Improved estimation of population parameter in the existence of non-response using auxiliary information

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Abstract

In many domains, including healthcare, economics, and weather forecasting, estimating the finite population mean using non-response is an essential and practically useful task. The mean estimation is heavily used by actuaries and insurance analysts for the purpose of making insightful conclusions. To improve the precision of estimators in survey samples, auxiliary variables play a crucial role. Based on simple random sampling, this study suggests a novel family of estimators that are better designed for determining the population mean using data which include non-response. Efficiency requirements are determined by comparing the mean squared errors of the suggested class of estimators with existing counterparts. The empirical study makes use of both real life data sets and a simulation study. The empirical results show clearly that the proposed estimators are better than the existing estimators. The biases and mean square errors of these estimators, among their other attributes, have been thoroughly examined and explored through numerical and simulation studies. The recommended estimators perform well for the population mean in terms of minimum MSEs and higher percentage relative efficiency.

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1. INTRODUCTION

The accessible computation and simple structure of the traditional ratio, regression, and exponential processes have led to their widespread use in recent years for estimating population means. By adding the populations mean to standard ratio and exponential estimators, along with a number of auxiliary variables and their related characteristics like skewness and kurtosis as well as coefficients of variation, researchers have been able to better estimate the unknown population's mean of the study variable. A prior understanding of the auxiliary variable's population characteristics is also required for the ratio, exponential, and regression estimate approaches. Best linear unbiased (BLU) estimates, a common regression estimator of the study variable's population mean, were proposed by [Watson 1937] assuming auxiliary information was considered. Cochran had suggested the conventional ratio

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estimate of the population mean using SRS [Cochran 1940]. It is well-known that estimator precision is enhanced during both the design and estimation stages when an auxiliary variable is appropriately used in survey sampling. Ratio, product, and regression-type procedures are among the various auxiliary variable integration strategies considered by the survey sampling technique. For these methods, there is an infinite number of possible combinations. One can choose from a wide variety of estimators, some of which combine ratio, product, or regression estimators with the addition of an additional variable all at once. Estimators that successfully interchange the auxiliary variables have been developed by a large number of scientists. [Raja and Maqbool 2024] recently developed an improved estimator using auxiliary information. [Qureshi et al. 2024] considered Ln-type estimator using auxiliary variable. Some more related work for estimation of population parameters includes [Ullah et al. 2022; Eleazer Kurbah and Khongji 2024; Ahmad et al. 2022a; Rueda et al. 2004; Semary et al. 2024; Singh and Garg 2024; Kumar Adichwal et al. 2022; Lone et al. 2021].

One of the most significant problems with surveys, from a practical point of view, is non-response, which may happen in many ways. Just a few instances include language barriers, response delays, incorrect return address entry, and input from another person, clustering, and censorship. Statistics has only recently come to terms with the possibility that data properties may change when the random component of incompleteness or non-response is ignored. Several variables affect the survey's non-response rate. These include, but are not limited to, the kind of data collected, the investigating agency's official status, the amount of publicity, the respondents' legal requirements, the enumerator's visit duration, and the withdrawal period. To improve estimator efficiency and decrease non-response bias, several authors have focused on estimating the population mean. Mail surveys are more likely to have a high rate of non-response in their sample than special interview surveys. Originally, the issue of incomplete samples in surveys conducted by mail or over the phone was initially addressed in [Hansen and Hurwitz 1946]. For relevant work, we may be seen in references [Ahmadini et al. 2022; Ahmad et al. 2022b; Basit et al. 2023; Gujarati 2004], [Hussain et al. 2020], [Hussain et al. 2020; Ismail 2017; Khare and Kumar 2015; Khare and Sinha 2009; Muneer et al. 2018; Raman Sinha and Khanna 2025] and [Yaqub et al. 2024] and the sources provided therein. Recently [Sajjad et al. 2025] and [Alamri and Aloraini 2025] developed improved classes of estimators for estimation of population mean and median using auxiliary information in the availability of non-response.

This study aims to improve the class of estimators used to determine the population mean from non-response under simple random sampling. We numerically compare the efficiency of the existing estimators to that of the proposed class of estimators. To demonstrate the suggested class of estimators, we offer two real-world data sets and simulated populations.

Looking at some real-life examples can help to highlight the practical importance of this research:

—Patient surveys are used in healthcare research to evaluate the effectiveness of medical treatments. However, not all patients may respond to these surveys, which might lead to measurement inaccuracies caused by self-reporting. In these situations, a reliable estimate of the population mean is necessary for making sensible choices about treatment approaches.

—A key part of market research, consumer preference surveys provide invaluable information for new product and advertising campaigns. Mistakes in the data collection process or a lack of participation from specific demographics could skew the assessment of market changes, influencing the decisions made by corporations.

—When evaluating the efficiency of educational programs or institutions, assessment findings could be skewed due to measurement mistakes and varying levels of student engagement with educational examinations. Precise population mean estimation is crucial for assuring high-quality education and making informed policy decisions.

—By addressing these issues in various domains, this new framework hopes to provide an efficient method for accurately determining the population mean, which will improve decision-making in practical situations. Using both real data sets and a simulation study, the proposed estimation approach can be utilized to determine the population mean under a simple random sample non-response, as shown in the next portion of the manuscript.

The following sections elaborate on the present paper design. The methodology and terminology are defined in Section 2. Section 3, discusses some existing adopted estimators and provides details on their bias, mean squared error (MSE), and formulations for the minimal MSE. In Section 4, we present a new class of estimators for population mean for non-response under SRS, together with the bias and minimum MSE expressions. In Section 5, we find the numerical analysis findings. Section 6 presents a simulation study. The article is discussed in Section 7. The paper concludes in Section 8.

2. METHODOLOGY

Take into consideration that $\mho = \{\Omega_1, \Omega_2, \cdots, \Omega_N\}$ denotes a population of N units split into two categories having sizes N_1 and N_2, where $N = N_1 + N_2$. Thus, we denote the group that responded as $\mho_1 = \{\Omega_1, \Omega_2, \cdots, N_1\}$ and the group that did not respond as $\mho_2 = \{\Omega_1, \Omega_2, \cdots, N_2\}$. In order to estimate the population mean, SRSWOR takes a sample of n from the underlying population, where n_1 are responding units and $n_2 = n - n_1$ are not. Furthermore, the sample sizes n_1 and n_2 are likely drawn from the groups $\mho_2$ (non-response) and $\mho_1$ (response), respectively. In addition, a sample of size $r = n_2/k$ units is created using a simple random sampling method without replacement from n_2, where $k > 1$.

Let Y, X, be the study and the auxiliary variable.

$\bar{Y} = \sum_{i=1}^{N} \frac{Y_i}{N}$, $\hat{\bar{y}} = \sum_{i=1}^{n} \frac{Y_i}{n}$, $\bar{X} = \sum_{i=1}^{N} \frac{X_i}{N}$, $\hat{\bar{X}} = \sum_{i=1}^{n} \frac{X_i}{n}$, are the population and sample mean of Y and X.

$\bar{Y}_2 = \sum_{i=1}^{N_2} \frac{Y_i}{N_2}$, and $\bar{X}_2 = \sum_{i=1}^{N_2} \frac{X_i}{N_2}$, are the population mean of Y and X for non-response group.

$\hat{\bar{y}}_1 = \sum_{i=1}^{n_1} \frac{Y_i}{n_1}$, and $\hat{\bar{x}}_1 = \sum_{i=1}^{n_1} \frac{X_i}{n_1}$, denote the sample mean based on n_1 responding units out of n units.

$\hat{\bar{y}}_{2r} = \sum_{i=1}^{r} \frac{Y_i}{r}$, and $\hat{\bar{x}}_{2r} = \sum_{i=1}^{r} \frac{X_i}{r}$, be the sample mean based on r reacting units out of n_2 non-response units.

$S_y^2 = \sum_{i=1}^{N} \frac{(Y_i - \bar{Y})^2}{(N-1)}$, and $S_x^2 = \sum_{i=1}^{N} \frac{(X_i - \bar{X})^2}{(N-1)}$ are the population variance of Y and X respectively.

$S_{Y_2}^2 = \sum_{i=1}^{N_2} \frac{(Y_i - \bar{Y}_2)^2}{(N_2-1)}, S_X^2 = \sum_{i=1}^{N_2)} \frac{(X_i - \bar{X}_2)^2}{(N_2-1)}$ are the population variance of Y and X for non-response group.

Let $C_Y = S_Y/\bar{Y}$, $C_X = S_X/\bar{X}$ are the population coefficient of variation of Y and X.

Let $C_{Y(2)} = S_{y(2)}/\bar{Y}_2$ and $C_{X(2)} = S_{x(2)}/\bar{X}_2$ be the population coefficients of variation of and X for non-response group respectively.

Let $S_{YX} = \sum_{i=1}^{N}(Y_i - \bar{Y})(X_i - \bar{X})/(N-1)$ be the population covariance between (Y, X).

Let $S_{Y_2X_2} = \sum_{i=1}^{N}(Y_i - \bar{Y}_2)(X_i - \bar{X}_2)/(N_2 - 1)$ be the population covariance between (Y, X) for non-response group.

Let $\rho_{yx} = S_{yx}/(S_y S_x)$ be the population correlation coefficients between (Y, X).

Let $\rho_{y_2x_2} = S_{y_2x_2}/(S_{y_2} S_{x_2})$ be the population correlation coefficients between (Y, X), for non-response group.

The populations mean Y may be written as such

$$\bar{Y} = W_1\hat{\bar{Y}}_1 + W_2\hat{\bar{Y}}_2 \tag{1}$$

$$\bar{X} = W_1\hat{\bar{X}}_1 + W_2\hat{\bar{X}}_2 \tag{2}$$

where

$W_j = N_j/N$, $\bar{Y}_j = \sum_{i=1}^{N_j} Y_i/N_j \ for \ \ j = 1,2.$, and $\bar{X}_j = \sum_{i=1}^{N_j} X_i/N_j$.

