

Vertically heated natural convection loop model

Goran Bulatović¹, Tadeáš Blunár² and Peter Bokes³

We present a comprehensive analytical model for laminar flow in a vertically heated rectangular convection loop, with applications to oil cooling in transformer systems. Starting from an integral equation, we apply a quasi-one-dimensional approach to derive equations for both loop velocities and temperature distributions. In this model, we account for temperature-dependent dynamic viscosity and introduce cooling at the loop top and in the fin. The analytical predictions are rigorously validated against finite volume method (FVM) simulations across different geometries and heat flux conditions, demonstrating strong agreement within a 2% discrepancy range for most parameters. This model provides a viable analytical alternative to computationally intensive simulations, offering insights into natural convection dynamics within closed-loop configurations.

Keywords: natural convection, loop, fin, channel, quasi-one-dimensional, analytical model, finite volume method

1 Introduction

Natural convection within closed-loop systems represents a significant area of study due to its applications across engineering and environmental systems, notably in heat transfer and fluid dynamics [1-4]. Analytical models for natural convection loops remain limited, with much early foundational work by Zvirin addressing circulation loops in systems such as pressurized water reactors [5, 6]. More recent studies have examined the dynamics within natural convection loops, focusing on the stability and flow characteristics of single-phase loops [7], as well as on the role of vertical and horizontal heated surfaces in influencing boundary layer structures and heat transfer behaviors [8]. While some studies utilize semi-analytical approaches, such as Fourier series expansions and dimensionless parameters [9], or combine control volume with particle-tracking methods [10], fully analytical solutions remain rare. In the present study, we develop a closed-loop analytical model that, similar to Zvirin's work, relies on solving an integral equation. However, we further integrate a quasi-one-dimensional approach from previous studies [11, 12], allowing for detailed analysis of temperature distribution within the loop.

In our prior work on oil transformers, we developed a quasi-one-dimensional framework to determine the temperature distribution within transformer windings [11, 12]. This approach, however, assumes temperature

uniformity along the winding height, represented here by the z -axis. To extend this model and capture temperature variations with respect to height, we initiated the development of an analytical loop model specifically for transformer oil distribution. An initial model hypothesized that natural convection within the loop is driven by the differential in lengths over which heat enters and exits the loop, with linear temperature gradients in both the channel and the fin [13]. Through extensive finite element (FEM) and finite volume (FVM) simulations, we determined that a more robust model would account for the temperature differential between the channel and fin as the primary driver of the loop circulation. Additionally, rather than presupposing a specific temperature profile along z , we compute the temperature distribution analytically across the loop channel, top and fin. Preliminary results employing this enhanced approach were published in [14]. In this article, we build upon those findings by incorporating temperature-dependent dynamic viscosity into the model. Furthermore, we extend the cooling boundary conditions to include the loop top, refining the model's physical accuracy. In contrast to our previous work [14], here we provide a comprehensive derivation of the governing equations for loop velocities and temperature distributions. The validity of the analytical model is assessed through comparisons with four FVM simulations, offering a rigorous basis for evaluating its accuracy.

Institute of Nuclear and Physical Engineering, Faculty of Electrical Engineering and Information Technology,
Slovak University of Technology, Bratislava 841 04, Slovakia

¹goran.bulatovic@stuba.sk, ²xblunar@stuba.sk, ³peter.bokes@stuba.sk

2 Natural convection loop model geometry and temperature distribution

We have a natural convection loop that consists of a vertical channel and a fin of length l_{ch} and widths h_{ch} , h_f respectively, horizontal top and bottom of width h_t and lengths l_t , l_b , respectively, Fig. 1. We consider that the loop has the same depth along the axis perpendicular to the fluid flow (x axis). The loop is heated at the channel, with the heat amount characterized by surface heat flux density $j_{Q, ch}$, and cooled at the fin and top, with the heat loss characterized by the fin and top heat transfer coefficients α_f , α_t . The rest of the loop – the bottom and loop two top corners – is considered insulated. The fluid is heated in the channel to reach temperature $\vartheta_{ch, t}$ which it retains in the left top corner of the loop. The fluid is then cooled in the top part of the loop to reach temperature $\vartheta_{f, t}$ which it retains in the right top corner of the loop. The fluid is then cooled in the fin to reach temperature ϑ_b , which it retains in the bottom part of the loop.

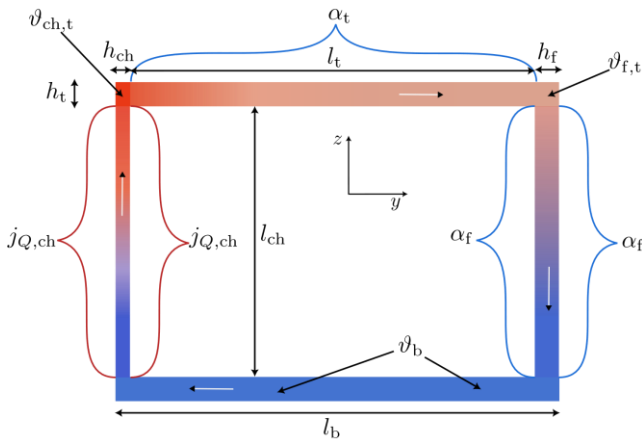


Fig. 1. Model of a natural convection loop. On the left, the heat enters the channel from two walls characterized by $j_{Q, ch}$. On the right, the heat leaves the fin through two walls characterized by α_f . On the top, the heat leaves the fin through one wall characterized by α_t .

We consider that the loop has one average density $\bar{\rho}$ – in the spirit of the Boussinesq approximation – assuming that density variations are small and primarily affect buoyancy forces in the momentum equations, rather than directly changing the density field across the flow. This approximation works as long as temperature differences are not too large. To find the loop temperatures we apply a quasi-one-dimensional energy balance from [11, 12] to fin, channel and top in order to obtain simple differential equations with analytical results. In doing so we choose the local vertical z coordinate to have its origin at the bottom of the channel and of the fin.

