

# Spectral properties of the Cartesian product of $K_3 \square K_g$ graphs.

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## Abstract

This study explores the Laplacian spectral characteristics of the Cartesian product graph of  $K_3 \square K_g$ , with particular emphasis on the trace, energy, and the coefficients of its characteristic polynomial. We establish general expressions for the trace of powers of the Laplacian matrix and develop recursive formulas for the Laplacian coefficients using trace-based identities. The findings are validated for various values of  $g$ , and generalized to arbitrary cases. Additionally, we derive upper bounds and lower bounds for the Laplacian eigenvalues of the graph  $K_3 \square K_g$ .

**Mathematics Subject Classification 2000:** 05C22, 05C50, 15A18

**Keywords:** Laplacian Matrix, Adjacency Matrix, Bounds.

## 1. INTRODUCTION

For all phrases and notes of graph theory, readers should refer to [2]. Here, we consider simple and loopless graphs, including a restricted number of nodes and edges.

**Definition 1.1.** Let  $H$  and  $K$  be two simple graphs. The *Cartesian product* of  $H$  and  $K$ , denoted by  $G = H \square K$ , is defined as the graph with vertex set  $V(G) = V(H) \square V(K)$ . Two vertices  $(a, b)$  and  $(c, d)$  in  $G$  are adjacent if and only if either  $a = c$  and  $b \sim d$  in  $K$ , or  $b = d$  and  $a \sim c$  in  $H$ , where  $\sim$  denotes adjacency of vertices. Let  $G = (V, E)$  be a connected, simple graph with  $n$  vertices and  $m$  edges. The *Laplacian matrix* (also called the *Kirchhoff matrix*) of  $G$  is given by  $L = D - A$ , where  $D$  is the diagonal degree matrix and  $A$  is the adjacency matrix of  $G$ .

Let  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{(n-1)} \geq \alpha_n = 0$  be, the eigenvalues of  $L$ . The Kirchhoff eigenvalues of a graph are important in graph theory since they are closely related to frequent graph invariants. Several new inequalities are obtained for the modules, the real part, and imaginary part of a linear arrangement of the ordered eigenvalues of a square matrix.

The spectrum of a graph  $G$  consists of its eigenvalues, with the largest eigenvalue  $\alpha_1$  referred to as the *spectral radius* or the *index* of  $G$ . In spectral graph theory, spectral

10.2478/jamsi-2026-0002

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inequalities typically involving one or more spectral invariants such as eigenvalues play a significant role in characterizing the structural properties of a graph. These inequalities have wide-ranging applications and are derived using various techniques, including the Rayleigh quotient, eigenvalue interlacing, spectral moments, eigenvector analysis, and established relationships between spectral and structural invariants. A specific type of spectral inequality involves establishing bounds on spectral invariants, which may hold for all graphs or for specific classes of graphs. In such cases, *extremal graphs* are those that attain the given bound, provided it exists. For example, it is well known that the largest Laplacian eigenvalue  $\alpha_1$  satisfies  $\alpha_1 \geq 0$ , with equality occurring if and only if the graph  $G$  is completely disconnected.

In this article, we present a new, improved upper bound for the Kirchhoff eigenvalues of the cross product of graphs. The graphs  $K_g$  and  $K_3$  denote complete graphs with  $g$  and 3 vertices, respectively and  $h = 3 + g$  where  $g \geq 2$ .

**THEOREM 1.** The greatest eigenvalue of  $K_3 \square K_g$  is  $h$ .

$$\begin{aligned}
 \text{PROOF} \quad \alpha_1(L) &= \alpha_1(D - A). \\
 &= \alpha_1(D) + \alpha_1(-A). \\
 &= d + h - d. \\
 &= h.
 \end{aligned}$$

## 2. LAPLACIAN COEFFICIENTS OF CHARACTERISTIC POLYNOMIAL OF $K_3 \square K_G$ GRAPH

The coefficients of a graph's characteristic polynomial play a significant role in analyzing the chemical properties of molecules. Various techniques have been developed to compute the characteristic polynomial of a graph ([13], [14], [1], [2], [3]). Among these, the Faddeev–LeVerrier–Frame method is widely regarded as one of the most efficient approaches, as highlighted in [14] and supported by earlier foundational works ([15], [16]). Furthermore, Balasubramanian ([17], [18]) has frequently applied a computerized version of this method in his studies.

The coefficients of the characteristic polynomial

$$p_G(\alpha) = \alpha^n + q_1 \alpha^{n-1} + q_2 \alpha^{n-2} + \cdots + q_n$$

of a graph  $G$  encode significant structural information about the graph.

For example, the number of edges in  $G$  is given by  $-q_2$ , while the number of triangles corresponds to  $-\frac{q_3}{2}$ . These interpretations follow from the combinatorial nature of the characteristic polynomial, as established in [19]. The following theorem from that work formally relates the coefficients of the characteristic polynomial to specific topological features of the graph.

A detailed theorem that establishes how these coefficients reflect graph structure is provided in [20].

THEOREM 2. [20]

The Laplacian characteristic polynomial of  $G = K_3 \square K_g$  is  $\psi(G, x) = x^n + q_1x^{n-1} + \dots + q_{n-1}x + q_n$  the sets  $E_i$ ,  $1 \leq i \leq n$  consisting of the elementary sub graphs of  $G$ , with  $i$  vertices and  $r(\wedge)$  and  $s(\wedge)$  the rank and co-rank of  $E_i$ , respectively. We then have.

$$(-1)^i q_i = \sum_{\wedge \in E_i} (-1)^{r(\wedge)} 2^{s(\wedge)}$$

Using the trace of the Laplacian matrix, Ivailo M. Mladenov et al. [19] have provided a very elegant approach to identify the coefficients of a characteristic polynomial. Samuel Jurkiewicz et al. have determined the Laplacian coefficients of a characteristic polynomial in [20].

We provide a new technique to discover Laplacian coefficients of characteristic polynomial using the trace of a Laplacian matrix of  $K_3 \square K_g$  based on the aforementioned theorem, Faddeev-Leverrier algorithm and using [19].

