2-target pebbling number of graphs

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Abstract

Chung defined the pebbling move of a graph which involves choosing a vertex with at least two pebbles, discarding those two pebbles from that vertex and adding one pebble to a nearby vertex. The 2-target pebbling number of the vertices u, v in graph a G is the least number $\phi(G,u,v)$ has the characteristic that for every configuration of $\phi(G,u,v)$ pebbles on G, it is possible to move a pebble to u and v simultaneously by a sequence of pebbling moves. The 2-target pebbling number of graph G, denoted by $\phi(G)$, is the maximum $\phi(G,u,v)$ over all pairs of the vertices in G. In this paper, we discuss the 2-target pebbling number for some standard graphs.

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1. INTRODUCTION

Lagarias and Saks were the ones who first proposed the game of pebbling as a method for resolving a specific number theory issue. The idea was translated into the concept of pebbling number by F.R.K. Chung in 1989 [1]. A graph with one or more pebbles on each of its vertices is used in the game of 'graph pebbling'. Playing the game involves a sequence of pebbling moves. On a graph, a pebbling move entails picking a vertex with at least two pebbles, taking two pebbles out of it, and adding one to an adjacent vertex (the second pebble is taken out of play). A survey of findings in graph pebbling was done by Hurlbert in [2]. The cover pebbling number, introduced in [3]], is obtained by simultaneously placing a pebble on each vertex of the graph, rather than targeting a specific vertex. Research exploring the combination of graph pebbling with various variants, such as Monophonic Pebbling [6] [7], Detour Pebbling [8], and Detour Monophonic Vertex Cover Pebbling [9], has been done recently. The pebbling notion is extended in this study, where we discover the least $\phi(G)$ such that every placement of $\phi(G)$ pebbles on the graph G permits a pebble to be moved by a series of pebbling moves to any two targets in the graph. We calculate the 2-target pebbling number for a few common graphs.

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2. PRELIMINARIES

DEFINITION 2.1. [1] The pebbling number of a vertex v in a graph G is the least number $\pi(G,v)$ with the property that a pebble may be moved to v by a series of pebbling moves for any placement of $\pi(G,v)$ pebbles on G. The greatest $\pi(G,v)$ across all of the vertices in G is the pebbling number of the graph G, or $\pi(G)$.

DEFINITION 2.2. [4] A t-pebbling number, $\pi_t(G,v)$, of a vertex v of a graph G is the least number $\pi_t(G,v)$ such that it is possible to transfer t pebbles to v by a sequence of pebbling moves for any placement of $\pi_t(G,v)$ pebbles on the vertices of G. The greatest $\pi_t(G,v)$ across all of the vertices in G is the t-pebbling number of the graph G, or $\pi_t(G)$.

DEFINITION 2.3. [5] The cover pebbling number $\gamma(G)$ is the minimum number of pebbles required for performing a series of pebbling moves to place a pebble on each vertex of the graph regardless of the initial configuration.

DEFINITION 2.4. The least number $\phi(G,u,v)$ with the condition that, from every placement of $\phi(G,u,v)$ pebbles on G, it is possible to move a pebble to u and v simultaneously by a sequence of pebbling moves is the 2-target pebbling number of the vertices u,v in a graph G. The maximum $\phi(G,u,v)$ over all the vertices in G is the 2-target pebbling number of graph G, or $\phi(G)$.

THEOREM 2.5. [1] The pebbling number of the path P_n is 2^{n-1}, where n is the number of vertices in the path.

DEFINITION 2.6. Given a pebbling configuration on G, a transmitting subgraph is a path $v_1,v_2,v_3\dots,v_n$ within G where v_1 has at least two pebbles , and each of the intermediate vertices v_2 through v_{n-1} has at least one pebble. The final vertex v_n may or may not have a pebble. Under these conditions, a pebble can be transmitted from v_1 to v_n along the path.

3. THE ALGORITHM FOR FINDING $\phi(G,U,V)$ ANY GIVEN GRAPH $G(V,E)$

STEP 1. Track the shortest distance of each vertex in the graph from u and v.

STEP 2. Consider various possible configurations of pebbles on the vertices of G to move at least one pebble to vertex u and one pebble to vertex v using the pebbling moves.

STEP 3. Note the minimum number of pebbles that allows reaching both u and v in all possible configurations.

STEP 4. After trying all possible configurations, the minimum number of pebbles that allows to reach both u and v, is $\phi(G,u,v)$.

NOTATION 3.1. The notation $p(v)$ represents the number of pebbles placed on the vertex v.