2.1 Temperature distribution in the fin

The fin steady state quasi-1D equation has the form of

$$-\lambda \frac{d^2 \vartheta_f(z)}{dz^2} - c_v \bar{\rho} \bar{v}_f \frac{d\vartheta_f(z)}{dz} + \frac{2\alpha_f k_f}{h_f} (\vartheta_f(z) - \vartheta_a) = 0 \quad (1)$$

where $\vartheta_f(z)$ is the temperature averaged over x , y and dependent on z axis, $k_f = \left(1 + \frac{\alpha_f h_f}{6\lambda}\right)^{-1}$, λ is the fluid thermal conductivity, c_v is the specific heat capacity at constant volume, $\bar{v}_f$ is the fin average velocity. The first term on the left side of the equation is the conduction term which is much smaller than the other two terms (flow and convection) and therefore can be neglected. Then Eqn. (1) has a solution

$$\vartheta_f(z) = (\vartheta_{f, t} - \vartheta_a) e^{c_f(z - l_{ch})} + \vartheta_a \quad (2)$$

where $\vartheta_{f, t}$ is the value of the temperature for $z = l_{ch}$ and $c_f = \frac{2\alpha_f k_f}{h_f c_v \bar{\rho} \bar{v}_f}$. When $z = 0$, we find the temperature of the loop bottom

$$\vartheta_b = (\vartheta_{f, t} - \vartheta_a) e^{-c_f l_{ch}} + \vartheta_a \quad (3)$$

The fin average temperature is given by

$$\bar{\vartheta}_f = \frac{(\vartheta_{f, t} - \vartheta_a)(1 - e^{-c_f l_{ch}})}{c_f l_{ch}} + \vartheta_a \quad (4)$$

Another equation for the fin average temperature is obtained from a wall-averaged Newton's law of cooling

$$\bar{\vartheta}_f = \frac{\bar{J}_{Q, f}}{\alpha_f k_f} + \vartheta_a \quad (5)$$

where $\bar{J}_{Q, f}$ is the average fin heat flux density.

2.2 Temperature distribution in the channel

The channel steady state quasi-1D equation has the form of

$$-\lambda \frac{d^2 \vartheta_{ch}(z)}{dz^2} + c_v \bar{\rho} \bar{v}_{ch} \frac{d\vartheta_{ch}(z)}{dz} - \frac{2j_{Q, ch}}{h_{ch}} = 0 \quad (6)$$

where $\vartheta_{ch}(z)$ is the temperature averaged over x , y and dependent on z axis, $\bar{v}_{ch}$ is the channel average velocity. The first term on the left side of the equation is much smaller than the other two and can be neglected. Then Eqn. (6) has a solution

$$\vartheta_{\text{ch}}(z) = c_{\text{ch}}z + \vartheta_{\text{b}} \quad (7)$$

where $c_{\text{ch}} = \frac{2j_{Q,\text{ch}}}{h_{\text{ch}}c_v\bar{\rho}\bar{v}_{\text{ch}}}$. The channel average temperature is given by

$$\bar{\vartheta}_{\text{ch}} = \frac{c_{\text{ch}}l_{\text{ch}}}{2} + \vartheta_{\text{b}} \quad (8)$$

2.3 Temperature distribution in the top

The top steady state quasi-1D equation has the form of

$$-\lambda \frac{d^2\vartheta_t(y)}{dy^2} + c_v\bar{\rho}\bar{v}_t \frac{d\vartheta_t(y)}{dy} + \frac{2\alpha_t k_t}{h_t}(\vartheta_t(y) - \vartheta_a) = 0 \quad (9)$$

where $\vartheta_t(y)$ is the temperature averaged over x, z and dependent on y axis, $k_t = \left(1 + \frac{\alpha_t h_t}{6\lambda}\right)^{-1}$, $\bar{v}_t$ is the top average velocity. The first term on the left side of the equation is the conduction term which is much smaller than the other two terms and can be neglected. Then Eqn. (9) has a solution

$$\vartheta_t(y) = (\vartheta_{f,t} - \vartheta_a)e^{c_t(l_t - y)} + \vartheta_a \quad (10)$$

where $c_t = \frac{2\alpha_t k_t}{h_t c_v \bar{\rho} \bar{v}_t}$. The top average temperature is given by

$$\bar{\vartheta}_t = \frac{(\vartheta_{f,t} - \vartheta_a)(e^{c_t l_t} - 1)}{c_t l_t} + \vartheta_a \quad (11)$$

Another equation for the fin average temperature is obtained from a wall-averaged Newton's law of cooling

$$\bar{\vartheta}_t = \frac{\bar{J}_{Q,t}}{\alpha_f k_t} + \vartheta_a \quad (12)$$

where $\bar{J}_{Q,t}$ is the average top heat flux density.