PROPOSITION 3. Trace of the Laplacian matrix of  $K_3 \square K_g$  is ,

$$tr(L) = 3g(g+1)$$

PROOF

$$tr(L) = \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n$$

$$= h(h+g-5) + (g-1)g + 6.$$

$$= (g+3)(2g-2) + g(g-1) + 6.$$

$$tr(L) = 3g(g+1)$$

PROPOSITION 4.  $tr(L^k) = (g+3)^k(2g-2) + (g-1)g^k + 2(3^k)$ .

PROOF  $tr(L) = 3g(g+1)$

$$\begin{aligned} tr(L^2) &= (g+3)^2(g+3+g-5) + (g-1)g^2 + 18 \\ &= (g+3)^2(2g-2) + (g-1)g^2 + 2(3^2) \end{aligned}$$

Similarly,

$$tr(L^3) = (g+3)^3(2g-2) + (g-1)g^3 + 2(3^3).$$

By induction, for any positive integer  $k$ ,

$$tr(L^k) = (g+3)^k(2g-2) + (g-1)g^k + 2(3^k).$$

Example:

If we put  $k = 1, 2$  and  $3$  in the above expression we have the following.

$$tr(L) = (g+3)(2g-2) + (g-1)g + 2(3).$$

$$tr(L^2) = (g+3)^2(2g-2) + (g-1)g^2 + 2(3^2).$$

$$tr(L^3) = (g+3)^3(2g-2) + (g-1)g^3 + 2(3^3).$$

Coefficients  $q_1, q_2$  and  $q_3$  of characteristic equation of a Laplacian matrix of a graph  $K_3 \square K_g$ , are calculated as follows:

$$q_1 = -tr(L) = -3g(g+1).$$

$$q_2 = \frac{-1}{2}tr\{S_1L\} \quad \text{where } S_1 = L + q_1I$$

$$= \frac{-1}{2}tr\{(L^2) + (-3g(g+1))L\}$$

$$q_2 = \frac{-1}{2}\{(g+3)^2(2g-2) + (g-1)g^2 + 2(3^2) - 9g^2(g+1)^2\}.$$

$$q_3 = \frac{-1}{3}tr\{S_2L\} \quad \text{where } S_2 = S_1L + q_2I$$

$$= \frac{-1}{3} \text{tr}\{(\mathbf{L} + q_1 \mathbf{L})\mathbf{L}^2 + q_2 \mathbf{L}\}$$

$$= \frac{-1}{3} \text{tr}\{\mathbf{L}^3 + q_1 \mathbf{L}^2 + q_2 \mathbf{L}\}$$

$$q_3 = \frac{-1}{3} \left[ ((g+3)^2(2g-2) + (g-1)g^2 + 18) \left( 1 - \frac{3g(g+1)}{2} \right) + \frac{27g^3(g+1)^3}{2} \right]$$

THEOREM 5. The Laplacian characteristic equation of  $K_3 \square K_g$  graph is

$\psi(\Gamma, p) = y^n + p_1 y^{n-1} + \dots + p_{n-1} y + p_n$  with  $p_0 = 1$  then,

$$p_h = \frac{-1}{h} \sum_{j=0}^{h-1} p_j \text{tr}(\mathbf{L}^{h-j}).$$

where,  $p_h$  are the coefficients of the characteristic polynomial and  $h \neq 0$ .

PROOF

Here  $p_1 = -\text{tr}(\mathbf{L})$ .

$$\begin{aligned} p_2 &= \frac{-1}{2} \text{tr}(p_1 \mathbf{L}) \\ &= \frac{-1}{2} \text{tr}((\mathbf{L} + p_1) \mathbf{L}) \\ &= \frac{-1}{2} \{ \text{tr}(\mathbf{L}^2) + p_1 \text{tr}(\mathbf{L}) \} \\ &= \frac{-1}{2} \{ p_0 \text{tr}(\mathbf{L}^2) + p_1 \text{tr}(\mathbf{L}) \}. \end{aligned}$$

Similarly for an integer  $k$ ,

$$p_k = \frac{-1}{k} \sum_{j=0}^{k-1} p_j \text{tr}(\mathbf{L}^{k-j}).$$

we have,

$$\begin{aligned} p_{k+1} &= \frac{-1}{k+1} \text{tr}(p_k \mathbf{L}) \\ &= \frac{-1}{k+1} \text{tr}((p_{k-1} \mathbf{L} + p_k) \mathbf{L}) \end{aligned}$$

$$\begin{aligned}
&= \frac{-1}{k+1} \text{tr}((\mathbf{L}p_{k-2} + p_{k-1})\mathbf{L}^2 + p_k\mathbf{L}) \\
&= \frac{-1}{k+1} (\text{tr}(\mathbf{L}^3 p_{k-2}) + p_{k-1} \text{tr}(\mathbf{L}^2) + p_k \text{tr}(\mathbf{L})) \\
&= \frac{-1}{k+1} (\text{tr}(\mathbf{L}^4 p_{k-3}) + \text{tr}(\mathbf{L}^3) p_{k-1} + \text{tr}(\mathbf{L}^2) p_k + \text{tr}(\mathbf{L}))
\end{aligned}$$

...

$$p_{k+1} = \frac{-1}{k+1} \{p_0 \text{tr}(\mathbf{L}^{k+1}) + p_1 \text{tr}(\mathbf{L}^k) + p_2 \text{tr}(\mathbf{L}^{k-1}) + \dots + p_k \text{tr}(\mathbf{L})\}$$

$$\text{i.e., } p_{k+1} = \frac{-1}{k+1} \sum_{j=0}^k p_j \text{tr}(\mathbf{L}^{k+1-j}).$$

Hence by induction,

$$p_h = \frac{-1}{h} \sum_{j=0}^{h-1} p_j \text{tr}(\mathbf{L}^{h-j}) \quad \text{where } h \neq 0.$$

**THEOREM 6.** For  $G = K_3 \square K_g$  graph, the Laplacian characteristic polynomial is  $\psi(G, y) = y^n + t_1 y^{n-1} + \dots + p_{n-1} y + p_n$  with  $p_0 = 1$  then,

$$p_t = \frac{-1}{t} \sum_{j=0}^{t-1} p_j [(g+3)^j (2g-2) + (g-1)g^j + 2.3^j] \quad \text{where } t \neq 0.$$

**PROOF** By Proposition 4, we have

$$\text{tr}(\mathbf{L}^{h-j}) = (g+3)^{h-j} (2g-2) + (g-1)g^{h-j} + 2(3^{h-j}).$$

.