Following [Ahmadini et al. 2022] have suggested an unbiased estimator of $\bar{Y}$ under non-response, which is given by :

$\hat{\bar{T}}^* = w_1\hat{\bar{Y}}_{(1)} + w_2\hat{\bar{Y}}_{(2r)}$.

and

$$Var(\bar{\hat{T}}^*) = \Lambda_1 S^2_{y(1)} + \Lambda_2 S^2_{y(2)} \tag{3}$$

where

$w_j = n_j/n$ for $j = 1,2$, $\Lambda_1 = \left(\frac{1}{n} - \frac{1}{N}\right)$ and $\Lambda_2 = W_2(k-1)$.

$\hat{\bar{X}}^* = w_1\hat{\bar{X}}_((1)) + w_2\hat{\bar{X}}_((2))$

are conversely, unbiased estimators of $\bar{X}$ respectively under non-response condition, with suitable variances

$Var(\bar{X}^*) = \Lambda_1 S_2^2 + \Lambda_2 S^2_{2(2)}$

To get expression of bias and MSEs of an estimators, we used the following error terms.

$e_0^* = (\hat{\bar{y}} - \bar{Y})/\bar{Y}$, $e_1^* = (\hat{\bar{x}} - \bar{X})/\bar{X}$, $e_1 = (\hat{\bar{x}} - \bar{X})/\bar{X}$

$V_{PQ} = E\left[e_0^P e_1^Q\right]$ and $V^*_{PQ} = [e_0^{*P} e_1^{*R}]$

$E[e_0^{*2}] = (\Lambda_1 S_y^2 + \Lambda_2 S_{y2}^2)/\bar{Y}^2 = V_{20}$, $E[e_1^{*2}] = (\Lambda_1 S_x^2 + \Lambda_2 S_{x2}^2)/\bar{X}^2 = V_{02}$,

$E[e_0^* e_1^*] = ((\Lambda_1 S_y S_x \rho_{yx}) + (\Lambda_2 S_y S_x \rho_{yx}))/(\bar{Y}\bar{X}) = V_{11}$,

$E[e_1^2] = (\Lambda_1 S_{x1}^2)/\bar{X}^2 = \psi_{02}$, $E[e_0^* e_1] = (\Lambda_1 S_y S_x \rho_{yx})/(\bar{Y}\bar{X}) = \psi_{11}$

$\Lambda_1 = (\frac{1}{n} - \frac{1}{N})$, $\Lambda_2 = (W_2.(K-1))/n$

3. EXISTING ESTIMATORS

Some of the best-known estimators are discussed here in the presence non-response. The clarification that follows includes their bias and MSE/minimum MSE expressions.

3.1. Situation–I : When non-response occurs only in the study variable

(1) The ratio estimator with his bias and MSEs are given by:

$$\hat{\hat{T}}^{*}_{Ratio} = \bar{y}^{*}(\bar{X}/\hat{\bar{x}}) \tag{4}$$

$Bias(\hat{\hat{T}}^{*}_{Ratio}) = \bar{Y}(\psi_{02} - \psi_{11})$

and

$$MSE(\hat{\hat{T}}^{*}_{Ratio}) = \bar{Y}^{2}[V_{20} + \psi_{02} - 2\psi_{11}] \tag{5}$$

(2) The ratio estimator with his bias and MSEs are given by:

$$\hat{\hat{T}}^{*}_{Product} = \hat{\bar{y}}^{*}(\hat{\bar{x}}/\bar{X}) \tag{6}$$

$Bias(\hat{\hat{T}}^{*}_{Product}) = \bar{Y}\psi_{11}$

and

$$MSE(\hat{\hat{T}}^{*}_{Product}) = \bar{Y}^{2}[V_{20} + \psi_{02} + 2\psi_{11}] \tag{7}$$

(3) The difference estimator is given by:

$$\hat{\hat{T}}^{*}_{Difference} = \hat{\bar{y}}^{*} + D(\bar{X} - \hat{\bar{x}}) \tag{8}$$

Where $D = \frac{\bar{Y}\psi_{11}}{\bar{X}\psi_{02}}$, the variance of $\hat{\hat{T}}^{*}_{Difference}$ at the optimum value of D, is given by:

$$Var(\hat{\hat{T}}^{*}_{Difference}) = \bar{Y}^{2}(V_{20}\psi_{02} - \psi_{11}^{2})/\psi_{02} \tag{9}$$

(4) [Rao 1991] suggested difference in difference estimator, which is given by:

$$\hat{\hat{T}}^{*}_{RD} = D_{1}\hat{\bar{y}}^{*} + D_{2}(\bar{X} - \hat{\bar{x}}) \tag{10}$$

The D_1 and D_2 are given by:

$D_{1(opt)} = \frac{\psi_{02}}{[\psi_{02}\{1+V_{2}0\} - \psi_{11}^{2}]}$, $D_{1(opt)} = \frac{\bar{Y}\psi_{11}}{(\bar{X}[\psi_{02}\{1+V_{2}0\} - \psi_{11}^{2}])}$

The minimum MSE at $D_1(opt)$ and $D_{2(opt)}$ are given by:

$$MSE(\hat{\hat{T}}^{*}_{RD}) = \frac{(\bar{Y}^{2}(V_{20}\psi_{02} - \psi_{11}^{2}))}{(\psi_{02}\{1+V_{2}0\} - \psi_{11}^{2})} \tag{11}$$

(5) The [Singh et al. 2008] recommended the following estimator:

$$\hat{\hat{T}}^{*}_{Singh} = \hat{\hat{y}}^{*} exp(\frac{a(\bar{X} - \hat{\hat{x}})}{a(\bar{X} + \hat{\hat{x}}) + 2b}) \tag{12}$$

The bias and MSE of $\hat{\hat{T}}^{*}_{Singh}$, are given as:
$Bias(\hat{\hat{T}}^{*}_{Singh}) \cong \bar{Y}(\frac{3}{8}\theta\psi_{02} - \frac{1}{2}\theta\psi_{11})$
and

$$MSE(\hat{\hat{T}}^{*}_{Singh}) \cong \frac{\bar{Y}^2}{4}(4V_{20} + \theta\psi_{02} - 4\theta\psi_{11}) \tag{13}$$

where $\theta = \frac{a\bar{X}}{(a\bar{X}+b)}$

(6) The suggested estimator of [Grover and Kaur 2014] is given as:

$$\hat{\hat{T}}^{*}_{Grover} = k_1\hat{\hat{y}}^{*} + k_2(\bar{X} - \hat{\hat{x}}) exp \frac{(a(\bar{X} - \hat{\hat{x}})}{a(\bar{X} + \hat{\hat{x}}) + 2b} \tag{14}$$

The bias and MSE of $\hat{\hat{T}}^{*}_{Grover}$, are given as:
$Bias(\hat{\hat{T}}^{*}_{Grover}) \cong \bar{Y}(k_1 - 1) + \frac{3}{8}\theta^2 k_1 \bar{Y} V_{20} + \frac{1}{2}\theta k_2 \bar{X}\psi_{02} - \frac{1}{2}\theta\bar{Y}\psi_{11}$
$k_{1(opt)} = \frac{(\psi_{02}(\psi_{02}-8))}{8(-V_{20}\psi_{02}-\psi_{11}^2-\psi_{02})}$
$k_{2(opt)} = \frac{(\bar{Y}(\psi_{02}^2-\psi_{02}\psi_{02}+4V_{20}\psi_{02}-4\psi_{11}^2-4\psi_{02}+8\psi_{11}))}{(8\bar{X}(V_{20}\psi_{02}-\psi_{11}^2+\psi_{02}))}$
The MSE of $\hat{\hat{T}}^{*}_{Grover}$ is given by:

$$(MSE)_{min}(\hat{\hat{T}}^{*}_{Grover}) \cong (Var)_{min}(\hat{\hat{T}}^{*}_{Difference}) - \frac{(\bar{Y}^2(\theta^2\psi_{02}^2 - 8\psi_{11}^2 + 8\psi_{02}V_{20})^2)}{(64\psi_{02}^2\left\{1 + V_{20}(1-\rho_{yx}^2)\right\})} \tag{15}$$

3.2. Situation –II: When non-response occur in both the study and auxiliary variable

—The ratio estimator with his bias and MSEs are given by:

$$\hat{\hat{T}}^{**}_{Ratio} = \hat{\hat{y}}^{*}(\bar{X}/\hat{\hat{x}}^{*}) \tag{16}$$

$$Bias(\hat{\hat{T}}^{**}_{Ratio}) = \bar{Y}(V_{02} - V_{11})$$

and

$$MSE(\hat{\hat{T}}^{**}_{Ratio}) = \bar{Y}^2[V_{20} + V_{02} - 2V_{11}] \tag{17}$$

—The ratio estimator with his bias and MSEs are given by:

$$\hat{\hat{T}}^{**}_{Product} = \bar{y}^{*}(\hat{\hat{x}}^{*}/\bar{X}) \tag{18}$$

$$Bias(\hat{\hat{T}}^{**}_{Product}) = \bar{Y}V_{11}$$

and

$$MSE(\hat{\hat{T}}^{**}_{Product}) = \bar{Y}^2[V_{20} + V_{02} + 2V_{11}] \tag{19}$$

—The difference estimator is given by:

$$\hat{\hat{T}}^{**}_{Difference} = \hat{\hat{y}}^* + D(\bar{X} - \hat{\hat{x}}^*) \tag{20}$$

Where D is a constant, and its value is given by:

$D = \frac{\bar{Y}V_{11}}{\bar{X}V_{02}}$, the variance of $\hat{\hat{T}}^{**}_{Product}$ at the optimum value of D, is given by:

$$Var(\hat{\hat{T}}^{**}_{Difference}) = \frac{\bar{Y}^2(V_{20}V_{02} - V_{11}^2)}{V_{02}} \tag{21}$$

Where D is a constant, and its value is given by:

$D = \frac{(\bar{Y}V_{11})}{(\bar{X}V_{02})}$, the variance of $\hat{\hat{T}}^{*}_{Difference}$ at the optimum value of D, is given by:

$$Var(\hat{\hat{T}}^{**}_{Difference}) = \frac{(\bar{Y}^2(V_{20}V_{02} - V_{11}^2))}{V_{02}} \tag{22}$$

—[Rao 1991] suggested difference in difference estimator, which is given by:

$$\hat{\hat{T}}^{**}_{RD} = D_1\hat{\hat{y}}^* + D_2(\bar{X} - \hat{\hat{x}}^*) \tag{23}$$