3 Force balance and average loop velocities

The force acting in the fluid on a chosen volume V^* bounded by a closed surface S^* is given by mechanical stress and buoyancy force

$$\vec{F} = \oint d\vec{S}^* \cdot \vec{\sigma} + \int \rho(\vartheta)\vec{g}dV^* \quad (13)$$

where $\vec{\sigma}$ is the stress tensor and $\vec{g}$ is gravitational acceleration vector. We use a linear approximation for density change with temperature

$$\rho(\vartheta) = \rho_{\text{ref}}(1 - \beta(\vartheta - \vartheta_{\text{ref}})) \quad (14)$$

where β is the coefficient of thermal expansion and the reference density ρ_{ref} corresponds to the reference temperature ϑ_{ref} . Considering values in direction of z as positive, then Eqn. (13) applied to the loop channel and fin can be expressed into pressure, viscous and buoyancy terms

$$F_{\text{ch},z} = (p_{\text{ch},b} - p_{\text{ch},t})S_{\text{ch}} - l_{\text{ch}}\Gamma_{\text{ch}}\langle\eta\frac{\partial v}{\partial y}|_{y=0}\rangle_{\text{ch}} - gS_{\text{ch}}\rho_{\text{ref}}l_{\text{ch}}(1 - \beta(\bar{\vartheta}_{\text{ch}} - \vartheta_{\text{ref}})) \quad (15)$$

$$F_{f,z} = (p_{f,b} - p_{f,t})S_f + l_{\text{ch}}\Gamma_f\langle\eta\frac{\partial v}{\partial y}|_{y=0}\rangle_f - gS_f\rho_{\text{ref}}l_{\text{ch}}(1 - \beta(\bar{\vartheta}_f - \vartheta_{\text{ref}})) \quad (16)$$

where the channel and fin are characterized by cross-sectional perimeter $\Gamma = 2(d + h) \approx 2d$ and cross-sectional area $S = h.d$, where h is the width (length along y axis) and d is the length along x axis. The pressure values at the bottom and top of the channel are $p_{\text{ch},b}$ and $p_{\text{ch},t}$ and $p_{f,b}$ and $p_{f,t}$ at the bottom and top of the fin. The viscous friction is characterized by dynamic viscosity η and symmetrical velocity gradient at the loop walls $|\partial v/\partial y|_{y=0}$ with the mean value of their product denoted by $\langle \rangle$. We consider that in the steady state flow velocity is constant within the four loop parts. The acting force must be then zero within these four parts. We also consider that the cross-sectional velocity has a parabolic profile. For the flow in the direction of z axis (channel and fin) the cross-sectional velocity is given by

$$v(y) = 4v_{\text{max}}\left(1 - \frac{y}{h}\right)\frac{y}{h} \quad (17)$$

which is a solution to a simplified Cartesian Navier-Stokes equation for laminar flow between two vertical walls and the boundary conditions $v(0) = v(h) = 0$, $v(h/2) = v_{\text{max}}$. The velocity gradient at the walls then equals

$$\frac{\partial v}{\partial y}|_{y=0} = \frac{4v_{\text{max}}}{h} = \frac{6\bar{v}}{h} \quad (18)$$

as $\bar{v} = \frac{2}{3}v_{\text{max}}$.

Dividing the channel and fin equations by the respective areas, introducing the formula for the velocity derivative for the laminar flow and introducing average values for viscosity and velocity we get

$$\frac{F_{\text{ch},z}}{S_{\text{ch}}} = (p_{\text{ch},b} - p_{\text{ch},t}) - l_{\text{ch}}\Gamma_{\text{ch}}\bar{\eta}_{\text{ch}}\frac{6\bar{v}_{\text{ch}}}{h_{\text{ch}}S_{\text{ch}}} - g\rho_{\text{ref}}l_{\text{ch}}(1 - \beta(\bar{\vartheta}_{\text{ch}} - \vartheta_{\text{ref}})) = 0 \quad (19)$$

$$\frac{F_{f,z}}{S_f} = (p_{f,b} - p_{f,t}) + l_{\text{ch}}\Gamma_f\bar{\eta}_f\frac{6\bar{v}_f}{h_f S_f} - g\rho_{\text{ref}}l_{\text{ch}}(1 - \beta(\bar{\vartheta}_f - \vartheta_{\text{ref}})) = 0 \quad (20)$$

The equations for the loop top and bottom are similar to Eqns. (19) and (20) but they do not contain the buoyancy term. Furthermore, we remove the viscous term from the top and bottom equations as the primary motivation for our loop model is the oil circulation in transformers which have wide top and bottom, so wall viscous force is negligible there. That means that the top and bottom equations are left only with the pressure difference terms which equal zero. This means that there is no change in pressure along the y coordinate, therefore, $p_{ch,b} = p_{f,b}$ and $p_{ch,t} = p_{f,t}$. Considering $\Gamma_{ch}/S_{ch} = 2/h_{ch}$, $\Gamma_f/S_f = 2/h_f$ and continuity equation

$$\bar{v}_{ch}S_{ch} = \bar{v}_fS_f \quad (21)$$

and by subtracting Eqns. (19) from (20) we arrive at the formula for the channel average velocity

$$\bar{v}_{ch} = \frac{g\rho_{ref}\beta(\bar{\vartheta}_{ch} - \bar{\vartheta}_f)}{12\left(\frac{\bar{\eta}_{ch}}{h_{ch}^2} + \frac{\bar{\eta}_f h_{ch}}{h_f^3}\right)} \quad (22)$$

The loop top and bottom velocities are easily obtained from the respective continuity equations. We see from Eqn. (22) that the loop average velocities correspond to temperature difference in the channel and fin.

