current
V_0	no-load voltage
V_t	terminal voltage
AH	ampere-hour rating
R_0	internal resistance
R_1	polarization resistance
C_1	parallel RC capacitance
C_Q	cell capacity (Ah)
T	temperature
V_1	polarization voltage over the RC network

A discrete-time model of the equivalent battery circuit can be obtained using a standard first-order Euler method as

$$SoC_{k+1} = SoC_k - \frac{i_k \Delta t}{C_Q} \quad (24)$$

$$V_{1,k+1} = V_{1,k} + \frac{\Delta t}{C_1(SoC_k, T_k)} \left(i_k - \frac{V_{1,k}}{R_1(SoC_k, T_k)} \right) \quad (25)$$

$$V_{t,k} = V_0(SoC_k, T_k) - i_k R_0 - V_{1,k} \quad (26)$$

Hence, the nonlinear model in Eqn. (3) becomes characterized by the functions f and h given by

$$f(x_k, i_k) = x_k + \begin{bmatrix} -\frac{i_k \Delta t}{C_Q} \\ \frac{\Delta t}{C_1(x_{k,1}, T_k)} \left(i_k - \frac{x_{k,2}}{R_1(x_{k,1}, T_k)} \right) \end{bmatrix} \quad (27)$$

$$h(x_k, i_k) = V_0(x_{k,1}, T_k) - i_k R_0 \quad (28)$$

4.2 Numerical example and performance comparison

This section describes the simulation outcomes of three SPKF implementations utilized for estimation of SoC for the case study.

In the equivalent circuit model, the values of the nonlinear parameters R_0 , R_1 , open circuit voltage, and C_1 are given by look-up tables as functions of SoC and cell temperature T and were extracted from MATLAB® "openExample ('simscapebattery/BatterySOCestimation Example')" and also used in [14].

In the description of the UKF, Eqns. (6 to 13), the common design choices for β and κ are $\beta = 2$ and $\kappa = 0$ and the UKF scaling factors are

$$\delta = \alpha^2(n + \kappa) - n, \alpha \in [0,1] \quad (29)$$

For the proposed reduced SPKF the term $(\sqrt{(n + \delta)P})_i$ is the i -th row or column of $\sqrt{cP}$ computed using the Cholesky decomposition algorithm.

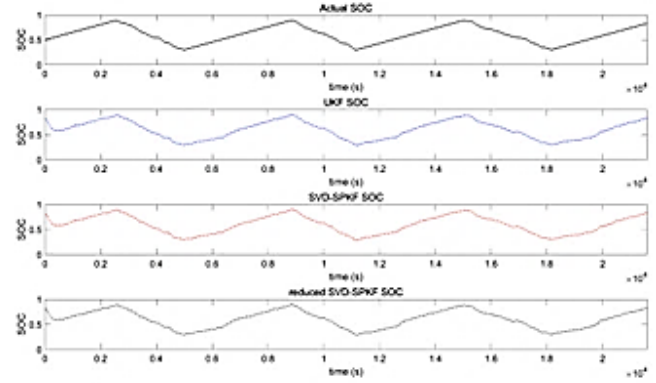


Fig. 5. The Actual SoC obtained with the model and the various estimates using UKF, SVD-SPKF SoC

Figure 5 displays the simulation results along with the actual (simulated) reference value for the UKF, SVD-SPKF, and the reduced sigma points SVD-SPKF models. The top figure corresponds to SoC as provided by the model. The following 3 curves are, respectively, the three estimates obtained by the UKF using Cholesky decomposition, the UKF using the complete sigma-point set $2n + 1$ and the UKF using the reduced set of sigma-points $n + 1$. The time axis was normalized to interval $(0, 1)$.

Figure 6 shows the root mean square errors calculated using the usual formula

$$RMSE = \sqrt{\frac{1}{N} \sum_{i=1}^N (\widehat{SoC}_i - SoC_i^{model})^2} \quad (30)$$

Although the UKF with reduced sigma points demonstrated slightly lower accuracy and precision, its errors remained within the same order of magnitude as those presented by other UKF implementations.

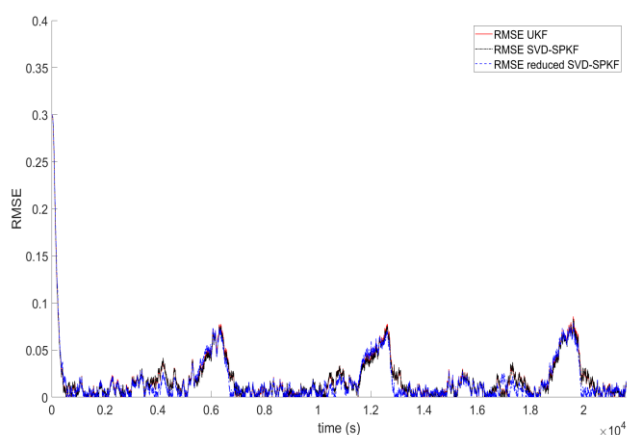


Fig. 6. Root Mean Square Error (RMSE) of SoC estimates obtained by the different UKF implementations

5 Conclusion

This paper presents an alternative implementation of SPKFs that requires only $n + 1$ sigma points as compared to the conventional one using $2n + 1$ points. The method utilizes the Singular Value Decomposition of the state covariance matrix, providing an alternative to the widely used Cholesky factorization in Unscented Kalman Filter (UKF) algorithms. An application example is provided to illustrate the proposed approach, namely, the estimation of the state of charge in electric vehicles. The results show that the accuracy of the proposed SVD-SPKF matches that of the conventional method, making it a viable alternative for a wide range of applications. The performance could possibly be improved by adopting adaptive techniques such as in [16].