The D_1 and D_2 are given by:

$D_{1(opt)} = \frac{V_{02}}{[V_{02}1 + V_{20} - V_{11}^2]}$, $D_{1(opt)} = \frac{(\bar{Y}V_{11})}{(\bar{X}[V_{02}1 + V_{20} - V_{11}^2])}$

The minimum MSE at $D_{1(opt)}$ and $D_{2(opt)}$ are given by:

—The [Singh et al. 2008] recommended the following estimator:

$$\hat{\hat{T}}^{**}_{Singh} = \frac{\hat{\hat{y}}^* + D_2 exp((a(\bar{X} - \hat{\hat{x}}^*))}{(a(\bar{X} + \hat{\hat{x}}^*) + 2b))} \tag{24}$$

The properties of $\hat{\hat{T}}^{**}_{Singh}$, are given as:

$$Bias(\hat{\hat{T}}^{**}_{Singh}) = \bar{Y}(\frac{3}{8}\theta V_{02} - \frac{1}{2}\theta V_{11})$$

and

$$MSE(\hat{\hat{T}}^{**}_{Singh}) \cong \frac{\bar{Y}^2}{4}(4V_{20} + \theta V_{02} - 4\theta V_{11}) \tag{25}$$

—Following [Grover and Kaur 2014], estimator of $\bar{Y}$ is given by:

$$\hat{\hat{Y}}^{**}_{Grover} = k_1\hat{\hat{y}}^* + k_2(\bar{X} - \hat{\hat{x}}^*)exp(\frac{a(\bar{X} - \hat{\hat{x}}^*)}{(a(\bar{X} + \hat{\hat{x}}^*) + 2b)}) \tag{26}$$

where k_1 and k_2 are unknown constants. The bias and MSE of $\hat{\hat{Y}}^{**}_{Grover}$, are given as:

$$Bias(\hat{\hat{Y}}^{**}_{Grover}) = \bar{Y}(k_1 - 1) + \frac{3}{8}\theta^2 k_1 \bar{Y} V_{20} + \frac{1}{2}\theta k_2 \bar{X} V_{02} - \frac{1}{2}\theta \bar{Y} V_{11}$$

$k_{1(opt)} = (V_{02}(V_{02} - 8))/8(V_{02}(1+V_{20}) - V_{11}^2)$, and $k_{2(opt)} = (\bar{Y}(V_{02}^2 - V_{02}V_{02} + 4V_{20}V_{02} - 4V_{11}^2 - 4V_{11} + 8V_{20}))/(8\bar{X}(\psi_{200}\psi_{020} - \psi_{110}^2 + \psi_{020}))$

The MSE of $\hat{\hat{Y}}^{**}_{Grover}$ at the optimal values of k_1 and k_2 is given by:

$$+MSE_{min}(\hat{\hat{Y}}^{**}_{Grover}) \cong Var_{min}(\hat{\hat{Y}}^{**}_{Difference}) - \frac{(\bar{Y}^2(\theta^2 V_{02}^2 - 8V_{11}^2 + 8V_{02}V_{20})^2)}{(64V_{02}^2 1 + V_{20}(1 - \rho^2_{yx(2)}))} \tag{27}$$

4. SUGGESTED CLASS OF ESTIMATORS

When the auxiliary variables are considered in both the design and estimation stages, estimators become more efficient. We provide a new family of estimators that draw on auxiliary variables and take advantage of non-response under simple random sampling, motivated by [Hussain et al. 2020]. Theoretical properties, including bias and mean square error, are obtained up to the first order of approximation. The key benefit of our generalized class of estimators is their superior efficiency and adaptability compared to the existing estimators, which are given by:

4.1. Situation –I: When non-response occurs only in the study variable

$$\hat{\hat{T}}^{*}_{prop} = \left[\Omega_1 \hat{\hat{y}}^{*}\left\{\frac{1}{4}\left(\frac{\bar{X}}{\hat{\hat{x}}} + \frac{\hat{\hat{x}}}{\bar{X}}\right) exp\left(\frac{\bar{X} - \hat{\hat{x}}}{\bar{X} + \hat{\hat{x}}}\right) + exp\left(\frac{\hat{\hat{x}} - \bar{X}}{\bar{X} + \hat{\hat{x}}}\right)\right\} + \Omega_2\left(\bar{X} - \hat{\hat{x}}\right)\right] exp\left(\frac{a(\bar{X} - \hat{\hat{x}})}{a(\bar{X} - \hat{\hat{x}}) + 2b}\right) \tag{28}$$

Where Ω_1 and Ω_2 are constants.

$$\hat{\hat{T}}^{*}_{prop} = \left[\Omega_1 \bar{Y}(1 + e_0^*)(1 + \frac{5}{8}e_1^2) - \Omega_2 \bar{X} e_1\right]\left[1 - \frac{1}{2}\theta e_1 + \frac{3}{8}\theta^2 e_1^2\right] \tag{29}$$

By expressing (29), we have

$$\hat{\hat{T}}^{*}_{prop} - \bar{Y} = \bar{Y} + \bar{Y}\left[(\Omega_1 - 1) + \Omega_1 e_0^* - \frac{1}{2}\theta e_1 - \frac{1}{2}\theta e_0^* e_1 + \frac{1}{8}(5 + 3\theta^2)e_1^2 - \Omega_2 R(e_1 - \frac{1}{2}\theta e_1^2)\right] \tag{30}$$

From (30), the bias of $\hat{\hat{T}}^{*}_{prop}$ is given by:

$$Bias(\hat{\hat{T}}^{*}_{prop}) = \bar{Y}[(\Omega_1 - 1) + \Omega_1\left\{\frac{1}{8}(5 + 3\theta^2)\psi_{02} - \frac{1}{2}\theta\psi_{11} + \frac{1}{2}\psi_2 R\psi_{02}\right\}$$

Squaring and taking expectation of (30), we have:

$$MSE(\hat{\hat{T}}^{*}_{prop}) = \bar{Y}^2\left[1 + \Omega_1^2 A_1 + \Omega_2^2 B_1 - 2\Omega_1 C_1 - 2\Omega_2 D_1 + 2\Omega_1\Omega_2 E_1\right] \tag{31}$$

Table I. Some members of the proposed and existing estimators for different choices of a and b

a	b	$\hat{\hat{T}}_{Singh}$	$\hat{\hat{T}}_{Grover}$	$\hat{\hat{T}}_{Proposed}$
1	C_x	$\hat{\hat{T}}_{Singh1}$	$\hat{\hat{T}}_{Grover1}$	$\hat{\hat{T}}_{Proposed1}$
1	β_x	$\hat{\hat{T}}_{Singh2}$	$\hat{\hat{T}}_{Grover2}$	$\hat{\hat{T}}_{Proposed2}$
β_x	C_x	$\hat{\hat{T}}_{Singh3}$	$\hat{\hat{T}}_{Grover3}$	$\hat{\hat{T}}_{Proposed3}$
C_x	β_x	$\hat{\hat{T}}_{Singh4}$	$\hat{\hat{T}}_{Grover4}$	$\hat{\hat{T}}_{Proposed4}$
1	ρ_{yx}	$\hat{\hat{T}}_{Singh5}$	$\hat{\hat{T}}_{Grover5}$	$\hat{\hat{T}}_{Proposed5}$
C_x	ρ_{yx}	$\hat{\hat{T}}_{Singh6}$	$\hat{\hat{T}}_{Grover6}$	$\hat{\hat{T}}_{Proposed6}$
ρ_{yx}	C_x	$\hat{\hat{T}}_{Singh7}$	$\hat{\hat{T}}_{Grover7}$	$\hat{\hat{T}}_{Proposed7}$
β_x	ρ_{yx}	$\hat{\hat{T}}_{Singh8}$	$\hat{\hat{T}}_{Grover8}$	$\hat{\hat{T}}_{Proposed8}$
ρ_{yx}	β_x	$\hat{\hat{T}}_{Singh9}$	$\hat{\hat{T}}_{Grover9}$	$\hat{\hat{T}}_{Proposed9}$
1	$N\bar{X}$	$\hat{\hat{T}}_{Singh10}$	$\hat{\hat{T}}_{Grover10}$	$\hat{\hat{T}}_{Proposed10}$

where

$$A_1 = 1 + \left[V_{20} + (\frac{5}{4} + \theta^2)\psi_{02} - 2\theta\psi_{11}\right]$$

$B_1 = R^2\psi_{02}$, where $R = \bar{Y}/\bar{X}$,
$C_1 = 1 + (\frac{5+3\theta^2}{8})\psi_{02} - \frac{1}{2}\theta\psi_{11}$,
$D_1 = R\left(\frac{\theta\psi_{02}}{2}\right), E_1 = R(\theta\psi_{02} - \psi_{11})$
Differentiate (31) w.r.t Ω_1 and Ω_1, we got the minimum MSEs, which are given by:
$\Omega_{1(opt)} = \frac{(B_1C_1 - D_1E_1)}{(A_1B_1 - E_1^2)}, \Omega_{2(opt)} = \frac{(A_1D_1 - C_1E_1)}{(A_1B_1 - E_1^2)}$
Putting the values of $\Omega_{1(opt)}$ and $\Omega_{1(opt)}$ in (31), we get the minimum MSEs, which are given by:

$$MSE(\hat{\hat{T}}^*_{prop})_{min} = \bar{Y}^2\left[1 - \frac{(A_1D_1^2 + B_1C_1^2 - 2C_1D_1E_1)}{(A_1B_1 - E_1^2)}\right] \tag{32}$$

4.2. Situation –II: When non-response occurs in both the study and auxiliary variable.

$$\hat{\hat{T}}^{**}_{prop} = \left[\Omega_1\hat{\bar{y}}^*\frac{1}{4}\left(\frac{\bar{X}}{\hat{\bar{x}}^*} + \frac{\hat{\bar{x}}^*}{\bar{X}}\right)exp\left(\frac{\bar{X} - \hat{\bar{x}}^*}{\bar{X} + \hat{\bar{x}}^*}\right) + exp\left(\frac{\hat{\bar{x}}^* - \bar{X}}{\hat{\bar{x}}^* + \bar{X}}\right) + \Omega_2(\bar{X} - \hat{\bar{x}}^*)\right]exp\left[\frac{a(\bar{X} - \hat{\bar{x}}^*)}{(a(\bar{X} - \hat{\bar{x}}^*) + 2b)}\right] \tag{33}$$