We want to have Eqn. (22) expressed only through known geometric, material and flux parameters. First, we use the continuity equations to replace the average fin and top velocity with channel velocity in parameters c_f and c_t in sections 2.1 and 2.3. Next, we find temperature $\vartheta_{f,t}$ that features in fin, top and bottom equations, Eqns. (3, 4, 11), by introducing an energy balance equation for the total heat amount entering the channel and exiting fin and top

$$j_{Q,ch}l_{ch} = \bar{j}_{Q,f}l_{ch} + \bar{j}_{Q,t}l_t \quad (23)$$

Combining Eqn. (23) with the Eqns. (4, 5, 11, 12) we arrive at the temperature

$$\vartheta_{f,t} = \frac{j_{Q,ch}c_f l_{ch}}{\alpha_f k_f (e^{c_t l_t} - e^{-c_f l_{ch}})} + \vartheta_a \quad (24)$$

Now Eqns. (3, 4, 8, 22, 24) contain five unknowns. Manipulating the system of equations, we arrive at the channel velocity equation

$$\bar{v}_{ch} - A \left[\frac{B \left(e^{\frac{C}{\bar{v}_{ch}}} + 1 \right)}{\bar{v}_{ch} \left(e^{\frac{C}{\bar{v}_{ch}}} - 1 \right)} - \frac{\left(e^{\frac{D}{\bar{v}_{ch}}} - 1 \right)}{\left(e^{\frac{C}{\bar{v}_{ch}}} - 1 \right)} \right] = 0 \quad (25)$$

which contains only one unknown – the channel average velocity – and is only dependent on the loop geometric, material and flux parameters, where

$$A = \frac{g\rho_{ref}\beta j_{Q,ch}}{12\alpha_f k_f \left(\frac{\bar{\eta}_{ch}}{h_{ch}^2} + \frac{\bar{\eta}_f h_{ch}}{h_f^3} \right)}$$

$$B = \frac{\alpha_f k_f l_{ch}}{h_{ch} c_v \bar{\rho}}$$

$$C = 2B \left(1 + \frac{\alpha_t k_t l_t}{\alpha_f k_f l_{ch}} \right)$$

$$D = 2B$$

The unknown average velocity in the channel appears in Eqn. (25) implicitly, hence the solution of this nonlinear algebraic equation must be obtained numerically. Note: we can model the loop without the top cooling if we set $\alpha_t = 0$.

4 Comparison of analytical model with 2D FVM simulations

4.1 FVM simulations parameters

We have two geometries which we label 1 and 2. The geometries are two-dimensional as we consider no velocity and temperature change along the x axis. Both geometries have common widths but different lengths, Tab. 1. All four FVM simulations have common fluid parameters, Tab. 2. Dynamic viscosity is a linear piecewise function of temperature, Tab. 3.

Table 1. Two loop geometries widths and lengths

	$h_{ch}(\text{mm})$	$h_f(\text{mm})$	$h_t(\text{mm})$	$l_{ch}(\text{mm})$	$l_t(\text{mm})$
Geometry 1	4.5	6	12	800	150
Geometry 2	4.5	6	12	500	500

Table 2. Mineral oil material parameters

β (K ⁻¹)	c_v (J·kg ⁻¹ ·K ⁻¹)	$\bar{\rho}$ (kg·m ⁻³)	λ (W·m ⁻¹ ·K ⁻¹)
6.6×10^{-4}	1900	873	0.131

Table 3. Mineral oil dynamic viscosity dependence on temperature

ϑ (K)	293.15	313.15	333.15	353.15	373.15
η (mPa·s)	16.65	7.95	4.52	2.86	1.99

The other common parameters are $\alpha_f = 5$ W/(m²·K), $\alpha_t = 10$ W/(m²·K) and ambient temperature $\vartheta_a = 289.65$ K. Please note that, while the fin has the heat transfer coefficient on both sides, the top has it only on the upper side, Fig. 1. We set two values for the channel heating, $j_{Q, ch} = 120$ W·m⁻² which we label as A and $j_{Q, ch} = 180$ W·m⁻² which we label as B. Two different geometries and two different channel heatings amount to four different setups: 1A, 1B, 2A and 2B.

The simulations are done in Ansys Fluent. The type of simulation is viscous laminar flow, pseudo transient, with density set to Boussinesq. To model zero velocity gradient at the top and bottom walls we set the value for the shear stress in the simulations to zero (see section 2). We find out that for the same simulation (same parameters and same mesh), when repeated, we get slightly different velocity and temperature results. This prevents us from performing the mesh refinement convergence test that determines the mesh refinement, which upon further refinement does not produce different values for the loop velocities and temperatures. Therefore, we refer to the mesh refinement convergence test in [14] in which we tested similar geometries, only there was no cooling at the top. In [14] it was determined that the cell size of 4×10^{-4} was fine enough for achieving convergence. However, our mesh differs from the mesh in [14] in that the cells in [14] are even and our mesh gets smaller at the channel and fin walls where the velocity gradient is most pronounced, Fig. 2, to capture the velocity change more accurately. This approach decreases the cell size at the walls but increases it in the middle. To compensate for the increased size in the middle, we set the average cell size to 3×10^{-4} for both geometries. This translates into 183404 elements for the first geometry and 224684 for the second. Another problem is that, unlike [14] where continuity convergence dropped below 10^{-7} on its own with pseudo timestep 1 s, here we are forced to repeatedly pause the simulation to gradually decrease the timestep from 1 s to 0.005 s to achieve continuity convergence below 10^{-4} for all four simulations, which is considered precise

enough for natural convection simulations [15, 16]. All four simulations are repeated seven times so that velocity and temperature values are recorded as the mean value. Velocity values are shown in the tables with the standard and relative standard deviation, Tab. 4 and Tab. 6. Standard deviation in the case of temperature values is negligible and is not shown in the tables.

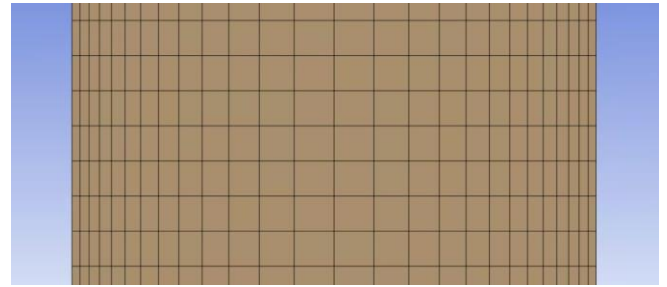


Fig. 2. Ansys Fluent mesh. The rectangular elements get smaller at the channel walls where the velocity change is maximum.