By Theorem 5, we get

$$p_t = \frac{-1}{t} \sum_{j=0}^{t-1} p_j [(g+3)^j (2g-2) + (g-1)g^j + 2.3^j] \quad \text{where } t \neq 0.$$

The energy  $E(G)$  of a graph  $G$  is defined to be the sum of the absolute values of its eigen values. Hence if  $A(G)$  is the adjacency matrix of  $G$  and  $\mu_1, \mu_2, \mu_3 \dots \mu_n$  are eigenvalues of Adjacency matrix  $A(G)$ , then  $E(G) = \sum_{i=1}^n |\mu_i|$ .

THEOREM 7. The energy of  $K_3 \square K_g$  is  $E[K_3 \square K_g] = 8(g-1)$

PROOF

The Adjacency matrix is  $A(K_3 \square K_g) = \begin{pmatrix} A(K_g) & I_g \\ I_g & A(K_g) \end{pmatrix}$

The characteristic polynomial of  $A(K_3 \square K_g)$  is given by:

$$Q(\mu) = (\mu - (g+1))(\mu - (g-2))^2((\mu - 1)^{(g-1)}(\mu + 2)^{2(g-1)}).$$

$$Spec[K_3 \square K_g] = \begin{pmatrix} -2 & 1 & (g+1) & (g-2) \\ 2(g-1) & (g-1) & 1 & 2 \end{pmatrix}$$

$$E[K_3 \square K_g] = |-2|(2(g-1)) + |1|(g-1) + |(g+1)|(1) + |g-2|(2).$$

$$= 4g - 4 + g - 1 + g + 1 + 2g - 4$$

$$= 8g(g-1).$$

### 3. UPPER BOUNDS FOR $\alpha_1$ IN $K_3 \square K_G$ GRAPH.

Bounds on the eigenvalues of a graph offer valuable insights into its structural and spectral properties. These bounds are instrumental in estimating characteristics such as connectivity, degree distribution, and overall graph topology. Numerous theorems and analytical techniques have been developed to derive such bounds, tailored to different classes of graphs and specific eigenvalues.

In this section, we focus on upper bounds for the largest Laplacian eigenvalue  $\alpha_1$  of the Cartesian product graph  $K_3 \square K_g$ . Several analytical expressions and inequalities are presented, offering comparative estimates for  $\alpha_1$  based on graph parameters.

PROPOSITION 8. The upper bound of eigen value

$$\alpha_1 \leq \frac{5}{3(g+1)} \left[ \frac{(g-1)\sqrt{\frac{2g}{5}} [(2\sqrt{\frac{2g}{5}} h \sqrt{\frac{2g}{5}} + (h-3)\sqrt{\frac{2g}{5}}] + 2(3\sqrt{\frac{2g}{5}})}{5} \right] \sqrt{\frac{1}{\frac{2g}{5}}} + 3 \quad (1)$$

PROOF

We have by the power mean inequality,

$$\left[ \frac{2(g-1)h + (g-1)(h-3) + 2 \times 3}{5} \right] \leq \left[ \frac{2\sqrt{\frac{2g}{5}} (g-1)\sqrt{\frac{2g}{5}} h \sqrt{\frac{2g}{5}} + (g-1)\sqrt{\frac{2g}{5}} (h-3)\sqrt{\frac{2g}{5}} + 2 \times 3\sqrt{\frac{2g}{5}}}{5} \right] \sqrt{\frac{1}{\frac{2g}{5}}}.$$

$$\frac{tr\{L\}}{5} \leq \left[ \frac{2\sqrt{\frac{2\pi}{5}}(g-1)\sqrt{\frac{2\pi}{5}}h\sqrt{\frac{2\pi}{5}} + (g-1)\sqrt{\frac{2\pi}{5}}(h-3)\sqrt{\frac{2\pi}{5}} + 2 \times 3\sqrt{\frac{2\pi}{5}}}{5} \right] \sqrt{\frac{1}{\frac{2\pi}{5}}}$$

By Proposition 3,

$$\frac{3(g+1)}{5} \leq \left[ \frac{2\sqrt{\frac{2\pi}{5}}(g-1)\sqrt{\frac{2\pi}{5}}h\sqrt{\frac{2\pi}{5}} + (g-1)\sqrt{\frac{2\pi}{5}}(h-3)\sqrt{\frac{2\pi}{5}} + 2 \times 3\sqrt{\frac{2\pi}{5}}}{5} \right] \sqrt{\frac{1}{\frac{2\pi}{5}}}$$

$$g \leq \frac{5}{3g(g+1)} \left[ \frac{2\sqrt{\frac{2\pi}{5}}(g-1)\sqrt{\frac{2\pi}{5}}h\sqrt{\frac{2\pi}{5}} + (g-1)\sqrt{\frac{2\pi}{5}}(h-3)\sqrt{\frac{2\pi}{5}} + 2 \times 3\sqrt{\frac{2\pi}{5}}}{5} \right] \sqrt{\frac{1}{\frac{2\pi}{5}}}$$

$$h \leq \frac{5}{3(g+1)} \left[ \frac{2\sqrt{\frac{2\pi}{5}}(g-1)\sqrt{\frac{2\pi}{5}}h\sqrt{\frac{2\pi}{5}} + (g-1)\sqrt{\frac{2\pi}{5}}(h-3)\sqrt{\frac{2\pi}{5}} + 2 \times 3\sqrt{\frac{2\pi}{5}}}{5} \right] \sqrt{\frac{1}{\frac{2\pi}{5}}} + 3$$

by Theorem 1,

$$\alpha_1 \leq \frac{5}{3(g+1)} \left[ \frac{2\sqrt{\frac{2\pi}{5}}(g-1)\sqrt{\frac{2\pi}{5}}h\sqrt{\frac{2\pi}{5}} + (g-1)\sqrt{\frac{2\pi}{5}}(h-3)\sqrt{\frac{2\pi}{5}} + 2 \times 3\sqrt{\frac{2\pi}{5}}}{5} \right] \sqrt{\frac{1}{\frac{2\pi}{5}}} + 3.$$

$$\alpha_1 \leq \frac{5}{3(g+1)} \left[ \frac{(g-1)\sqrt{\frac{2\pi}{5}}[(2\sqrt{\frac{2\pi}{5}}h\sqrt{\frac{2\pi}{5}} + (h-3)\sqrt{\frac{2\pi}{5}}] + 2(3\sqrt{\frac{2\pi}{5}})}{5} \right] \sqrt{\frac{1}{\frac{2\pi}{5}}} + 3$$

The following theorem presents a bound which is better than (1).