Where Ω_1 and Ω_2 are constants.

$$\hat{\hat{T}}^{**}_{prop} = [\Omega_1\bar{Y}(1 + e^*_\theta)(1 + \frac{5}{8}e_1^{*2}) - \Omega_2\bar{X}e_1^*][1 - \frac{1}{2}\theta e_1^* + \frac{3}{8}\theta^2 e_1^{*2}] \tag{34}$$

By expressing (34), we have

$$\hat{\hat{T}}^{**}_{prop} - \bar{Y} = \bar{Y} + \bar{Y}\left[(\Omega_1 - 1) + \Omega_1\left\{e_0^* - \frac{1}{2}\theta e_1^* - 1/2\theta e_0^* e_1^* + \frac{1}{8}(5 + 3\theta^2)e_1^{*2} - \Omega_2 R(e_1^* - \frac{1}{2}\theta e_1^{*2})\right\}\right] \tag{35}$$

From (35), the bias of $\hat{\hat{T}}_p rop^{**}$ is given by:

$$Bias(\hat{\hat{T}}^{**}_{prop}) = \bar{Y}\left[(\Omega_1 - 1) + \Omega_1\left\{\frac{1}{8}(5+3\theta^2)V_{02} - \frac{1}{2}\theta V_{11} + 1/2\Omega_2 RV_{02}\right\}\right]$$

Squaring and taking expectation of (35), we have:

$$MSE(\hat{\hat{T}}^{**}_{prop}) = \bar{Y}^2[1 + \Omega_1^2 A_1 + \Omega_2^2 B_1 - 2\Omega_1 C_1 - 2\Omega_2 D_1 + 2\Omega_1\Omega_2 E_1] \tag{36}$$

where

$A_1 = 1 + \left[V_{20} + (\frac{5}{4} + \theta^2)V_{02} - 2\theta V_{11}\right]$

$B_1 = R^2 V_{02}$, where $R = \frac{\bar{Y}}{\bar{X}}$

$C_1 = 1 + (\frac{5+3\theta^2}{8})V_{02} - \frac{1}{2}\theta V_{11}$

$D_1 = R\left(\frac{\theta V_{02}}{2}\right), E_1 = R(\theta V_{02} - V_{11})$.

Differentiate (36) w.r.t Ω_1 and Ω_1, we got the minimum MSEs, which are given by:

$W_{1(opt)} = \frac{(B_1 C_1 - D_1 E_1)}{(A_1 B_1 - E_1^2)}, W_{2(opt)} = \frac{(A_1 D_1 - C_1 E_1)}{(A_1 B_1 - E_1^2)}$

Putting the values of $\Omega_{1(opt)}$ and $\Omega_{1(opt)}$ in (36), we get the minimum MSEs, which are given by:

$$MSE(\hat{\hat{T}}^{**}_{prop})_{min} = \bar{Y}^2\left[1 - \frac{(A_1 D_1^2 + B_1 C_1^2 - 2C_1 D_1 E_1)}{(A_1 B_1 - E_1^2)}\right] \tag{37}$$

5. NUMERICAL STUDY

This section presents the mathematical results of all considered estimators. We used four different datasets. To assess the efficiency of estimators, we used mean squared error and percentage relative efficiency. Here is a numerical representation of the PRE:

$PRE(\hat{\hat{T}}_i, \hat{\hat{T}}_U) = \frac{Var(\hat{\hat{T}}_U)}{MSE(\hat{\hat{T}}_i)}100$

Population-I:

Y : The egg assembles in 1990, X : value per dozen in 1991

$N = 50, n = 15, \lambda = 0.046670, \bar{Y} = 1357.622, \bar{X} = 75.8720, \rho_{yx} = -0.3022287, s_y = 1661.242$

$s_X = 22.18052, N_2 = 12, w_2 = 0.240000, \lambda_2 = 0.016000, S_{y(2)} = 940.7629, S_{x(2)} = 19.53920.$

Population-II:

Y : Duration of sleep for individuals aged 50 and up, X : Number of years that a person has lived.

$N = 30, n = 8, \lambda = 0.09166667, \bar{Y} = 384.2, \bar{X} = 67.26667, \rho_{yx} = -0.8552412, s_y = 59.85465,$

$s_X = 9.232377, N_2 = 8, w_2 = 0.2400, \lambda_2 = 0.03, S_{y(2)} = 40.43845, S_{x(2)} = 9.387492.$

Table II. MSE of estimators using Population-I, Situation -I

Estimators	MSEs	Estimators	MSEs	Estimators	MSEs	Estimators	MSEs
$\hat{\hat{T}}_1$	142947.8	$\hat{\hat{T}}_{Singh1}$	154482.5	$\hat{\hat{T}}_{\text{Grover1}}$	122552.2	$\hat{\hat{T}}_{\text{Proposed1}}$	121918.6
$\hat{\hat{T}}_2$	170101.9	$\hat{\hat{T}}_{Singh2}$	153870.3	$\hat{\hat{T}}_{\text{Grover2}}$	122564.0	$\hat{\hat{T}}_{\text{Proposed2}}$	121930.8
$\hat{\hat{T}}_3$	131756.0	$\hat{\hat{T}}_{Singh3}$	154517.8	$\hat{\hat{T}}_{\text{Grover3}}$	122551.5	$\hat{\hat{T}}_{\text{Proposed3}}$	121917.9
$\hat{\hat{T}}_4$	131433.2	$\hat{\hat{T}}_{Singh4}$	152423.3	$\hat{\hat{T}}_{\text{Grover4}}$	122590.4	$\hat{\hat{T}}_{\text{Proposed4}}$	121958.0
$\hat{\hat{T}}_5$	122684.6	$\hat{\hat{T}}_{Singh5}$	154579.8	$\hat{\hat{T}}_{\text{Grover5}}$	122550.3	$\hat{\hat{T}}_{\text{Proposed5}}$	121916.6
		$\hat{\hat{T}}_{Singh6}$	154716.4	$\hat{\hat{T}}_{\text{Grover6}}$	122547.6	$\hat{\hat{T}}_{\text{Proposed6}}$	121913.8
		$\hat{\hat{T}}_{Singh7}$	154695.3	$\hat{\hat{T}}_{\text{Grover7}}$	122548.0	$\hat{\hat{T}}_{\text{Proposed7}}$	121914.2
		$\hat{\hat{T}}_{Singh8}$	154542.0	$\hat{\hat{T}}_{\text{Grover8}}$	122551.0	$\hat{\hat{T}}_{\text{Proposed8}}$	121917.4
		$\hat{\hat{T}}_{Singh9}$	157563.5	$\hat{\hat{T}}_{\text{Grover9}}$	122486.9	$\hat{\hat{T}}_{\text{Proposed9}}$	121851.2
		$\hat{\hat{T}}_{Singh10}$	143136.5	$\hat{\hat{T}}_{\text{Grover10}}$	122684.5	$\hat{\hat{T}}_{\text{Proposed10}}$	122055.2

Table III. PREs of different estimators

Estimators	PREs	Estimators	PREs	Estimators	PREs	Estimators	PREs
$\hat{\hat{T}}_1$	100.00000	$\hat{\hat{T}}_{\text{Singh1}}$	92.53331	$\hat{\hat{T}}_{\text{Grover1}}$	116.64236	$\hat{\hat{T}}_{\text{Proposed1}}$	117.24854
$\hat{\hat{T}}_2$	84.03655	$\hat{\hat{T}}_{\text{Singh2}}$	92.90144	$\hat{\hat{T}}_{\text{Grover2}}$	116.63109	$\hat{\hat{T}}_{\text{Proposed2}}$	117.23679
$\hat{\hat{T}}_3$	108.49430	$\hat{\hat{T}}_{\text{Singh3}}$	92.51216	$\hat{\hat{T}}_{\text{Grover3}}$	116.64302	$\hat{\hat{T}}_{\text{Proposed3}}$	117.24923
$\hat{\hat{T}}_4$	108.76080	$\hat{\hat{T}}_{\text{Singh4}}$	93.78342	$\hat{\hat{T}}_{\text{Grover4}}$	116.60604	$\hat{\hat{T}}_{\text{Proposed4}}$	117.21065
$\hat{\hat{T}}_5$	116.51650	$\hat{\hat{T}}_{\text{Singh5}}$	92.47504	$\hat{\hat{T}}_{\text{Grover5}}$	116.64419	$\hat{\hat{T}}_{\text{Proposed5}}$	117.25044
		$\hat{\hat{T}}_{\text{Singh6}}$	92.39342	$\hat{\hat{T}}_{\text{Grover6}}$	116.64676	$\hat{\hat{T}}_{\text{Proposed6}}$	117.25313
		$\hat{\hat{T}}_{\text{Singh7}}$	92.40601	$\hat{\hat{T}}_{\text{Grover7}}$	116.64636	$\hat{\hat{T}}_{\text{Proposed7}}$	117.25272
		$\hat{\hat{T}}_{\text{Singh8}}$	92.49769	$\hat{\hat{T}}_{\text{Grover8}}$	116.64347	$\hat{\hat{T}}_{\text{Proposed8}}$	117.24970
		$\hat{\hat{T}}_{\text{Singh9}}$	90.72388	$\hat{\hat{T}}_{\text{Grover9}}$	116.70453	$\hat{\hat{T}}_{\text{Proposed9}}$	117.31339
		$\hat{\hat{T}}_{\text{Singh10}}$	99.86814	$\hat{\hat{T}}_{\text{Grover10}}$	116.51654	$\hat{\hat{T}}_{\text{Proposed10}}$	117.11727