4.2 Analytical model and FVM simulations differences

The values we input into the analytical model are exactly the same as the values for the four FVM simulations with one exception, $\alpha_t = 5$ W/(m²·K), which is the half of the FVM simulation value. The reason for this is the parameter k_t formula in section 2.3 below Eqn. (9) which requires symmetrical cooling on both sides. Since the loop top is cooled only on the upper side, in the model we simulate this as if loop top was cooled from, both, upper and lower side with the transfer coefficient set to the half of the value. Viscosity values change fluidly in FVM simulations. In the analytical model the viscosity change is achieved in the python script in the way that the average value for channel and fin viscosity is tied to the average value for channel and fin temperatures, Eqns. (4) and (8).

4.3 Comparison of loop velocities and temperatures

The analytical model provides the average velocity in the direction of the flow which is responsible for increased heat transfer due to the fluid flow. This value is compared to FVM corresponding average velocity component in the direction of the flow which is z for the channel and fin and y for the top and bottom. FVM average velocity component values are extracted from simulations and model average velocity values are

obtained from Eqns. (21) and (25). To solve Eqn. (25) we write a script in Python which makes use of function `fsolve`, see the supplement. FVM average temperatures are extracted from simulations and model average temperature values are obtained from Eqns. (3, 4, 8, 11). First, as an illustration, we compare the analytical model and FVM simulation results for the 2B setup, Tabs. 4 and 5. The quantities of interest are the channel and fin average velocities, and temperature rises above the ambient temperature.

Table 4. Comparison of the analytical model results with the FVM simulation for the channel and fin average velocities for the 2B setup. Velocity values include the standard deviation.

	Model (mm/s)	FVM (mm/s)	Relative difference (%)
$\bar{v}_{ch}$	2.155	2.170 ± 0.018	0.69
$\bar{v}_f$	1.616	1.628 ± 0.012	0.73

Table 5. Comparison of the analytical model results with the FVM simulation for the channel and fin average temperature rises for the 2B setup.

	Model (K)	FVM (K)	Relative difference (%)
$\Delta\vartheta_{ch}$	19.57	19.84	1.36
$\Delta\vartheta_f$	16.29	16.28	0.06

In the next two tables we summarize the comparison of the analytical model and four FVM simulations for channel, fin, top and bottom average velocities and temperature rises.

Table 6. Comparison of the analytical model results with the FVM simulations for the channel, fin, top and bottom average velocities. The left part of the table is the relative difference to FVM, the right part is the FVM relative standard deviation.

	Relative difference (%)				Relative standard deviation (FVM) (%)			
	1A	1B	2A	2B	1A	1B	2A	2B
$\bar{v}_{ch}$	0.29	1.11	0.16	0.69	0.42	0.96	1.44	0.80
$\bar{v}_f$	0.27	1.09	0.15	0.73	0.38	0.87	1.30	0.72
$\bar{v}_t$	0.27	1.08	0.14	0.69	0.38	0.87	1.30	0.72
$\bar{v}_b$	0.27	1.08	0.15	0.69	0.38	0.87	1.30	0.72

Table 7. Comparison of the analytical model results with the FVM simulations for the channel, fin, top and bottom average temperature rises. Comparison is made with relative difference to FVM.

	1A (%)	1B (%)	2A (%)	2B (%)
$\Delta\vartheta_{ch}$	2.09	1.85	0.39	1.36
$\Delta\vartheta_f$	0.60	0.03	2.07	0.06
$\Delta\vartheta_t$	0.18	0.89	2.08	0.55
$\Delta\vartheta_b$	2.72	0.21	5.02	2.31

Finally, we compare FVM and analytical model temperature rises (above the ambient temperature) for the channel and fin. The temperature rises are dependent on z , Fig. 3. The FVM values are extracted from simulations and model values are calculated with Eqns. (2 and 7). Simulation 2B is chosen as a representative simulation.

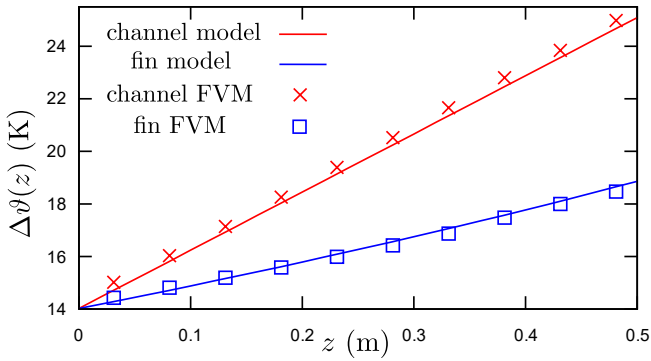


Fig. 3. Comparison of the analytical model and FVM values for the temperature change (above the ambient temperature) in the channel and the fin

5 Discussion

We see from Tabs. 6, 7 and Fig. 3 that we get very good agreement between the analytical model and FVM simulations for the loop velocities and temperatures. Almost all model values exhibit less than 2% discrepancy with the FVM simulations and that is including the standard deviation in the case of velocity. A few values go above the 2% discrepancy, notably bottom temperature for 2A simulation, which shows 5.02% discrepancy. The actual temperatures are 298.41 K (FVM) and 297.97 K (model) and they are closer to the ambient temperature which is $\vartheta_a = 289.65$ K than the other three simulations so the relative difference to FVM for the temperature rise is worst here. Other discrepancies we explain with the following six factors:

1) The numerical instability of the FVM simulations. Not only would the values obtained from the seven repetitions of each of the four FVM simulations slightly differ in velocity and temperature values but repetitions themselves would converge to a value in an oscillating manner so even that value is characterized by a deviation which however is significantly smaller than the deviations between the seven repetitions and is therefore not explicitly included in the tables. This shows that simulating a 2D natural convection rectangular loop heated at the channel and cooled at the fin and top is numerically challenging.