**THEOREM 9.** Let  $\alpha_1$  be the maximum Laplacian eigenvalue of  $K_3 \square K_g$  graph then

$$\alpha_1 \leq h + \frac{1}{g}. \quad (2)$$

**PROOF** By the Theorem 1,

$$\alpha_1(K_3 \square K_g) = h = g + 3$$

since  $g > 1$ ,

let us take  $u(g) = g + 3 + \frac{1}{g} > g + 3$

$$\lim_{g \rightarrow \infty} U(g) = g + 3 = h.$$

hence  $\alpha_1 \leq h + \frac{1}{g}$ .

For  $v \in V$ , the degree of vertex  $v$ , the set of neighbors of  $v$  and the average of the degrees of the vertices adjacent to  $v$  are denoted by  $d_v$ ,  $N_v$  and  $m_v$ , respectively.

Among the known upper bounds for  $\alpha_1$  are the following:

—**Anderson and Morley's bound** [10]:

$$\alpha_1 \leq \max\{d_u + d_v \mid uv \in E\} \quad (3)$$

—**Li and Zhang's bound** [11]:

$$\alpha_1 \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} \quad (4)$$

—**Another Li and Zhang's bound** [11]:

$$\alpha_1 \leq 2 + \sqrt{(r - 2)(s - 2)} \quad (5)$$

where  $r = d_x + d_y$  for all  $xy \in E$ , and  $s = \max\{d_u + d_v \mid uv \in E \setminus \{xy\}\}$ .

—**Merris's bound** [12]:

$$\alpha_1 \leq \max\{d_v + m_v \mid v \in V\} \quad (6)$$

Upper bounds of  $\alpha_1$  of Laplacian matrix of  $K_3 \square K_g$  graph as follows:

Table I: Upper bounds for  $\alpha_1$  in  $K_3 \square K_g$  graph using different inequalities

| $g$ | $\alpha_1$ | Eq(1)   | Eq(2)   | Eq(3) | Eq(4) | Eq(5) | Eq(6) |
|-----|------------|---------|---------|-------|-------|-------|-------|
| 1   | 4          | 4.1040  | 5.0000  | 4     | 4.0   | 4.0   | 4     |
| 2   | 5          | 5.1035  | 5.5000  | 6     | 6.0   | 6.0   | 6     |
| 3   | 6          | 6.2242  | 6.3333  | 8     | 8.0   | 8.0   | 8     |
| 4   | 7          | 7.3519  | 7.2500  | 10    | 10.0  | 10.0  | 10    |
| 5   | 8          | 8.4782  | 8.2000  | 12    | 12.0  | 12.0  | 12    |
| 6   | 9          | 9.6024  | 9.1667  | 14    | 14.0  | 14.0  | 14    |
| 7   | 10         | 10.7245 | 10.1429 | 16    | 16.0  | 16.0  | 16    |
| 8   | 11         | 11.8450 | 11.1250 | 18    | 18.0  | 18.0  | 18    |
| 9   | 12         | 12.9642 | 12.1111 | 20    | 20.0  | 20.0  | 20    |
| 10  | 13         | 14.0824 | 13.1000 | 22    | 22.0  | 22.0  | 22    |
| 11  | 14         | 15.1997 | 14.0909 | 24    | 24.0  | 24.0  | 24    |
| 12  | 15         | 16.3162 | 15.0833 | 26    | 26.0  | 26.0  | 26    |
| 13  | 16         | 17.4322 | 16.0769 | 28    | 28.0  | 28.0  | 28    |
| 14  | 17         | 18.5478 | 17.0714 | 30    | 30.0  | 30.0  | 30    |
| 15  | 18         | 19.6629 | 18.0667 | 32    | 32.0  | 32.0  | 32    |
| 16  | 19         | 20.7776 | 19.0625 | 34    | 34.0  | 34.0  | 34    |
| 17  | 20         | 21.8921 | 20.0588 | 36    | 36.0  | 36.0  | 36    |
| 18  | 21         | 23.0063 | 21.0556 | 38    | 38.0  | 38.0  | 38    |
| 19  | 22         | 24.1202 | 22.0526 | 40    | 40.0  | 40.0  | 40    |
| 20  | 23         | 25.2340 | 23.0500 | 42    | 42.0  | 42.0  | 42    |
| 21  | 24         | 26.3475 | 24.0476 | 44    | 44.0  | 44.0  | 44    |
| 22  | 25         | 27.4609 | 25.0455 | 46    | 46.0  | 46.0  | 46    |
| 23  | 26         | 28.5742 | 26.0435 | 48    | 48.0  | 48.0  | 48    |
| 24  | 27         | 29.6873 | 27.0417 | 50    | 50.0  | 50.0  | 50    |
| 25  | 28         | 30.8003 | 28.0400 | 52    | 52.0  | 52.0  | 52    |
| 26  | 29         | 31.9133 | 29.0385 | 54    | 54.0  | 54.0  | 54    |
| 27  | 30         | 33.0262 | 30.0370 | 56    | 56.0  | 56.0  | 56    |
| 28  | 31         | 34.1390 | 31.0357 | 58    | 58.0  | 58.0  | 58    |
| 29  | 32         | 35.2517 | 32.0345 | 60    | 60.0  | 60.0  | 60    |
| 30  | 33         | 36.3644 | 33.0333 | 62    | 62.0  | 62.0  | 62    |