Table IV. Mean Squared Errors (MSEs) of different estimators

Estimators	MSEs	Estimators	MSEs	Estimators	MSEs	Estimators	MSEs
$\hat{\hat{T}}_1$	142947.8	$\hat{\hat{T}}_{\text{Singh1}}$	152779.7	$\hat{\hat{T}}_{\text{Grover1}}$	123249.5	$\hat{\hat{T}}_{\text{Proposed1}}$	122742.9
$\hat{\hat{T}}_2$	165879.7	$\hat{\hat{T}}_{\text{Singh2}}$	152262.5	$\hat{\hat{T}}_{\text{Grover2}}$	123259.0	$\hat{\hat{T}}_{\text{Proposed2}}$	122752.7
$\hat{\hat{T}}_3$	132770.0	$\hat{\hat{T}}_{\text{Singh3}}$	152809.6	$\hat{\hat{T}}_{\text{Grover3}}$	123248.9	$\hat{\hat{T}}_{\text{Proposed3}}$	122742.4
$\hat{\hat{T}}_4$	132203.8	$\hat{\hat{T}}_{\text{Singh4}}$	151038.4	$\hat{\hat{T}}_{\text{Grover4}}$	123280.1	$\hat{\hat{T}}_{\text{Proposed4}}$	122774.3
$\hat{\hat{T}}_5$	123355.8	$\hat{\hat{T}}_{\text{Singh5}}$	152861.9	$\hat{\hat{T}}_{\text{Grover5}}$	123247.9	$\hat{\hat{T}}_{\text{Proposed5}}$	122741.3
		$\hat{\hat{T}}_{\text{Singh6}}$	152977.2	$\hat{\hat{T}}_{\text{Grover6}}$	123245.8	$\hat{\hat{T}}_{\text{Proposed6}}$	122739.1
		$\hat{\hat{T}}_{\text{Singh7}}$	152959.4	$\hat{\hat{T}}_{\text{Grover7}}$	123246.1	$\hat{\hat{T}}_{\text{Proposed7}}$	122739.5
		$\hat{\hat{T}}_{\text{Singh8}}$	152830.0	$\hat{\hat{T}}_{\text{Grover8}}$	123248.5	$\hat{\hat{T}}_{\text{Proposed8}}$	122742.0
		$\hat{\hat{T}}_{\text{Singh9}}$	155377.4	$\hat{\hat{T}}_{\text{Grover9}}$	123197.1	$\hat{\hat{T}}_{\text{Proposed9}}$	122689.2
		$\hat{\hat{T}}_{\text{Singh10}}$	143110.7	$\hat{\hat{T}}_{\text{Grover10}}$	123355.7	$\hat{\hat{T}}_{\text{Proposed10}}$	122851.9

6. SIMULATION ANALYSIS

Simulation study is used to study the efficiency of the proposed classes of estimators. For this study, we take two populations of different sizes for both situations i.e, in situation-I 10 % non-response occurs only in study variable whereas in situation-II, 10% non-response occurs on both study and auxiliary variables. The size of population-I and population-II is 1000 and 2000 respectively. The sample size for population-I and population-II is 200 and 400 respectively. The characteristics of both populations are given below.

Table V. PRE of estimators using Population-II, Situation -II

Estimators	PREs	Estimators	PREs	Estimators	PREs	Estimators	PREs
$\hat{\hat{T}}_1$	100.00000	$\hat{\hat{T}}_{\text{Singh1}}$	93.56461	$\hat{\hat{T}}_{\text{Grover1}}$	115.98245	$\hat{\hat{T}}_{\text{Proposed1}}$	116.46110
$\hat{\hat{T}}_2$	86.17567	$\hat{\hat{T}}_{\text{Singh2}}$	93.88242	$\hat{\hat{T}}_{\text{Grover2}}$	115.97351	$\hat{\hat{T}}_{\text{Proposed2}}$	116.45186
$\hat{\hat{T}}_3$	107.6657	$\hat{\hat{T}}_{\text{Singh3}}$	93.54634	$\hat{\hat{T}}_{\text{Grover3}}$	115.98297	$\hat{\hat{T}}_{\text{Proposed3}}$	116.46165
$\hat{\hat{T}}_4$	108.127	$\hat{\hat{T}}_{\text{Singh4}}$	94.64333	$\hat{\hat{T}}_{\text{Grover4}}$	115.95364	$\hat{\hat{T}}_{\text{Proposed4}}$	116.43131
$\hat{\hat{T}}_5$	115.8825	$\hat{\hat{T}}_{\text{Singh5}}$	93.51430	$\hat{\hat{T}}_{\text{Grover5}}$	115.98390	$\hat{\hat{T}}_{\text{Proposed5}}$	116.46260
		$\hat{\hat{T}}_{\text{Singh6}}$	93.44381	$\hat{\hat{T}}_{\text{Grover6}}$	115.98594	$\hat{\hat{T}}_{\text{Proposed6}}$	116.46472
		$\hat{\hat{T}}_{\text{Singh7}}$	93.45469	$\hat{\hat{T}}_{\text{Grover7}}$	115.98562	$\hat{\hat{T}}_{\text{Proposed7}}$	116.46439
		$\hat{\hat{T}}_{\text{Singh8}}$	93.53386	$\hat{\hat{T}}_{\text{Grover8}}$	115.98333	$\hat{\hat{T}}_{\text{Proposed8}}$	116.46202
		$\hat{\hat{T}}_{\text{Singh9}}$	92.00034	$\hat{\hat{T}}_{\text{Grover9}}$	116.03173	$\hat{\hat{T}}_{\text{Proposed9}}$	116.51208
		$\hat{\hat{T}}_{\text{Singh10}}$	99.88616	$\hat{\hat{T}}_{\text{Grover10}}$	115.88256	$\hat{\hat{T}}_{\text{Proposed10}}$	116.35778

Table VI. MSE of estimators using Population-II, Situation -I

Estimators	MSEs	Estimators	MSEs	Estimators	MSEs	Estimators	MSEs
$\hat{\hat{T}}_1$	377.46110	$\hat{\hat{T}}_{\text{Singh1}}$	769.47341	$\hat{\hat{T}}_{\text{Grover1}}$	99.77899	$\hat{\hat{T}}_{\text{Proposed1}}$	99.06265
$\hat{\hat{T}}_2$	1334.0010	$\hat{\hat{T}}_{\text{Singh2}}$	755.12927	$\hat{\hat{T}}_{\text{Grover2}}$	99.78382	$\hat{\hat{T}}_{\text{Proposed2}}$	99.07475
$\hat{\hat{T}}_3$	103.19300	$\hat{\hat{T}}_{\text{Singh3}}$	770.01154	$\hat{\hat{T}}_{\text{Grover3}}$	99.77880	$\hat{\hat{T}}_{\text{Proposed3}}$	99.06219
$\hat{\hat{T}}_4$	99.91624	$\hat{\hat{T}}_{\text{Singh4}}$	680.34860	$\hat{\hat{T}}_{\text{Grover4}}$	99.80612	$\hat{\hat{T}}_{\text{Proposed4}}$	99.13244
$\hat{\hat{T}}_5$	99.84865	$\hat{\hat{T}}_{\text{Singh5}}$	776.62019	$\hat{\hat{T}}_{\text{Grover5}}$	99.77651	$\hat{\hat{T}}_{\text{Proposed5}}$	99.05650
		$\hat{\hat{T}}_{\text{Singh6}}$	820.16350	$\hat{\hat{T}}_{\text{Grover6}}$	99.76042	$\hat{\hat{T}}_{\text{Proposed6}}$	99.01729
		$\hat{\hat{T}}_{\text{Singh7}}$	771.59118	$\hat{\hat{T}}_{\text{Grover7}}$	99.77826	$\hat{\hat{T}}_{\text{Proposed7}}$	99.06083
		$\hat{\hat{T}}_{\text{Singh8}}$	773.18116	$\hat{\hat{T}}_{\text{Grover8}}$	99.77771	$\hat{\hat{T}}_{\text{Proposed8}}$	99.05947
		$\hat{\hat{T}}_{\text{Singh9}}$	789.95390	$\hat{\hat{T}}_{\text{Grover9}}$	99.77177	$\hat{\hat{T}}_{\text{Proposed9}}$	99.04481
		$\hat{\hat{T}}_{\text{Singh10}}$	387.47575	$\hat{\hat{T}}_{\text{Grover10}}$	99.84859	$\hat{\hat{T}}_{\text{Proposed10}}$	99.25417

Table VII. Percent Relative Efficiencies (PREs) of different estimators

Estimators	PREs	Estimators	PREs	Estimators	PREs	Estimators	PREs
$\hat{\hat{T}}_1$	100.00000	$\hat{\hat{T}}_{\text{Singh1}}$	49.05447	$\hat{\hat{T}}_{\text{Grover1}}$	378.29722	$\hat{\hat{T}}_{\text{Proposed1}}$	381.03275
$\hat{\hat{T}}_2$	28.29543	$\hat{\hat{T}}_{\text{Singh2}}$	49.98629	$\hat{\hat{T}}_{\text{Grover2}}$	378.27890	$\hat{\hat{T}}_{\text{Proposed2}}$	380.98622
$\hat{\hat{T}}_3$	365.78180	$\hat{\hat{T}}_{\text{Singh3}}$	49.02019	$\hat{\hat{T}}_{\text{Grover3}}$	378.29793	$\hat{\hat{T}}_{\text{Proposed3}}$	381.03453
$\hat{\hat{T}}_4$	377.77760	$\hat{\hat{T}}_{\text{Singh4}}$	55.48055	$\hat{\hat{T}}_{\text{Grover4}}$	378.19439	$\hat{\hat{T}}_{\text{Proposed4}}$	380.76449
$\hat{\hat{T}}_5$	378.03330	$\hat{\hat{T}}_{\text{Singh5}}$	48.60305	$\hat{\hat{T}}_{\text{Grover5}}$	378.30661	$\hat{\hat{T}}_{\text{Proposed5}}$	381.05641
		$\hat{\hat{T}}_{\text{Singh6}}$	46.02267	$\hat{\hat{T}}_{\text{Grover6}}$	378.36763	$\hat{\hat{T}}_{\text{Proposed6}}$	381.20729
		$\hat{\hat{T}}_{\text{Singh7}}$	48.91984	$\hat{\hat{T}}_{\text{Grover7}}$	378.29999	$\hat{\hat{T}}_{\text{Proposed7}}$	381.03973
		$\hat{\hat{T}}_{\text{Singh8}}$	48.81924	$\hat{\hat{T}}_{\text{Grover8}}$	378.30207	$\hat{\hat{T}}_{\text{Proposed8}}$	381.04499
		$\hat{\hat{T}}_{\text{Singh9}}$	47.78268	$\hat{\hat{T}}_{\text{Grover9}}$	378.32460	$\hat{\hat{T}}_{\text{Proposed9}}$	381.10139
		$\hat{\hat{T}}_{\text{Singh10}}$	97.41542	$\hat{\hat{T}}_{\text{Grover10}}$	378.03351	$\hat{\hat{T}}_{\text{Proposed10}}$	380.29750

Population I.