2) The nature of the flow in the loop top. We purposefully chose the width of the loop top and bottom only somewhat bigger than that of the channel and fin in order to keep the flow in the simulations laminar. Although the flow in the FVM simulations is seen as laminar in most of the loop when velocity vectors are charted, there are parts of the loop top which exhibit curls as shown in Fig. 4. This points at transitional or weak turbulent flow.

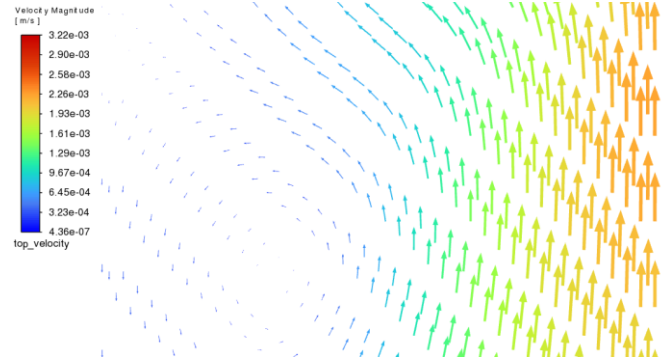


Fig. 4. Noticeable curling of the velocity vector in the loop top

We calculate the Rayleigh number $Ra_h = \rho^2 g \beta c_v (\vartheta_2 - \vartheta_1) h_i^3 / (\eta \lambda) = 3.6 \times 10^4$ for the loop top, according to the approach in [17] for a horizontal rectangular enclosure, where ϑ_2 , ϑ_1 are the top wall temperatures at the position where the velocity curling occurs. The cited reference states that, for thusly calculated Rayleigh number, the natural convection laminar flow ends and the turbulent flow regime begins at $Ra_h = 5 \times 10^4$ without referring to the transitional regime. In [18] it is stated that when Prandtl number is very high (as in our case, $Pr = 126$) the flow regime is either laminar or transitional and not fully turbulent. This leads us to conclude that the parts of the loop top exhibit transitional flow regime. This does not seem to affect the model's validity to any significant degree. Nevertheless, it remains to be seen how well the model can fare when there is pronounced turbulent flow ($Ra_h > 5 \times 10^4$) in the loop and can it be adjusted to accommodate the turbulence.

3) The formula for parameter k used for calculation of k_f and k_t in sections 2.1 and 2.3. We derived the formula in [11, 12] for the case of a solid where there is only thermal conduction with convective boundaries. Here in the fin and top we have thermal conduction with convective boundaries plus a moving fluid. We have not studied as how the formula for k is affected by a moving fluid. We use it because the temperature results are more in accordance with FVM results when k is computed

than set to 1. When $k=1$, it means that there is no difference between the average temperature and surface temperature (both dependent on the coordinate in the direction of the flow) which is obviously not true and is true less as the width of the loop increases.

4) Velocity gradient. In Eqns. (15) and (16) we consider only the velocity gradient at the walls which is a valid approximation for a fully developed laminar flow. Nevertheless, velocity gradients exist within the fluid too and this can lead to a small error in the analytical model values.

5) Average density. It would be more precise to consider an average density for each of the three loop parts in Eqn. (1), (6) and (9). Instead, we consider one density for the whole loop, see the explanation at the beginning of section 2.

6) The cross-sectional velocity profile. They are not identical as seen in Fig. 5.

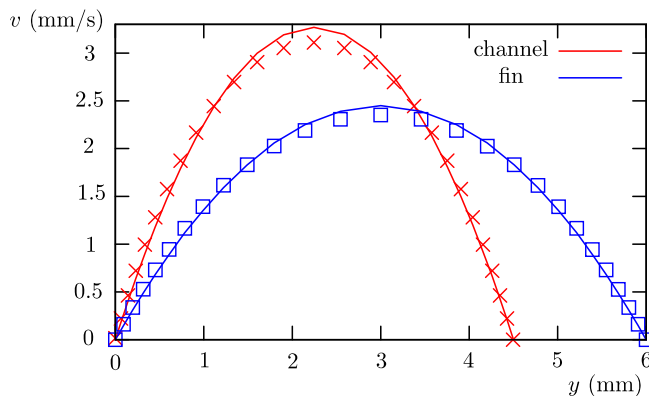


Fig. 5. Comparison of the model (line) and FVM cross-sectional velocity profile (crosses and squares)

Model values are computed from Eqn. (17) and FVM simulation values are extracted from 2B. For both, the channel and fin, model cross-sectional velocity is slightly lower near the walls and slightly higher in the middle of the flow. The relative difference to simulation 2B for the average cross-sectional velocity is 0.76% for the channel and 0.43% for fin which is quite small and substantiates the use of Eqn. (18) for the purpose of calculating loop average velocities.

Despite the analytical model simplifications, overall, the model values are largely in accordance with the FVM values. In fact, in light of the numerical instability of FVM simulations, we would like to comment on how important is to have an analytical model and we believe that we have one – despite the fact that the equation for channel velocity is solved with a python script which runs instantly as opposed FVM simulations that can take considerable time. It should be noted that neither the analytical model nor the FVM simulations deal with

viscous dissipation which is considered negligible in natural convection flows due to small fluid velocities and small viscosities [19]. Typically, for the loop flow we get the Brinkman number $Br \approx 10^{-8} \ll 1$, which is the requirement for neglecting the viscous dissipation [20].

6 Conclusion

We have derived equations for a laminar flow in a natural convection loop heated and cooled vertically, with additional cooling at the loop top. We have tested the analytical results for the loop velocities and temperatures with four FVM simulations. The comparisons show that the model is in very good accordance with the FVM simulations with discrepancies less than 2% for most values. We consider this an important step toward modeling circulation in oil transformers, which is our next goal.