Acknowledgement

T. Yoneyama is indebted to CNPq – Conselho Nacional de Desenvolvimento Científico e Tecnológico of Brazil for the Research Grant 309370/2023-0.

References

- [1] B. Chen and G. Chuang, "State of charge estimation for lithium-ion batteries using extended Kalman filter with local linearization," in *13th International Conference on Automotive Engineering*, 2017. <https://doi.org/10.4271/2017-01-1734>
- [2] P. Dheenadhayalan, A. Nair, M. Manalikandy, and A. Reghu, "A novel method for estimation of state of charge of lithium-ion battery using extended Kalman filter," in *SAE 2015 World Congress and Exhibition*, 2015. <https://doi.org/10.4271/2015-01-1183>
- [3] S. J. Julier and J. K. Uhlmann, "A new extension of the Kalman filter to nonlinear systems," in *Proc. of AeroSense: The 11th Int. Symp. on Aerospace/Defense Sensing, Simulations and Controls*, 1997. <https://doi.org/10.1117/12.280797>
- [4] H. M. T. Menegaz, J. Y. Ishihara, G. A. Borges, and A. N. Vargas, "A systematization of the unscented Kalman filter theory," *IEEE Transactions on Automatic Control*, vol. 60, no. 10, pp. 2583–2598, 2015. <https://doi.org/10.1109/TAC.2015.2404511>
- [5] R. E. Kalman, "A new approach to linear filtering and prediction problems," *Transactions of the ASME—Journal of Basic Engineering*, vol. 82 (Series D), pp. 35–45, 1960. <https://doi.org/10.1115/1.3662552>
- [6] D. Simon, *Optimal State Estimation: Kalman, H Infinity, and Nonlinear Approaches*. Hoboken, NJ: Wiley-Interscience, 2006.
- [7] S. J. Julier and J. K. Uhlmann, "A general method for approximating nonlinear transformation of probability distribution," Technical report, University of Oxford, 1996.
- [8] Y. Ma, Z. Wang, X. Zhao, J. Han, and Y. He, "A UKF algorithm based on the singular value decomposition of state covariance," in *2010 8th World Congress on Intelligent Control and Automation*, pp. 5830–5835, 2010. <https://doi.org/10.1109/WCICA.2010.5554585>
- [9] J. Hu, S. Wu, Y. Wang, F. Lu, and D. Liu, "Lithium battery SoC estimation based on improved unscented Kalman filter," in *2020 15th IEEE Conference on Industrial Electronics and Applications (ICIEA)*, pp. 511–515, 2020. <https://doi.org/10.1109/ICIEA48937.2020.9248230>
- [10] Y. Tian, Z. Huang, J. Tian, and X. Li, "State of charge estimation of lithium-ion batteries based on cubature Kalman filters with different matrix decomposition strategies," *Energy*, vol. 238, p. 121917, 2022. <https://doi.org/10.1016/j.energy.2021.121917>
- [11] G. M. S. and A. P. Andrews, *Kalman Filtering: Theory and Practice Using MATLAB*. Hoboken, NJ: John Wiley & Sons, 2008. <https://doi.org/10.1002/9780470377819>
- [12] H. He, R. Xiong, X. Zhang, F. Sun, and J. Fan, "State-of-charge estimation of the lithium-ion battery using an adaptive extended Kalman filter based on an improved Thevenin model," *IEEE Transactions on Vehicular Technology*, vol. 60, no. 4, pp. 1461–1469, 2011. <https://doi.org/10.1109/TVT.2011.2132812>
- [13] G. L. Plett, *Battery Management Systems Vol 1: Battery Modeling*. London: Artech House Publishers, 2015.
- [14] T. Huria, M. Ceraolo, J. Gazzarri, and R. Jackey, "High fidelity electrical model with thermal dependence for characterization and simulation of high power lithium battery cells," in *2012 IEEE International Electric Vehicle Conference*, pp. 1–8, 2012. <https://doi.org/10.1109/IEVC.2012.6183271>
- [15] S. Nejad, D. T. Gladwin, and D. A. Stone, "A systematic review of lumped-parameter equivalent circuit models for real-time estimation of lithium-ion battery states," *Journal of Power Sources*, vol. 316, pp. 183–196, 2016. <https://doi.org/10.1016/j.jpowsour.2016.03.042>
- [16] J. Moura, M. Camboim, T. Nunes, and M. Rosolem, "State of charge and state of health online estimation algorithm for lithium-ion batteries applied to hybrid electric vehicles based on adaptive filtering," in *SAE Brasil 2023 Congress*, 2024. <https://doi.org/10.4271/2023-36-0073>

Received 8 February 2026