$X = N(5,10), Y = X + N(0,1), N = 1000, n = 200$

Population II.

$X = N(3,7), Y = X + N(0,1), N = 2000, n = 200$

The populations are used in simulation study algorithm to investigate the performance of the proposed estimator over the existing estimators.

Table VIII. MSE of estimators using Population-II, Situation -II

Estimators	MSEs	Estimators	MSEs	Estimators	MSEs	Estimators	MSEs
$\hat{\hat{T}}_1$	377.4611	$\hat{\hat{T}}_{\text{Singh1}}$	687.8601	$\hat{\hat{T}}_{\text{Grover1}}$	137.0615	$\hat{\hat{T}}_{\text{Proposed1}}$	136.5269
$\hat{\hat{T}}_2$	1127.2300	$\hat{\hat{T}}_{\text{Singh2}}$	676.6134	$\hat{\hat{T}}_{\text{Grover2}}$	137.0658	$\hat{\hat{T}}_{\text{Proposed2}}$	136.5352
$\hat{\hat{T}}_3$	137.4726	$\hat{\hat{T}}_{\text{Singh3}}$	688.2819	$\hat{\hat{T}}_{\text{Grover3}}$	137.0613	$\hat{\hat{T}}_{\text{Proposed3}}$	136.5266
$\hat{\hat{T}}_4$	137.2548	$\hat{\hat{T}}_{\text{Singh4}}$	617.8819	$\hat{\hat{T}}_{\text{Grover4}}$	137.0860	$\hat{\hat{T}}_{\text{Proposed4}}$	136.5752
$\hat{\hat{T}}_5$	137.1273	$\hat{\hat{T}}_{\text{Singh5}}$	693.4615	$\hat{\hat{T}}_{\text{Grover5}}$	137.0593	$\hat{\hat{T}}_{\text{Proposed5}}$	136.5226
		$\hat{\hat{T}}_{\text{Singh6}}$	727.5602	$\hat{\hat{T}}_{\text{Grover6}}$	137.0452	$\hat{\hat{T}}_{\text{Proposed6}}$	136.4957
		$\hat{\hat{T}}_{\text{Singh7}}$	689.5201	$\hat{\hat{T}}_{\text{Grover7}}$	137.0609	$\hat{\hat{T}}_{\text{Proposed7}}$	136.5256
		$\hat{\hat{T}}_{\text{Singh8}}$	690.7663	$\hat{\hat{T}}_{\text{Grover8}}$	137.0604	$\hat{\hat{T}}_{\text{Proposed8}}$	136.5247
		$\hat{\hat{T}}_{\text{Singh9}}$	703.9083	$\hat{\hat{T}}_{\text{Grover9}}$	137.0551	$\hat{\hat{T}}_{\text{Proposed9}}$	136.5146
		$\hat{\hat{T}}_{\text{Singh10}}$	385.5094	$\hat{\hat{T}}_{\text{Grover10}}$	137.1272	$\hat{\hat{T}}_{\text{Proposed10}}$	136.6607

Table IX. PRE of estimators using Population-II, Situation -II

Estimators	PREs	Estimators	PREs	Estimators	PREs	Estimators	PREs
$\hat{\hat{T}}_1$	100.0000	$\hat{\hat{T}}_{\text{Singh1}}$	54.8747	$\hat{\hat{T}}_{\text{Grover1}}$	275.3955	$\hat{\hat{T}}_{\text{Proposed1}}$	276.4739
$\hat{\hat{T}}_2$	33.4857	$\hat{\hat{T}}_{\text{Singh2}}$	55.7868	$\hat{\hat{T}}_{\text{Grover2}}$	275.3868	$\hat{\hat{T}}_{\text{Proposed2}}$	276.4569
$\hat{\hat{T}}_3$	274.5719	$\hat{\hat{T}}_{\text{Singh3}}$	54.8411	$\hat{\hat{T}}_{\text{Grover3}}$	275.3958	$\hat{\hat{T}}_{\text{Proposed3}}$	276.4745
$\hat{\hat{T}}_4$	275.0077	$\hat{\hat{T}}_{\text{Singh4}}$	61.0895	$\hat{\hat{T}}_{\text{Grover4}}$	275.3462	$\hat{\hat{T}}_{\text{Proposed4}}$	276.3760
$\hat{\hat{T}}_5$	275.2634	$\hat{\hat{T}}_{\text{Singh5}}$	54.4315	$\hat{\hat{T}}_{\text{Grover5}}$	275.3999	$\hat{\hat{T}}_{\text{Proposed5}}$	276.4825
		$\hat{\hat{T}}_{\text{Singh6}}$	51.8804	$\hat{\hat{T}}_{\text{Grover6}}$	275.4281	$\hat{\hat{T}}_{\text{Proposed6}}$	276.5371
		$\hat{\hat{T}}_{\text{Singh7}}$	54.7426	$\hat{\hat{T}}_{\text{Grover7}}$	275.3968	$\hat{\hat{T}}_{\text{Proposed7}}$	276.4764
		$\hat{\hat{T}}_{\text{Singh8}}$	54.6438	$\hat{\hat{T}}_{\text{Grover8}}$	275.3977	$\hat{\hat{T}}_{\text{Proposed8}}$	276.4783
		$\hat{\hat{T}}_{\text{Singh9}}$	53.6236	$\hat{\hat{T}}_{\text{Grover9}}$	275.4083	$\hat{\hat{T}}_{\text{Proposed9}}$	276.4988
		$\hat{\hat{T}}_{\text{Singh10}}$	97.9123	$\hat{\hat{T}}_{\text{Grover10}}$	275.2635	$\hat{\hat{T}}_{\text{Proposed10}}$	276.2032

Table X. MSE of estimators using simulation Population-I, Situation -I

Estimators	MSEs	Estimators	MSEs	Estimators	MSEs	Estimators	MSEs
$\hat{\hat{T}}_1$	0.48120136	$\hat{\hat{T}}_{\text{Singh1}}$	0.23602164	$\hat{\hat{T}}_{\text{Grover1}}$	0.10019671	$\hat{\hat{T}}_{\text{Proposed1}}$	0.09411668
$\hat{\hat{T}}_2$	0.10626294	$\hat{\hat{T}}_{\text{Singh2}}$	0.26007159	$\hat{\hat{T}}_{\text{Grover2}}$	0.10026662	$\hat{\hat{T}}_{\text{Proposed2}}$	0.09434396
$\hat{\hat{T}}_3$	1.80928641	$\hat{\hat{T}}_{\text{Singh3}}$	0.19731322	$\hat{\hat{T}}_{\text{Grover3}}$	0.10003177	$\hat{\hat{T}}_{\text{Proposed3}}$	0.09361148
$\hat{\hat{T}}_4$	0.10084423	$\hat{\hat{T}}_{\text{Singh4}}$	0.22327985	$\hat{\hat{T}}_{\text{Grover4}}$	0.10015095	$\hat{\hat{T}}_{\text{Proposed4}}$	0.09397251
$\hat{\hat{T}}_5$	0.10046551	$\hat{\hat{T}}_{\text{Singh5}}$	0.20797620	$\hat{\hat{T}}_{\text{Grover5}}$	0.10008561	$\hat{\hat{T}}_{\text{Proposed5}}$	0.09377218
		$\hat{\hat{T}}_{\text{Singh6}}$	0.19202703	$\hat{\hat{T}}_{\text{Grover6}}$	0.10000203	$\hat{\hat{T}}_{\text{Proposed6}}$	0.09352424
		$\hat{\hat{T}}_{\text{Singh7}}$	0.23625120	$\hat{\hat{T}}_{\text{Grover7}}$	0.10019748	$\hat{\hat{T}}_{\text{Proposed7}}$	0.09411912
		$\hat{\hat{T}}_{\text{Singh8}}$	0.18623461	$\hat{\hat{T}}_{\text{Grover8}}$	0.09996677	$\hat{\hat{T}}_{\text{Proposed8}}$	0.09342212
		$\hat{\hat{T}}_{\text{Singh9}}$	0.26036044	$\hat{\hat{T}}_{\text{Grover9}}$	0.10026735	$\hat{\hat{T}}_{\text{Proposed9}}$	0.09434638
		$\hat{\hat{T}}_{\text{Singh10}}$	0.48077615	$\hat{\hat{T}}_{\text{Grover10}}$	0.10046551	$\hat{\hat{T}}_{\text{Proposed10}}$	0.09505282

7. DISCUSSION

As was noted earlier, the efficiency of the suggested class of estimators was evaluated by employing two real datasets in addition to a simulation study for both Situation I and Situation II. Additionally, we take into consideration the different population sample sizes. Our comparison of the suggested generalized class of estimators to the estimators that are now accessible was conducted with regard to the mean square error and the percentage of relative efficiency. The many alternatives for a and b are listed in Table 1, along with some of the estimators that belong to the proposed