Acknowledgement

This work was supported by the Slovak Research and Development Agency, Slovakia under the contract No. APVV-22-0115.

References

- [1] C. Cheng, D. Lu, Q. Su, Y. Huang, and G. Zhang, "Experimental study on flow rate distortion of natural circulation reduced-height scaling", *Nuclear Engineering and Design*, vol. 402, pp. 112-122, 2023.
- [2] S. Zhao, Q. Liu, M. Wilkinson, G. Wilson, and Z. Wang, "A reduced radiator model for simplification of onan transformer cfd simulation," *IEEE Transactions on Power Delivery*, vol. 37, no. 5, pp. 4007–4018, 2022.
- [3] S. K. Rai, P. Kumar, and V. Panwar, "Numerical investigation of steady state characteristics and stability of supercritical water natural circulation loop of a heater and cooler arrangements," *Nuclear Engineering and Technology*, vol. 53, no. 11, pp. 3597–3611, 2021.
- [4] P. Vijayan, M. Sharma, and D. Saha, "Steady state and stability characteristics of single-phase natural circulation in a rectangular loop with different heater and cooler orientations," *Experimental Thermal and Fluid Science*, vol. 31, no. 8, pp. 925–945, 2007.
- [5] Y. Zvirin, "A review of natural circulation loops in pressurized water reactors and other systems," *Nucl. Eng. Des.*, vol. 67, no. 2, pp. 203–225, 1981.
- [6] Y. Zvirin, *Experimental and Analytical Investigation of a PWR Natural Circulation Loop*. Electric Power Research Institute, 1981.
- [7] P. Ehrhard and U. Müller, "Dynamical behavior of natural convection in a single-phase loop," *Journal of Fluid Mechanics*, vol. 217, pp. 487–518, 1990.
- [8] Y. Fan, Y. Zhao, J. F. Torres, F. Xu, C. Lei, Y. Li, and J. Carmeliet, "Natural convection over vertical and horizontal heated flat surfaces: A review of recent progress focusing on underpinnings and implications for heat transfer and environmental applications," *Physics of Fluids*, vol. 33, no. 10, pp. 101301, 2021.

- [9] A. Dass and S. Gedupudi, "1-D semi-analytical modeling and parametric study of a single phase rectangular coupled natural circulation loop," *Chemical Engineering Science*, vol. 207, pp. 105–129, 2019.
- [10] B. Knížat, M. Vach, M. Mlkvik, P. Mlynár, F. Világi, R. Olšiak, and F. Urban, "Numerical and experimental study of a large-scale natural circulation helium loop," *Applied Sciences*, vol. 13, 2023.
- [11] P. Bokes and G. Bulatović, "Quasi-one-dimensional approach to heat transfer: general formulation and its application to transformer windings," *Applied Thermal Engineering*, vol. 248, 2023.
- [12] P. Bokes, "Average temperature of oil-filled transformer windings with partial cooling ducts," *Journal of Electrical Engineering*, vol. 72, pp. 35–39, 2021.
- [13] M. Z. Diešková, S. Kotorová, G. Bulatović, and P. Bokes, "Simple model for heat transfer in magnetic nanofluid-enhanced oil-filled transformers," *AIP Conf. Proc.*, vol. 3251, 2024.
- [14] G. Bulatović and P. Bokes, "FEM verification of natural convection loop model," to be published in IOP Publishing.
- [15] Ansys Fluent 12.0 User's Guide, 12th ed. Ansys, 2022.
- [16] H. K. Versteeg and W. Malalasekera, *An Introduction to Computational Fluid Dynamics, the finite volume method*, 2nd ed. Pearson, 2007.
- [17] Natural Convection Course. Ansys Innovation Space, 2020. [Online]. Available: <https://innovationspace.ansys.com/product/natural-convection>
- [18] U. Madanan and R. Goldstein, "Experimental investigation on very-high Rayleigh-number thermal convection in tilted rectangular enclosures," *International Journal of Heat and Mass Transfer*, vol. 139, pp. 121–129, 2019.
- [19] E. Guazzelli, J. F. Morris, and S. Pic, *Basic concepts in viscous flow*. Cambridge University Press, 2011.
- [20] S. Mukherjee and P. Mondal, "An Analytical Approach to the Effect of Viscous Dissipation on Shear-Driven Flow between two parallel plates with Constant Heat Flux Boundary Conditions," *International Journal of Engineering*, vol. 26, pp. 533–542, 2013.

Received 15 September 2024

Supplement

Python code for calculating the loop velocities and temperatures

Code for computing loop average velocities and temperatures

Author: Goran Bulatovic

Date: 15 November 2024

Supplement for the article "Vertically Heated Natural Convection Loop Model" to be published in Journal of Electrical Engineering

import numpy as np

from scipy.optimize import fsolve

Constants

$g = 9.81$

$\rho_{av} = 873$ # average density

$\rho_{ref} = \rho_{av}$ # reference density

$c_v = 1900$ # specific heat capacity

$\beta = 0.00066$ # thermal expansion coefficient

$h_{ch} = 0.0045$ # channel width

$h_t = 0.012$ # top width

$h_b = 0.012$ # bottom width

$h_f = 0.006$ # fin width

$l_t = 0.5$ # top and bottom length

$l_{ch} = 0.5$ # channel and fin length

$j_Q = 120$ # channel surface heat flux density

$\vartheta_a = 289.65$ # ambient temperature

$\alpha_f = 5$ # fin heat transfer coefficient

$\alpha_t = 5$ # top heat transfer coefficient

$\lambda = 0.131$ # fluid thermal conductivity

def calculate_viscosity(temperature):