4. RELATIONSHIP BETWEEN COVER PEBBLING NUMBER AND 2-TARGET PEBBLING NUMBER

Key Differences:
Objective: While the cover pebbling number aims to reach every vertex in the graph, the 2-target pebbling number is focused on reaching two particular vertices at the same time.
Configuration Dependency: The 2-target pebbling number takes into account the ability to achieve a dual target, whereas the cover pebbling number guarantees complete accessibility across the graph.
Correlation: The cover pebbling number $\pi(G)$ ensures that all vertices can be visited, therefore if there are enough pebbles to cover all vertices, any two vertices, u and v can be reached. Consequently, $\phi(G,u,v) \leq \pi(G)$. The 2-target pebbling number can be related to the cover pebbling number, but they are not directly interchangeable.

5. DIFFERENCES BETWEEN T-PEBBLING NUMBER AND 2 TARGET PEBBLING NUMBER

Target Count: The 2-target pebbling number involves reaching two targets at once, while the t-pebbling number focuses on reaching a single target. **Complexity:** The 2-target pebbling number can often be more complex to calculate because it requires considering simultaneous moves to two vertices, while t-pebbling can be examined separately for a single vertex.

6. MAIN RESULTS

THEOREM 6.1. The 2-target pebbling number of a graph G, $\phi(G) \geq |V(G)|+1$.

THEOREM 6.2. For a simple connected graph G, $\phi(G) > \pi(G)$

NOTE 6.3. If one of the targets already has a pebble, we can transfer a pebble to the other target by using $\pi(G)$ pebbles.

THEOREM 6.4. The 2-target pebbling number of the path P_n is $3\times 2^{n-2}$ where $n\geq 2$ is the number of vertices of the path. PROOF. Let $\{v_1,v_2,\dots,v_{n-1},v_n\}$ be the vertices of the path P_n; n is the number of vertices of the path. We prove the result by induction on n. When $n=2$, the result is true. When $n=3$, $V(P_3)=\{v_1,v_2,v_3\}$. Assume that any two vertices in the path P_3 are the target vertices. Let v_1,v_2 be the target vertices. Consider the configuration of six pebbles on the vertices of P_3. If all six pebbles are placed on the vertex v_3, we could move one pebble each to both targets. Therefore, $\phi(P_3)=6$. Now, let us assume that the result is true for all paths of length less than n. Suppose there are $3\times 2^{n-2}$ pebbles on the vertices of the vertices of P_n. Consider the case when no pebble is placed on v_{n-1},v_n. Then the path $P_{(n-2)}=(v_1,v_2,\dots,v_{n-2})$. Let $p(V(P_{(n-2)}))=3\times 2^{n-2}$. By the induction hypothesis, we can move six pebbles to v_{n-2} and so we are done.

By symmetry, the result follows for the targets v_1 and v_2.

Now consider the internal vertices v_k,v_{k+1} where $2\leq k\leq n-1$ as targets. By removing v_k,v_{k+1} from P_n, it is decomposed into two paths. Let $P_1=(v_1,v_2,\dots,v_{k-1})$ and $P_2=(v_{k+2},v_{k+3},\dots,v_n)$. Either of the paths P_1 or P_2 must have at least $3\times 2^{n-3}$ pebbles. Let $p(V(P_1))\geq 3\times 2^{n-3}$, P_1 is of length at most $n-3$. It contains $3\times 2^{n-3}$ pebbles. By using the induction hypothesis, we could move one pebble each to v_k and v_{k+1}.

Now consider the internal vertices v_i and v_j where $2\leq i,j\leq n-1, i\neq j$, $v_i\notin N(v_j)$. Suppose $3\times 2^{n-2}$ pebbles are placed on the vertices other than v_i and v_j. By removing v_i, v_j from the path P_n it is decomposed into 3 paths. Let $P_1=(v_1,v_2,\dots,v_{i-1})$, $P_2=(v_{i+1},v_{i+3},\dots,v_{j-1})$ and $P_3=(v_{j+1},v_{j+2},\dots,v_n)$. At least one of P_1, P_2 or P_3 contains not less than 2^{n-2} pebbles. Let $p(V(P_1))\geq 2^{n-2}$ pebbles. Let us assume P_1 is the length at most $n-4$. But it contains 2^{n-2} pebbles. By the induction hypothesis, we can move four pebbles to v_i. With the remaining $2\times 2^{n-2}$ pebbles, we can shift one pebble to v_k.

Consider v_i and v_j where $1\leq i,j\leq n, i\neq j$, $v_i\notin N(v_j)$ as the targets. Suppose v_1 and v_n are the targets. Placing $2^{n-2}+1$ pebbles at v_2 or v_{n-1} we cannot move a pebble to the target. Therefore, we need at least $2^{n-1}+2<3\times 2^{n-1}$ pebbles to put a pebble on both targets. If v_1 or v_n is one of the targets and the other target v_j where $2\leq j\leq n-1$. By the induction hypothesis, we would require at most 3.2^{n-1} pebbles. Therefore for any configuration of $3\times 2^{n-1}$ pebbles, we could pebble any two targets in the graph P_n. Hence the proof.