Table XI. PRE of estimators using Simulation Population-I Situation -I

Estimators	PREs	Estimators	PREs	Estimators	PREs	Estimators	PREs
$\hat{\hat{T}}_1$	100.0000	$\hat{\hat{T}}_{\text{Singh1}}$	203.8803	$\hat{\hat{T}}_{\text{Grover1}}$	480.2566	$\hat{\hat{T}}_{\text{Proposed1}}$	511.2817
$\hat{\hat{T}}_2$	452.8403	$\hat{\hat{T}}_{\text{Singh2}}$	185.0265	$\hat{\hat{T}}_{\text{Grover2}}$	479.9218	$\hat{\hat{T}}_{\text{Proposed2}}$	510.0500
$\hat{\hat{T}}_3$	26.5962	$\hat{\hat{T}}_{\text{Singh3}}$	243.8769	$\hat{\hat{T}}_{\text{Grover3}}$	481.0485	$\hat{\hat{T}}_{\text{Proposed3}}$	514.0410
$\hat{\hat{T}}_4$	477.1729	$\hat{\hat{T}}_{\text{Singh4}}$	215.5149	$\hat{\hat{T}}_{\text{Grover4}}$	480.4761	$\hat{\hat{T}}_{\text{Proposed4}}$	512.0661
$\hat{\hat{T}}_5$	478.9717	$\hat{\hat{T}}_{\text{Singh5}}$	231.3733	$\hat{\hat{T}}_{\text{Grover5}}$	480.7898	$\hat{\hat{T}}_{\text{Proposed5}}$	513.1600
		$\hat{\hat{T}}_{\text{Singh6}}$	250.5904	$\hat{\hat{T}}_{\text{Grover6}}$	481.1916	$\hat{\hat{T}}_{\text{Proposed6}}$	514.5205
		$\hat{\hat{T}}_{\text{Singh7}}$	203.6821	$\hat{\hat{T}}_{\text{Grover7}}$	480.2530	$\hat{\hat{T}}_{\text{Proposed7}}$	511.2685
		$\hat{\hat{T}}_{\text{Singh8}}$	258.3845	$\hat{\hat{T}}_{\text{Grover8}}$	481.3613	$\hat{\hat{T}}_{\text{Proposed8}}$	515.0829
		$\hat{\hat{T}}_{\text{Singh9}}$	184.8212	$\hat{\hat{T}}_{\text{Grover9}}$	479.9183	$\hat{\hat{T}}_{\text{Proposed9}}$	510.0369
		$\hat{\hat{T}}_{\text{Singh10}}$	100.0884	$\hat{\hat{T}}_{\text{Grover10}}$	478.9717	$\hat{\hat{T}}_{\text{Proposed10}}$	506.2463

Table XII. MSE of estimators using Situation –II, Population-I

Estimators	MSEs	Estimators	MSEs	Estimators	MSEs	Estimators	MSEs
$\hat{\hat{T}}_1$	0.48120136	$\hat{\hat{T}}_{\text{Singh1}}$	0.22937696	$\hat{\hat{T}}_{\text{Grover1}}$	0.05495973	$\hat{\hat{T}}_{\text{Proposed1}}$	0.05076436
$\hat{\hat{T}}_2$	0.05521629	$\hat{\hat{T}}_{\text{Singh2}}$	0.25499530	$\hat{\hat{T}}_{\text{Grover2}}$	0.05499857	$\hat{\hat{T}}_{\text{Proposed2}}$	0.05092962
$\hat{\hat{T}}_3$	1.75823976	$\hat{\hat{T}}_{\text{Singh3}}$	0.18727475	$\hat{\hat{T}}_{\text{Grover3}}$	0.05486476	$\hat{\hat{T}}_{\text{Proposed3}}$	0.05039602
$\hat{\hat{T}}_4$	0.05521617	$\hat{\hat{T}}_{\text{Singh4}}$	0.21565407	$\hat{\hat{T}}_{\text{Grover4}}$	0.05493381	$\hat{\hat{T}}_{\text{Proposed4}}$	0.05065937
$\hat{\hat{T}}_5$	0.05510243	$\hat{\hat{T}}_{\text{Singh5}}$	0.19900404	$\hat{\hat{T}}_{\text{Grover5}}$	0.05489621	$\hat{\hat{T}}_{\text{Proposed5}}$	0.05051332
		$\hat{\hat{T}}_{\text{Singh6}}$	0.18141463	$\hat{\hat{T}}_{\text{Grover6}}$	0.05484723	$\hat{\hat{T}}_{\text{Proposed6}}$	0.05033228
		$\hat{\hat{T}}_{\text{Singh7}}$	0.22962233	$\hat{\hat{T}}_{\text{Grover7}}$	0.05496016	$\hat{\hat{T}}_{\text{Proposed7}}$	0.05076613
		$\hat{\hat{T}}_{\text{Singh8}}$	0.17495468	$\hat{\hat{T}}_{\text{Grover8}}$	0.05482629	$\hat{\hat{T}}_{\text{Proposed8}}$	0.05025762
		$\hat{\hat{T}}_{\text{Singh9}}$	0.25530101	$\hat{\hat{T}}_{\text{Grover9}}$	0.05499897	$\hat{\hat{T}}_{\text{Proposed9}}$	0.05093138
		$\hat{\hat{T}}_{\text{Singh10}}$	0.48077614	$\hat{\hat{T}}_{\text{Grover10}}$	0.05510243	$\hat{\hat{T}}_{\text{Proposed10}}$	0.05144312

Table XIII. PRE of estimators using Situation –II, Population-I

Estimators	PREs	Estimators	PREs	Estimators	PREs	Estimators	PREs
$\hat{\hat{T}}_1$	100.00000	$\hat{\hat{T}}_{\text{Singh1}}$	209.78626	$\hat{\hat{T}}_{\text{Grover1}}$	875.55263	$\hat{\hat{T}}_{\text{Proposed1}}$	947.91183
$\hat{\hat{T}}_2$	871.48444	$\hat{\hat{T}}_{\text{Singh2}}$	188.70989	$\hat{\hat{T}}_{\text{Grover2}}$	874.93431	$\hat{\hat{T}}_{\text{Proposed2}}$	944.83591
$\hat{\hat{T}}_3$	27.36836	$\hat{\hat{T}}_{\text{Singh3}}$	256.94942	$\hat{\hat{T}}_{\text{Grover3}}$	877.06817	$\hat{\hat{T}}_{\text{Proposed3}}$	954.84009
$\hat{\hat{T}}_4$	871.48639	$\hat{\hat{T}}_{\text{Singh4}}$	223.13577	$\hat{\hat{T}}_{\text{Grover4}}$	875.96578	$\hat{\hat{T}}_{\text{Proposed4}}$	949.87627
$\hat{\hat{T}}_5$	873.28516	$\hat{\hat{T}}_{\text{Singh5}}$	241.80483	$\hat{\hat{T}}_{\text{Grover5}}$	876.56576	$\hat{\hat{T}}_{\text{Proposed5}}$	952.62270
		$\hat{\hat{T}}_{\text{Singh6}}$	265.24948	$\hat{\hat{T}}_{\text{Grover6}}$	877.34857	$\hat{\hat{T}}_{\text{Proposed6}}$	956.04921
		$\hat{\hat{T}}_{\text{Singh7}}$	209.56124	$\hat{\hat{T}}_{\text{Grover7}}$	875.54580	$\hat{\hat{T}}_{\text{Proposed7}}$	947.87877
		$\hat{\hat{T}}_{\text{Singh8}}$	275.04343	$\hat{\hat{T}}_{\text{Grover8}}$	877.68354	$\hat{\hat{T}}_{\text{Proposed8}}$	957.46942
		$\hat{\hat{T}}_{\text{Singh9}}$	188.48932	$\hat{\hat{T}}_{\text{Grover9}}$	874.92794	$\hat{\hat{T}}_{\text{Proposed9}}$	944.80327
		$\hat{\hat{T}}_{\text{Singh10}}$	100.08845	$\hat{\hat{T}}_{\text{Grover10}}$	873.28517	$\hat{\hat{T}}_{\text{Proposed10}}$	935.40474

collection of estimators. In Table 2-9, the MSE and PRE values for the real data sets are presented for your scrutiny. In the same vein, Table 10-17 presents the mean squared error (MSE) and the relative error (PRE) of the proposed and existing estimators, which are derived from simulated data sets. As we noticed that all the numerical results which are available in Tables, $\hat{\hat{T}}_2$ provides less efficient results compared to all adopted estimators. For the comparison purpose we used classes of estimators in adopted estimators as well. In existing classes of estimators which are developed by [Singh et al. 2008] and [Grover and Kaur 2014] are included and

Table XIV. MSE of estimators using Population-II, Situation -I

Estimators	MSEs	Estimators	MSEs	Estimators	MSEs	Estimators	MSEs
$\hat{\hat{T}}_1$	0.23821998	$\hat{\hat{T}}_{\text{Singh1}}$	0.13759789	$\hat{\hat{T}}_{\text{Grover1}}$	0.05122479	$\hat{\hat{T}}_{\text{Proposed1}}$	0.04736908
$\hat{\hat{T}}_2$	0.05428679	$\hat{\hat{T}}_{\text{Singh2}}$	0.14728860	$\hat{\hat{T}}_{\text{Grover2}}$	0.05127373	$\hat{\hat{T}}_{\text{Proposed2}}$	0.04745507
$\hat{\hat{T}}_3$	0.88968767	$\hat{\hat{T}}_{\text{Singh3}}$	0.10819243	$\hat{\hat{T}}_{\text{Grover3}}$	0.05112396	$\hat{\hat{T}}_{\text{Proposed3}}$	0.04698180
$\hat{\hat{T}}_4$	0.05163091	$\hat{\hat{T}}_{\text{Singh4}}$	0.12009283	$\hat{\hat{T}}_{\text{Grover4}}$	0.05118449	$\hat{\hat{T}}_{\text{Proposed4}}$	0.04716632
$\hat{\hat{T}}_5$	0.05135881	$\hat{\hat{T}}_{\text{Singh5}}$	0.11339197	$\hat{\hat{T}}_{\text{Grover5}}$	0.05115258	$\hat{\hat{T}}_{\text{Proposed5}}$	0.04706805
		$\hat{\hat{T}}_{\text{Singh6}}$	0.09977071	$\hat{\hat{T}}_{\text{Grover6}}$	0.05106864	$\hat{\hat{T}}_{\text{Proposed6}}$	0.04681961
		$\hat{\hat{T}}_{\text{Singh7}}$	0.13791471	$\hat{\hat{T}}_{\text{Grover7}}$	0.05124882	$\hat{\hat{T}}_{\text{Proposed7}}$	0.04737215
		$\hat{\hat{T}}_{\text{Singh8}}$	0.09708082	$\hat{\hat{T}}_{\text{Grover8}}$	0.05104881	$\hat{\hat{T}}_{\text{Proposed8}}$	0.04676087
		$\hat{\hat{T}}_{\text{Singh9}}$	0.14763025	$\hat{\hat{T}}_{\text{Grover9}}$	0.05127454	$\hat{\hat{T}}_{\text{Proposed9}}$	0.04745781
		$\hat{\hat{T}}_{\text{Singh10}}$	0.23801139	$\hat{\hat{T}}_{\text{Grover10}}$	0.05135881	$\hat{\hat{T}}_{\text{Proposed10}}$	0.04775561