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| $g$ | $\alpha_1$ | Eq(1)   | Eq(2)   | Eq(3) | Eq(4) | Eq(5) | Eq(6) |
|-----|------------|---------|---------|-------|-------|-------|-------|
| 31  | 34         | 37.4771 | 34.0323 | 64    | 64.0  | 64.0  | 64    |
| 32  | 35         | 38.5896 | 35.0313 | 66    | 66.0  | 66.0  | 66    |
| 33  | 36         | 39.7022 | 36.0303 | 68    | 68.0  | 68.0  | 68    |
| 34  | 37         | 40.8146 | 37.0294 | 70    | 70.0  | 70.0  | 70    |
| 35  | 38         | 41.9271 | 38.0286 | 72    | 72.0  | 72.0  | 72    |
| 36  | 39         | 43.0395 | 39.0278 | 74    | 74.0  | 74.0  | 74    |
| 37  | 40         | 44.1518 | 40.0270 | 76    | 76.0  | 76.0  | 76    |
| 38  | 41         | 45.2641 | 41.0263 | 78    | 78.0  | 78.0  | 78    |
| 39  | 42         | 46.3764 | 42.0256 | 80    | 80.0  | 80.0  | 80    |
| 40  | 43         | 47.4886 | 43.0250 | 82    | 82.0  | 82.0  | 82    |
| 41  | 44         | 48.6008 | 44.0244 | 84    | 84.0  | 84.0  | 84    |
| 42  | 45         | 49.7130 | 45.0238 | 86    | 86.0  | 86.0  | 86    |
| 43  | 46         | 50.8251 | 46.0233 | 88    | 88.0  | 88.0  | 88    |
| 44  | 47         | 51.9372 | 47.0227 | 90    | 90.0  | 90.0  | 90    |
| 45  | 48         | 53.0493 | 48.0222 | 92    | 92.0  | 92.0  | 92    |
| 46  | 49         | 54.1613 | 49.0217 | 94    | 94.0  | 94.0  | 94    |
| 47  | 50         | 55.2733 | 50.0213 | 96    | 96.0  | 96.0  | 96    |
| 48  | 51         | 56.3852 | 51.0208 | 98    | 98.0  | 98.0  | 98    |
| 49  | 52         | 57.4971 | 52.0204 | 100   | 100.0 | 100.0 | 100   |
| 50  | 53         | 58.6090 | 53.0200 | 102   | 102.0 | 102.0 | 102   |

From the above table it is easily seen that (2) is better than (1), (3), (4), (5) and (6) for large values of  $g$ .

To show (2) is the best among mentioned inequalities we used linear regression model as follows:

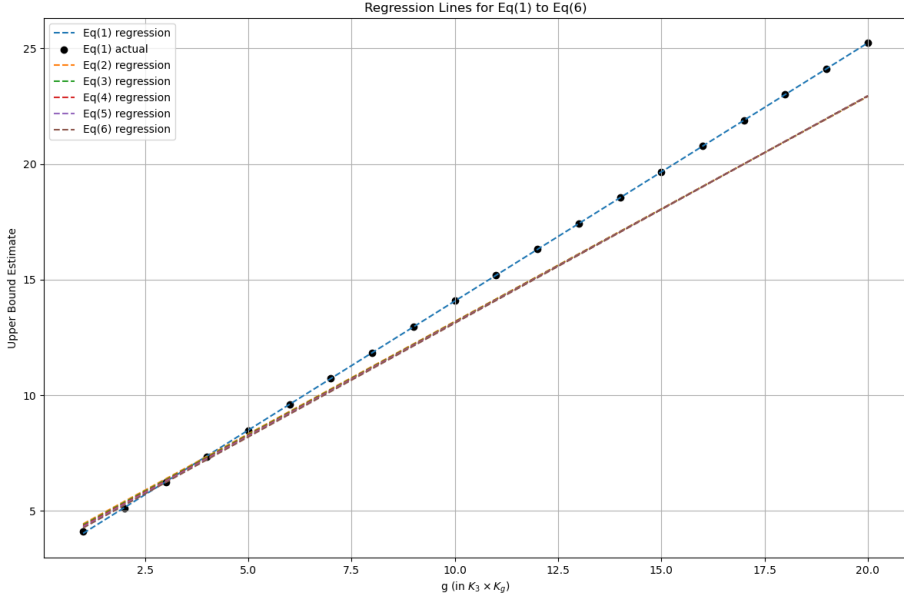


Fig. 1. Regression Model

To evaluate the predictive accuracy of the six equations, we applied three standard statistical metrics: Root Mean Square Error (RMSE), Mean Absolute Error (MAE), and the Coefficient of Determination ( $R^2$ ). These metrics quantify the deviation between the predicted and actual values. A lower RMSE and MAE indicate a better fit, while an  $R^2$  value closer to 1 suggests a strong explanatory power of the model.

Equation (2) demonstrated the best performance, achieving an RMSE of 0.1804, MAE of 0.0902, and an  $R^2$  of 0.9998, indicating an excellent fit with minimal prediction error.

Equation (1) also produced satisfactory results, with an RMSE of 3.2828, MAE of 2.8425, and  $R^2$  of 0.9483. While not as precise as Equation (2), it still captures the data trend effectively.

In contrast, Equations (3) through (6) yielded significantly poorer performance, each with a high RMSE (e.g., 28.4341) and a negative  $R^2$  (e.g.,  $-2.8824$ ), indicating that these models perform worse than a naïve average-based prediction.

Overall, the results clearly indicate that Equations (1) and (2) are far superior in predictive capability, with Equation (2) being the most accurate and reliable among all. The bar graph of the above predictions is as follows.