 # Calculates viscosity based on temperature using linear interpolation

 viscosity_data = [(293.15, 0.01665), (313.15, 0.00795), (333.15, 0.00452), (353.15, 0.00286), (373.15, 0.00199)]

```

# Find the temperature range that the input temperature falls into
for i in range(1, len(viscosity_data)):
    if viscosity_data[i][0] > temperature:
        t1, v1 = viscosity_data[i-1]
        t2, v2 = viscosity_data[i]
        break
else:
    # If the input temperature is outside the data range, use the last data point
    t1, v1 = viscosity_data[-1]
    t2, v2 = viscosity_data[-1]

# Calculate viscosity using linear interpolation
return v1 + (v2 - v1) * (temperature - t1) / (t2 - t1)

# Main loop
initial_guess = 0.001 # initial guess for the channel averager temperature
eta_ch = 0.005 # Initial guess for channel average viscosity
eta_f = 0.005 # Initial guess for fin average viscosity
convergence_tolerance = 1e-6
max_iterations = 100
bar_vartheta_ch_previous = 320
bar_vartheta_f_previous = 320

for iteration in range(max_iterations):

    # Calculates parameters
    k_f = 1/(1 + alpha_f*h_f/lam/6)
    k_t = 1/(1 + alpha_t*h_t/lam/6)
    A = (g * rho_ref * beta * j_Q) / (k_f*alpha_f**2*(eta_ch/h_ch**2 + eta_f*h_ch/h_f**3))
    B = k_f*alpha_f*l_ch/(h_ch*c_v*rho_av)
    C = 2*B*(1 + k_t*alpha_t*l_t/(k_f*alpha_f*l_ch))
    D = 2*B

    # Entering the average channel velocity equation
    def equation(v_av_ch):
        return v_av_ch - A * (B*(np.exp(C/v_av_ch)+ 1)/(v_av_ch*(np.exp(C/v_av_ch)- 1)) - (np.exp(D/v_av_ch)-
1)/(np.exp(C/v_av_ch)- 1)))

    # Solution for channel average velocity
    v_av_ch_solution, = fsolve(equation, initial_guess)

    # Calculates average fin, top and bottom temperatures
    vel_av_f = v_av_ch_solution*h_ch/h_f
    vel_av_t = v_av_ch_solution*h_ch/h_t
    vel_av_b = v_av_ch_solution*h_ch/h_b

    # Calculates parameters
    c_f = (2 * k_f * alpha_f) / (h_ch * c_v * rho_av * v_av_ch_solution)
    c_t = (2 * k_t * alpha_t) / (h_ch * c_v * rho_av * v_av_ch_solution)
    c_ch = (2*j_Q)/(h_ch*c_v*rho_av*v_av_ch_solution)

    # Calculates top fin temperature
    vartheta_t_f = c_f * l_ch * j_Q / (k_f * alpha_f*(np.exp(c_t * l_t) - np.exp(-c_f * l_ch))) + vartheta_a
    # Calculates average fin temperature
    bar_vartheta_f = (vartheta_t_f - vartheta_a)*(1 - np.exp(-c_f * l_ch))/(c_f*l_ch) + vartheta_a

```

```

# Calculates bottom temperature
vartheta_b = (vartheta_t_f - vartheta_a)*np.exp(-c_f * l_ch) + vartheta_a
# Calculates average channel temperature temperature
bar_vartheta_ch = c_ch*l_ch/2 + vartheta_b
# Calculates average top temperature temperature
bar_vartheta_t = ((vartheta_t_f - vartheta_a)*(np.exp(c_t * l_t) - 1))/(c_t*l_t) + vartheta_a
# Calculates average loop temperature temperature
bar_vartheta = (bar_vartheta_ch*l_ch*h_ch + bar_vartheta_f*l_ch*h_f + vartheta_b*l_t*h_t +
bar_vartheta_t*l_t*h_t)/(l_ch*h_ch + l_ch*h_f + l_t*h_t + l_t*h_t)

# Calculate average channel and fin viscosities based on the current channel and fin temperature
eta_ch = calculate_viscosity( bar_vartheta_ch)
eta_f = calculate_viscosity(bar_vartheta_f)

# Check for convergence
if abs( bar_vartheta_ch_previous - bar_vartheta_ch) < convergence_tolerance and \
abs( bar_vartheta_f_previous - bar_vartheta_f) < convergence_tolerance:
    break

bar_vartheta_ch_previous = bar_vartheta_ch
bar_vartheta_f_previous = bar_vartheta_f

# Calculate average top and bottom viscosities
eta_vartheta_t = calculate_viscosity(bar_vartheta_t)
eta_vartheta_b = calculate_viscosity(vartheta_b)

print("eta_vartheta_t =", eta_vartheta_t)
print("eta_vartheta_b =", eta_vartheta_b)
print("eta_ch =", eta_ch)
print("eta_f =", eta_f)
print("alpha_f =", alpha_f)
print("alpha_t =", alpha_t)
print("k_f =", k_f)
print("k_t =", k_t)
print("A =", A)
print("B =", B)
print("C =", C)
print(f"The solution for v_av_ch is: {v_av_ch_solution:.6f}")
print("vel_av_f =", vel_av_f)
print("vel_av_t =", vel_av_t)
print("vel_av_b =", vel_av_b)
print("c_f =", c_f)
print("c_t =", c_t)
print("vartheta_t_f =", vartheta_t_f)
print("bar_vartheta_t =", bar_vartheta_t)
print("bar_vartheta_f =", bar_vartheta_f)
print("vartheta_b =", vartheta_b)
print("c_ch =", c_ch)
print("bar_vartheta_ch =", bar_vartheta_ch)
print("bar_vartheta =", bar_vartheta)

```