Table XV. PRE of estimators using Population-II, Situation -I

Estimators	PREs	Estimators	PREs	Estimators	PREs	Estimators	PREs
$\hat{\hat{T}}_1$	100.00000	$\hat{\hat{T}}_{\text{Singh1}}$	173.12763	$\hat{\hat{T}}_{\text{Grover1}}$	464.83864	$\hat{\hat{T}}_{\text{Proposed1}}$	502.90187
$\hat{\hat{T}}_2$	438.81757	$\hat{\hat{T}}_{\text{Singh2}}$	161.73658	$\hat{\hat{T}}_{\text{Grover2}}$	464.60437	$\hat{\hat{T}}_{\text{Proposed2}}$	501.99062
$\hat{\hat{T}}_3$	26.77569	$\hat{\hat{T}}_{\text{Singh3}}$	220.18174	$\hat{\hat{T}}_{\text{Grover3}}$	465.96546	$\hat{\hat{T}}_{\text{Proposed3}}$	507.04738
$\hat{\hat{T}}_4$	461.39026	$\hat{\hat{T}}_{\text{Singh4}}$	198.36319	$\hat{\hat{T}}_{\text{Grover4}}$	465.41436	$\hat{\hat{T}}_{\text{Proposed4}}$	505.06373
$\hat{\hat{T}}_5$	463.83466	$\hat{\hat{T}}_{\text{Singh5}}$	210.08541	$\hat{\hat{T}}_{\text{Grover5}}$	465.70474	$\hat{\hat{T}}_{\text{Proposed5}}$	506.11827
		$\hat{\hat{T}}_{\text{Singh6}}$	238.76746	$\hat{\hat{T}}_{\text{Grover6}}$	466.47017	$\hat{\hat{T}}_{\text{Proposed6}}$	508.80382
		$\hat{\hat{T}}_{\text{Singh7}}$	172.72992	$\hat{\hat{T}}_{\text{Grover7}}$	464.83017	$\hat{\hat{T}}_{\text{Proposed7}}$	502.86925
		$\hat{\hat{T}}_{\text{Singh8}}$	245.38314	$\hat{\hat{T}}_{\text{Grover8}}$	466.65751	$\hat{\hat{T}}_{\text{Proposed8}}$	509.44302
		$\hat{\hat{T}}_{\text{Singh9}}$	161.36258	$\hat{\hat{T}}_{\text{Grover9}}$	464.59699	$\hat{\hat{T}}_{\text{Proposed9}}$	501.96158
		$\hat{\hat{T}}_{\text{Singh10}}$	100.08764	$\hat{\hat{T}}_{\text{Grover10}}$	463.83467	$\hat{\hat{T}}_{\text{Proposed10}}$	498.83142

Table XVI. MSE of estimators using Population-II, Situation -II

Estimators	MSEs	Estimators	MSEs	Estimators	MSEs	Estimators	MSEs
$\hat{\hat{T}}_1$	0.23821998	$\hat{\hat{T}}_{\text{Singh1}}$	0.13553558	$\hat{\hat{T}}_{\text{Grover1}}$	0.02910097	$\hat{\hat{T}}_{\text{Proposed1}}$	0.02642512
$\hat{\hat{T}}_2$	0.02924755	$\hat{\hat{T}}_{\text{Singh2}}$	0.14567696	$\hat{\hat{T}}_{\text{Grover2}}$	0.02911514	$\hat{\hat{T}}_{\text{Proposed2}}$	0.02648877
$\hat{\hat{T}}_3$	0.86464843	$\hat{\hat{T}}_{\text{Singh3}}$	0.10415564	$\hat{\hat{T}}_{\text{Grover3}}$	0.02903031	$\hat{\hat{T}}_{\text{Proposed3}}$	0.02614235
$\hat{\hat{T}}_4$	0.02924749	$\hat{\hat{T}}_{\text{Singh4}}$	0.11698610	$\hat{\hat{T}}_{\text{Grover4}}$	0.02906532	$\hat{\hat{T}}_{\text{Proposed4}}$	0.02627722
$\hat{\hat{T}}_5$	0.02915996	$\hat{\hat{T}}_{\text{Singh5}}$	0.10978756	$\hat{\hat{T}}_{\text{Grover5}}$	0.02904697	$\hat{\hat{T}}_{\text{Proposed5}}$	0.02620542
		$\hat{\hat{T}}_{\text{Singh6}}$	0.09493158	$\hat{\hat{T}}_{\text{Grover6}}$	0.02997640	$\hat{\hat{T}}_{\text{Proposed6}}$	0.02602358
		$\hat{\hat{T}}_{\text{Singh7}}$	0.13586841	$\hat{\hat{T}}_{\text{Grover7}}$	0.02910148	$\hat{\hat{T}}_{\text{Proposed7}}$	0.02642736
		$\hat{\hat{T}}_{\text{Singh8}}$	0.09195443	$\hat{\hat{T}}_{\text{Grover8}}$	0.02898539	$\hat{\hat{T}}_{\text{Proposed8}}$	0.02598052
		$\hat{\hat{T}}_{\text{Singh9}}$	0.14603281	$\hat{\hat{T}}_{\text{Grover9}}$	0.02911558	$\hat{\hat{T}}_{\text{Proposed9}}$	0.02648973
		$\hat{\hat{T}}_{\text{Singh10}}$	0.23801139	$\hat{\hat{T}}_{\text{Grover10}}$	0.02915996	$\hat{\hat{T}}_{\text{Proposed10}}$	0.02670606

compared to proposed generalized class of estimators as well. The numerical outcomes presented in Tables 2–17 clearly show that the recommended estimators provide the minimum mean squared error and higher percentage relative efficiency, both on real-life data and in simulation analysis.

Table XVII. PRE of estimators using Population-II, Situation -II

Estimators	MSEs	Estimators	MSEs	Estimators	MSEs	Estimators	MSEs
$\hat{\hat{T}}_1$	0.23822	$\hat{\hat{T}}_{Singh1}$	0.13554	$\hat{\hat{T}}_{Grover1}$	0.02910	$\hat{\hat{T}}_{Proposed1}$	0.02643
$\hat{\hat{T}}_2$	0.02925	$\hat{\hat{T}}_{Singh2}$	0.14568	$\hat{\hat{T}}_{Grover2}$	0.02912	$\hat{\hat{T}}_{Proposed2}$	0.02649
$\hat{\hat{T}}_3$	0.86465	$\hat{\hat{T}}_{Singh3}$	0.10416	$\hat{\hat{T}}_{Grover3}$	0.02903	$\hat{\hat{T}}_{Proposed3}$	0.02614
$\hat{\hat{T}}_4$	0.02925	$\hat{\hat{T}}_{Singh4}$	0.11699	$\hat{\hat{T}}_{Grover4}$	0.02907	$\hat{\hat{T}}_{Proposed4}$	0.02628
$\hat{\hat{T}}_5$	0.02916	$\hat{\hat{T}}_{Singh5}$	0.10979	$\hat{\hat{T}}_{Grover5}$	0.02905	$\hat{\hat{T}}_{Proposed5}$	0.02621
		$\hat{\hat{T}}_{Singh6}$	0.09493	$\hat{\hat{T}}_{Grover6}$	0.02998	$\hat{\hat{T}}_{Proposed6}$	0.02602
		$\hat{\hat{T}}_{Singh7}$	0.13587	$\hat{\hat{T}}_{Grover7}$	0.02910	$\hat{\hat{T}}_{Proposed7}$	0.02643
		$\hat{\hat{T}}_{Singh8}$	0.09195	$\hat{\hat{T}}_{Grover8}$	0.02899	$\hat{\hat{T}}_{Proposed8}$	0.02598
		$\hat{\hat{T}}_{Singh9}$	0.14603	$\hat{\hat{T}}_{Grover9}$	0.02912	$\hat{\hat{T}}_{Proposed9}$	0.02649
		$\hat{\hat{T}}_{Singh10}$	0.23801	$\hat{\hat{T}}_{Grover10}$	0.02916	$\hat{\hat{T}}_{Proposed10}$	0.02671

8. CONCLUSION

In real-life scenarios, non-response causes practically all investigators to encounter difficulties. Non-responsiveness might in two forms: item non-response and unit non-response. Item non-response suggests that certain units' gathered data is lacking. Unit non-response is a problem resulting from some of the sampled units failing to supply their information, hence failing to measure certain of them. We have examined unit non-response in this paper. Usually, we find ourselves in this scenario with a survey when the respondents are human persons. Respondents may be busy, indifferent, uncooperative, uneducated, physically or mentally impaired, or terrified for various different reasons. This study presents an improved class of estimators for estimation of population mean under simple random sampling based on non-response scenario. By using different choices of a and b , we generate ten new estimators. We can find the bias and mean square error to the first order approximation. To check efficiency of estimators, we used real life data sets and a simulation study. To obtain minimum mean squared error, we compute the optimum values of unknown constant, and by putting these values in the mean squared equation we got the minimized mean squared error equation. By comparing the suggested class of estimators to the existing estimators, the empirical results show that the suggested class is more efficient with minimum MSE and higher PRE. One simple way to extend the current method and estimate the population mean is to include auxiliary variables that are based on measurement error, nonresponse distribution function, and population proportion. Consequently, out of all the significant estimators discussed in this work, we strongly recommend our suggested class of estimators for experimental surveys. The recommended work just addresses the homogeneous population. Other sampling strategies help to improve this work. The present work may be extended to produce an improved estimator based on non-response with measurement error, two-stage sampling, PPS sampling and estimation of finite proportion using auxiliary variables.

DECLARATIONS

Conflict of interest: The authors declare no conflict of interest.

Data availability: The datasets generated and/or analyzed during the current study are available in the current study are available from the corresponding author on reasonable request.

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