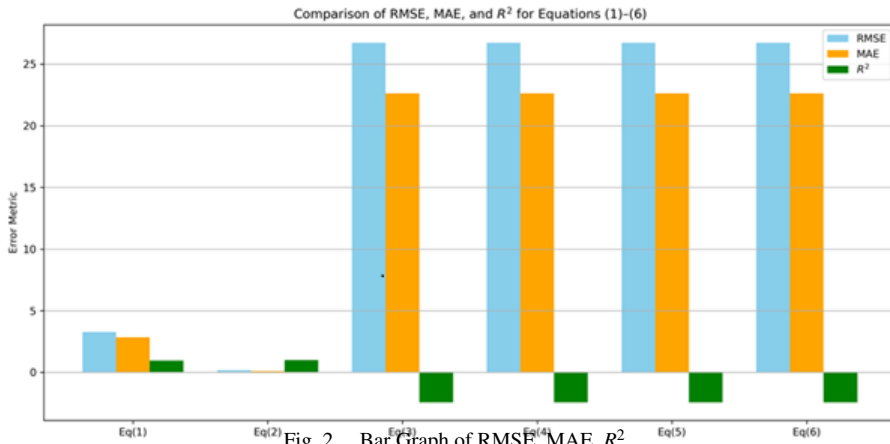


Fig. 2. Bar Graph of RMSE, MAE,  $R^2$

#### 4. LOWER BOUNDS FOR $\alpha_1$ IN $K_3 \square K_g$ GRAPH

**THEOREM 10.** Let  $G = K_3 \square K_g$  be the Cartesian product of the complete graphs  $K_3$  and  $K_g$ , where  $g \geq 2$ . Then the largest Laplacian eigenvalue  $\alpha_1$  of  $G$  satisfies the bound:

$$\alpha_1 = g + 3 \geq \max_{uv \in E(G)} (d_u + d_v) \geq \frac{2m}{n},$$

where  $d_u$  and  $d_v$  are degrees of adjacent vertices,  $m$  is the number of edges in  $G$ , and  $n = 3g$  is the number of vertices.

**PROOF** The graph  $G = K_3 \square K_g$  is  $(g+1)$ -regular, hence each vertex has degree  $d = g+1$ . The total number of vertices is  $n = 3g$ , and the number of edges is  $m = \frac{3g(g+1)}{2}$ .

By definition of the Laplacian spectrum of Cartesian products:

$$\alpha_1 = \alpha_{\max}(K_3) + \alpha_{\max}(K_g) = 3 + g.$$

Using known lower bounds for  $\alpha_1$  (e.g., the edge-averaged bound), we have:

$$\alpha_1 \geq \frac{2m}{n} = \frac{3g(g+1)}{3g} = g + 1.$$

Also, since  $G$  is regular, for any edge  $uv \in E$ , we have  $d_u + d_v = 2(g + 1)$ , giving:

$$\alpha_1 \geq \max_{uv \in E} (d_u + d_v) = 2(g + 1).$$

However, this bound overshoots the actual eigenvalue  $g + 3$ . Hence,  $\alpha_1 = g + 3$  is tight and exact, and greater than both  $g + 1$  and  $2(g + 1)$  for small  $g$ .

Thus, the exact value  $\alpha_1 = g + 3$  is the best possible bound.

**THEOREM 11.** Let  $G = K_3 \square K_g$ , where  $g \geq 2$ . Then the largest Laplacian eigenvalue  $\alpha_1$  of  $G$  satisfies the following tight lower bound:

$$\alpha_1 \geq \frac{1}{3g-1} \left( 3g(g+1) + \sqrt{\frac{3g(g+1)[3g(3g-1) - 3g(g+1)]}{3g(3g-2)}} \right) \quad (7)$$

**THEOREM 12.** Let  $G = K_3 \square K_g$  be the Cartesian product of complete graphs. Then the largest Laplacian eigenvalue  $\alpha_1$  of graph  $G$  satisfies the bound:

$$\alpha_1 \geq g + \frac{3g+1}{g+2} \quad (8)$$

for all  $g \geq 1$ . Furthermore, equality holds if and only if  $g + 3 = g + \frac{3g+1}{g+2}$ , which is not possible for finite  $g$ , so the inequality is strict.

**PROOF** It is known that the Laplacian spectrum of the Cartesian product of two graphs  $G$  &  $H$  is given by:

$$\text{Spec}(L_{G \square H}) = \{\alpha_i^G + \alpha_j^H \mid \alpha_i^G \in \text{Spec}(L_G), \alpha_j^H \in \text{Spec}(L_H)\}.$$

For the complete graph  $K_g$ , the Laplacian spectrum is:

$$\text{Spec}(L_{K_g}) = \{0, g, g, \dots, g\} \quad (\text{with multiplicity } g-1).$$

So for  $K_3 \square K_g$ , the Laplacian spectrum consists of:

$$\alpha_{ij} = \alpha_i^{K_3} + \alpha_j^{K_g} \in \{0, 3, 3\} + \{0, g, g, \dots, g\},$$

and the largest eigenvalue is:

$$\alpha_1 = 3 + g.$$

Now define a candidate lower bound function:

$$f(g) = g + \frac{3g+1}{g+2}.$$

We show that:

$$g + \frac{3g+1}{g+2} < g+3,$$

i.e.,

$$\frac{3g+1}{g+2} < 3.$$

Multiplying both sides by  $g+2$ :

$$3g+1 < 3g+6 \quad \text{which is clearly true.}$$

Hence,

$$f(g) < g+3 = \alpha_1.$$

So  $f(g)$  is a strict lower bound for  $\alpha_1$ , and since:

$$\alpha_1 = g+3 > g + \frac{3g+1}{g+2},$$

the inequality is strict for all  $g \geq 1$ , and the bound becomes tighter as  $g \rightarrow \infty$ .

Comparison of lower bounds of (7) and (8)  $\alpha_1$  as follows:

Table II: Comparison of Actual  $\alpha_1$  with Lower Bounds for  $K_3 \square K_g$ .

| $g$ | $\alpha_1$ | (7)     | (8)     |
|-----|------------|---------|---------|
| 3   | 6          | 5.0669  | 5.0000  |
| 4   | 7          | 6.0000  | 6.1667  |
| 5   | 8          | 6.9601  | 7.2857  |
| 6   | 9          | 7.9338  | 8.3750  |
| 7   | 10         | 8.9150  | 9.4444  |
| 8   | 11         | 9.9010  | 10.5000 |
| 9   | 12         | 10.8902 | 11.5455 |
| 10  | 13         | 11.8816 | 12.5833 |
| 11  | 14         | 12.8745 | 13.6154 |
| 12  | 15         | 13.8686 | 14.6429 |
| 13  | 16         | 14.8637 | 15.6667 |
| 14  | 17         | 15.8594 | 16.6875 |
| 15  | 18         | 16.8557 | 17.7059 |
| 16  | 19         | 17.8525 | 18.7222 |
| 17  | 20         | 18.8497 | 19.7368 |