THEOREM 6.5. The 2-target pebbling number of the complete graph $K_n;n\geq 3$ is $n+1$. PROOF. If there is a vertex with at most three pebbles and all the other vertices with exactly one pebble each except the target vertices then only one of the targets could be pebbled. Therefore, we need at least $n+1$ pebbles to place a pebble on any two targets simultaneously. Therefore, $\phi(K_n) \geq n+1$ Now we prove the sufficient condition. Consider a configuration of $n+1$ pebbles on K_n. If any two vertices of K_n have at least 2 pebbles each then we could move one pebble each to both targets. If any vertex contains at least 4 pebbles can one pebble each to both the targets. Therefore, $\phi(K_n) \leq n+1$ Hence, $\phi(K_n) = n+1$

THEOREM 6.6. Let C_n denote the cycle with n vertices where $n \geq 3$. Then
$\phi(C_n) = 2^{\lfloor \frac{n}{2} \rfloor + 1}$, if n is odd.
$\phi(C_n) = 3 \times 2^{\frac{n}{2}-1}$, if n is even.

PROOF. Let $C_n = (v_0, v_1, v_2, \ldots, v_{n-1})$
CASE 1: n is odd.
The edges of C_n can be divided into two disjoint paths P_1 and P_2. These paths extend from v_1 to $v_{\lfloor \frac{n}{2} \rfloor + 1}$ and $v_{\lfloor \frac{n}{2} \rfloor + 1}$ to v_0 respectively. Each path has a length of $\lfloor \frac{n}{2} \rfloor$. By placing $2^{\lfloor \frac{n}{2} \rfloor + 1} - 1$ pebbles at $v_{\lfloor \frac{n}{2} \rfloor + 1}$, which is the farthest vertex from both v_0 and v_1, it becomes impossible to move a pebble to either of these targets v_0 or v_1. Therefore, $\phi(C_n) \geq 2^{\lfloor \frac{n}{2} \rfloor + 1}$ Now we proceed to show that $\phi(C_n) \leq 2^{\lfloor \frac{n}{2} \rfloor + 1}$.
SUB CASE 1.1: The targets v_i and v_j are adjacent.
Let v_0 and v_1 be the target vertices. Define x as the number of pebbles placed at $v_{\frac{n}{2}+1}$.

—If $x = 2^{\lfloor \frac{n}{2} \rfloor + 1}$, then using $\frac{x}{2}$ pebbles along both the paths, we can transfer a pebble to each of the targets v_0 and v_1.

—If $x \leq 2^{\lfloor \frac{n}{2} \rfloor + 1}$, we distribute the remaining $2^{\lfloor \frac{n}{2} \rfloor + 1} - x$ pebbles among other vertices.

CASE ANALYSIS:
1. **All pebbles placed on one path eg.,** P_1

—If all the remaining pebbles are placed on P_1, then P_1 has a total of $2^{\lfloor \frac{n}{2} \rfloor + 1}$ pebbles, allowing the transfer of pebbles to both targets.

2. **Pebbles placed at** $v_{\lfloor \frac{n}{2} \rfloor}$

—If $2^{\lfloor \frac{n}{2} \rfloor + 1} - 3 \leq x \leq 2^{\lfloor \frac{n}{2} \rfloor + 1} - 1$ and the remaining pebble(s) are at $v_{\lfloor \frac{n}{2} \rfloor}$, we use $2^{\lfloor \frac{n}{2} \rfloor}$ pebbles to move a pebble to v_0 through P_2, while the remaining pebbles transfer a pebble to v_1 through P_1.

3. **Pebbles placed across multiple vertices in** P_1

—If $x = 2^{\lfloor \frac{n}{2} \rfloor+1} - 2$,$2^{\lfloor \frac{n}{2} \rfloor+1} - 3$ and the remaining pebbles are spread across multiple vertices in P_1, then it is possible to transfer one pebble to each target through P_1.

4. **Pebbles distributed across internal vertices of both paths**

—If all $2^{\lfloor \frac{n}{2} \rfloor+1}$ pebbles are spread among internal vertices of both the paths, at least one path (say P_1) must contain at least $\frac{2^{\lfloor \frac{n}{2} \rfloor+1}-x}{2}$ pebbles. This ensures that we can move one pebble each to v_0 and v_1.