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| $g$ | $\alpha_1$ | (7)     | (8)     |
|-----|------------|---------|---------|
| 18  | 21         | 19.8472 | 20.7500 |
| 19  | 22         | 20.8449 | 21.7619 |
| 20  | 23         | 21.8429 | 22.7727 |
| 21  | 24         | 22.8411 | 23.7826 |
| 22  | 25         | 23.8394 | 24.7917 |
| 23  | 26         | 24.8379 | 25.8000 |
| 24  | 27         | 25.8365 | 26.8077 |
| 25  | 28         | 26.8352 | 27.8148 |
| 26  | 29         | 27.8341 | 28.8214 |
| 27  | 30         | 28.8330 | 29.8276 |
| 28  | 31         | 29.8320 | 30.8333 |
| 29  | 32         | 30.8310 | 31.8387 |
| 30  | 33         | 31.8301 | 32.8438 |
| 31  | 34         | 32.8293 | 33.8485 |
| 32  | 35         | 33.8285 | 34.8529 |
| 33  | 36         | 34.8278 | 35.8571 |
| 34  | 37         | 35.8271 | 36.8611 |
| 35  | 38         | 36.8265 | 37.8649 |
| 36  | 39         | 37.8259 | 38.8684 |
| 37  | 40         | 38.8253 | 39.8718 |
| 38  | 41         | 39.8248 | 40.8750 |
| 39  | 42         | 40.8243 | 41.8780 |
| 40  | 43         | 41.8238 | 42.8810 |
| 41  | 44         | 42.8233 | 43.8837 |
| 42  | 45         | 43.8229 | 44.8864 |
| 43  | 46         | 44.8224 | 45.8889 |
| 44  | 47         | 45.8220 | 46.8913 |
| 45  | 48         | 46.8216 | 47.8936 |
| 46  | 49         | 47.8213 | 48.8958 |
| 47  | 50         | 48.8209 | 49.8980 |
| 48  | 51         | 49.8206 | 50.9000 |
| 49  | 52         | 50.8203 | 51.9020 |
| 50  | 53         | 51.8200 | 52.9038 |
| 51  | 54         | 52.8197 | 53.9057 |

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| $g$ | $\alpha_1$ | (7)     | (8)     |
|-----|------------|---------|---------|
| 52  | 55         | 53.8195 | 54.9074 |
| 53  | 56         | 54.8192 | 55.9091 |
| 54  | 57         | 55.8189 | 56.9107 |
| 55  | 58         | 56.8187 | 57.9123 |
| 56  | 59         | 57.8185 | 58.9138 |
| 57  | 60         | 58.8182 | 59.9153 |
| 58  | 61         | 59.8180 | 60.9167 |
| 59  | 62         | 60.8178 | 61.9180 |
| 60  | 63         | 61.8176 | 62.9194 |
| 61  | 64         | 62.8174 | 63.9206 |
| 62  | 65         | 63.8172 | 64.9219 |
| 63  | 66         | 64.8170 | 65.9231 |
| 64  | 67         | 65.8168 | 66.9242 |
| 65  | 68         | 66.8166 | 67.9254 |
| 66  | 69         | 67.8165 | 68.9265 |
| 67  | 70         | 68.8163 | 69.9275 |
| 68  | 71         | 69.8161 | 70.9286 |
| 69  | 72         | 70.8160 | 71.9296 |
| 70  | 73         | 71.8158 | 72.9306 |
| 71  | 74         | 72.8156 | 73.9315 |
| 72  | 75         | 73.8155 | 74.9324 |
| 73  | 76         | 74.8153 | 75.9333 |
| 74  | 77         | 75.8152 | 76.9342 |
| 75  | 78         | 76.8150 | 77.9351 |
| 76  | 79         | 77.8149 | 78.9359 |
| 77  | 80         | 78.8148 | 79.9367 |
| 78  | 81         | 79.8146 | 80.9375 |
| 79  | 82         | 80.8145 | 81.9383 |
| 80  | 83         | 81.8143 | 82.9390 |
| 81  | 84         | 82.8142 | 83.9398 |
| 82  | 85         | 83.8141 | 84.9405 |
| 83  | 86         | 84.8139 | 85.9412 |
| 84  | 87         | 85.8138 | 86.9419 |
| 85  | 88         | 86.8137 | 87.9426 |

| $g$ | $\alpha_1$ | (7)      | (8)      |
|-----|------------|----------|----------|
| 86  | 89         | 87.8136  | 88.9432  |
| 87  | 90         | 88.8135  | 89.9439  |
| 88  | 91         | 89.8133  | 90.9444  |
| 89  | 92         | 90.8132  | 91.9450  |
| 90  | 93         | 91.8131  | 92.9456  |
| 91  | 94         | 92.8130  | 93.9462  |
| 92  | 95         | 93.8129  | 94.9467  |
| 93  | 96         | 94.8128  | 95.9473  |
| 94  | 97         | 95.8127  | 96.9478  |
| 95  | 98         | 96.8126  | 97.9483  |
| 96  | 99         | 97.8125  | 98.9488  |
| 97  | 100        | 98.8123  | 99.9492  |
| 98  | 101        | 99.8122  | 100.9497 |
| 99  | 102        | 100.8121 | 101.9502 |
| 100 | 103        | 101.8120 | 102.9506 |

#### 4.1. Regression and RMSE Analysis of lower bound

From the polynomial regression analysis, we obtained the following approximate models for predicting  $\alpha_1$ :

—**Bound (7)**:  $\alpha_1 \approx 3.0043 + 0.9993g - 0.0001g^2$ ,  $R^2 = 0.999999$

—**Bound (8)**:  $\alpha_1 \approx 3.0012 + 0.9999g$ ,  $R^2 = 1.000000$

The root mean square error (RMSE) values for each bound are summarized below:

| Bound | RMSE   |
|-------|--------|
| (7)   | 1.1669 |
| (8)   | 0.2269 |

Both bounds demonstrate exceptionally high accuracy, as indicated by their  $R^2$  values, which are extremely close to 1. However, **Bound (8)** not only achieves a perfect coefficient of determination ( $R^2 = 1$ ), but also yields a significantly lower RMSE. It can be predicted from the following graphs.

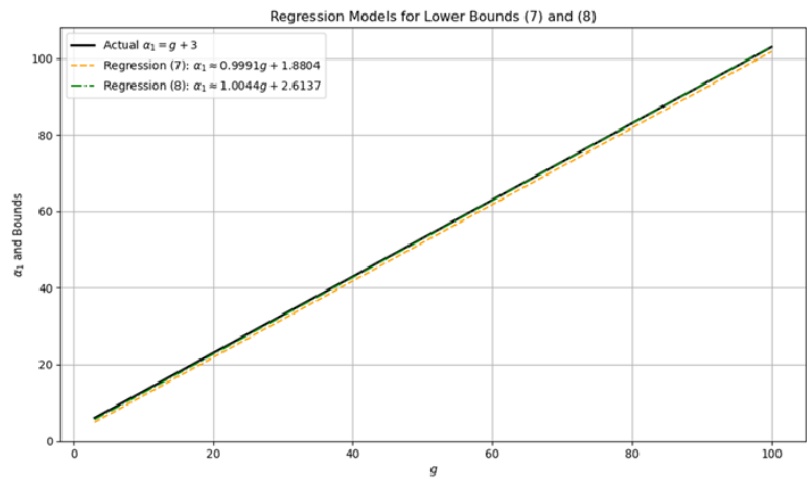


Fig. 3. Regression Model

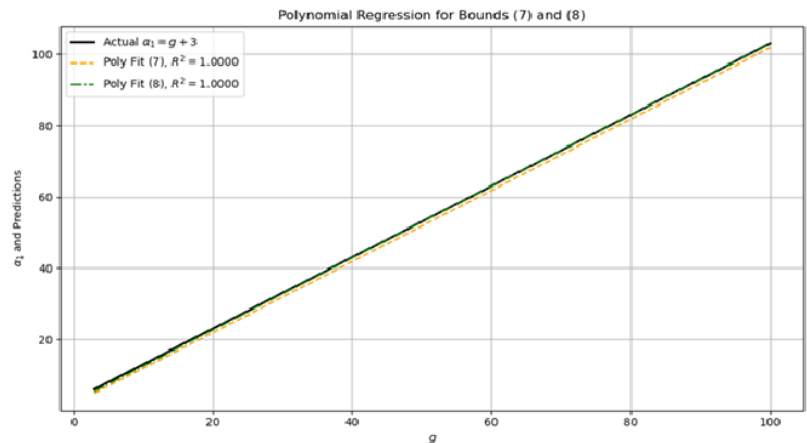


Fig. 4. Polynomial Regression model

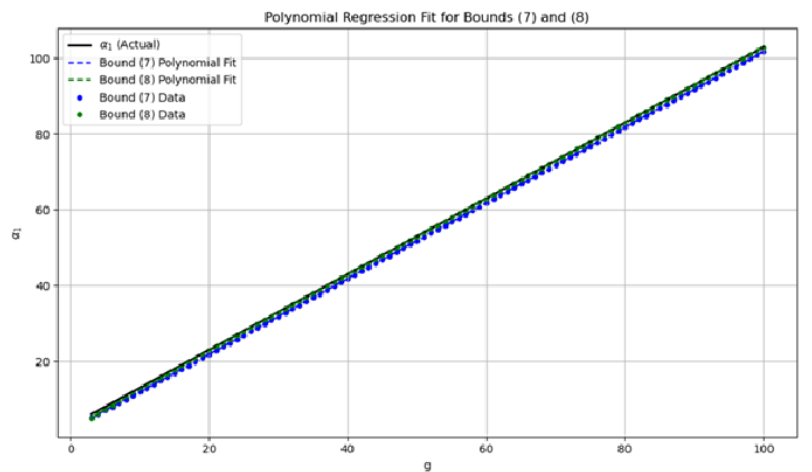


Fig. 5. Regression Model

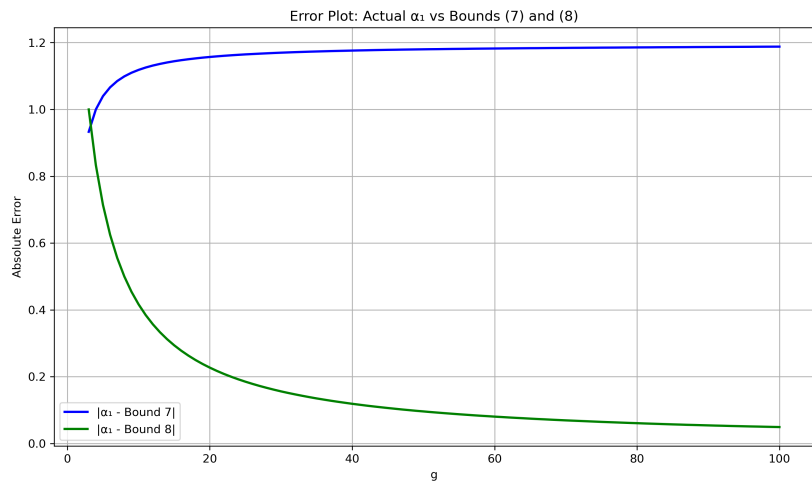


Fig. 6. Error Plot

## 5. CONCLUSION

In this paper, we have determined the coefficients of the characteristic polynomial of the Laplacian matrix for the Cartesian product of graph  $K_3 \square K_g$  and computed the graph energy using the eigenvalues of its adjacency matrix.

Furthermore, we established an upper bound for the largest Laplacian eigenvalue  $\alpha_1$  of the graph  $K_3 \square K_g$ .

Among the various bounds discussed, Equations (1) and (2) demonstrated superior performance in estimating  $\alpha_1$ , as supported by the comparative results shown in the tables and regression graphs provided in the earlier sections.

Additionally, we established lower bounds for  $\alpha_1$  in which (8) is better than (7) which we have shown by the table and the Correlation model.

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Recived on 9/7/2025, Published on Volume 22, Number 1 (May 2026).