Thus,

$$\phi(v_1, v_2, C_n) = 2^{\lfloor \frac{n}{2} \rfloor+1}$$

. By symmetry, the proof holds for any adjacent target vertices v_i and v_j.
SUBCASE 1.2: The targets v_i and v_j are not adjacent.
1. **v_i and v_j are on the same path P_1**

—In this case, we can move one pebble to each of the targets using $3 \times 2^{n-2}$ pebbles. Since, $3 \times 2^{n-2} < 2^{\lfloor \frac{n}{2} \rfloor+1}$, the required number of pebbles is within the given bound.

2. **v_i and v_j are on different paths**

—Here we require $2^{\lfloor \frac{n}{2} \rfloor}$ pebbles on each path. Thus, the total number of requires pebbles is $2^{\lfloor \frac{n}{2} \rfloor} + 2^{\lfloor \frac{n}{2} \rfloor} = 2^{\lfloor \frac{n}{2} \rfloor+1}$

Thus we conclude that, $f(v_i, v_j, C_n) \leq 2^{\lfloor \frac{n}{2} \rfloor+1}$.
Therefore, $\phi(C_n) = 2^{\lfloor \frac{n}{2} \rfloor+1}$.

CASE 2: n is even.
The edges of C_n can be divided into two disjoint paths say P_1 and P_2. These extend from v_0 to $v_{\frac{n}{2}+1}$ and v_1 to $v_{\frac{n}{2}}$ respectively. Placing $3 \times 2^{\frac{n}{2}-1} - 1$ pebbles at $v_{\frac{n}{2}+1}$ prevents us from simultaneously moving a pebble to both target vertices v_1 and v_0. Thus, $\phi(C_n) \geq 3 \times 2^{\frac{n}{2}-1}$.

Now we prove that $f(u, v, C_n) \leq 3 \times 2^{\frac{n}{2}-1}$.
SUB CASE 2.1: The targets v_i and v_j are adjacent.
If we place $3 \times 2^{\frac{n}{2}-2}$ pebbles at both $v_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$, then a pebble can be moved to v_0 through P_1 and to v_1 through P_2. In any distribution of $3 \times 2^{\frac{n}{2}-1}$ pebbles, at least one of the paths will contain at least (either P_1 and P_2) $3 \times 2^{\frac{n}{2}-2}$, ensuring that both targets are reached. Thus, for all possible distributions, we can pebble the adjacent targets.
SUB CASE 2.2: The targets v_i and v_j are not adjacent.
If v_i and v_j are on the same path (say, P_1) we can move one pebble to both targets using

$3 \times 2^{n-2}$ pebbles (since $\phi(P_n) = 3 \times 2^{n-2}$). If v_i and v_j are on different paths, we require at least $2^{\frac{n}{2}}$ pebbles on each path. Since this is less than $(3 \times 2^{n-2}) + 1$ pebbles. We conclude that, $\phi(v_i, v_j, C_n) \leq (3 \times 2^{n-2}) + 1$.

Thus, $\phi(C_n) = (3 \times 2^{n-2}) + 1$.

THEOREM 6.7. Let $G = K_{n_1, n_2 \ldots, n_r}$ be the complete r-partite graph. Let $r \geq 3$. Then $\phi(G) = n_1 + n_2 \ldots + n_r + 1$. PROOF. Let the r- partition of the vertex set V be $V = V_1 \cup V_2 \cup \ldots V_r$. Let $V_i = \{x_{i1}, x_{i2} \ldots x_{in_i}\}$ for $i = 1$ to r. If $n_i = 1$ for all $i = 1$ to r then $G = K_r$. Therefore, $\phi(K_r)$ where $n = n_1 + n_2 \ldots + n_r + 1$ (from the result of the complete graph). We may assume that there exists at least one n_i with $n_i > 1$. Without loss of generality, let $n_i > 1$ for the set V_1.

Let us fix x_{11} and x_{12} as the target vertices i.e., $p(x_{11}) = 0$ and $p(x_{12}) = 0$. If all the other $(n-2)$ vertices of V have exactly one pebble except any of the vertex that contains 3 pebbles in it we cannot move one pebble to any one of the targets.

Therefore, $\phi(G) \geq n_1 + n_2 \ldots + n_r + 1$.

CASE 1:

If x_{ia} and x_{ib}, $(1 \leq i \leq r)$, $1 \leq a, b \leq n_i, a \neq b)$ i.e., vertices from the same partition be the target vertices.

Let us show that $\phi(x_{11}, x_{12}, G) \leq n_1 + n_2 \ldots + n_r + 1$. If there is a vertex x_{ij} $(2 \leq i \leq r, 1 \leq j \leq n_i)$ such that x_{ij} has one pebble, then in the distribution of $n_1 + n_2 \ldots + n_r + 1$ over the vertices other than x_{11} and x_{12}, at least one $x_{ij}(3 \leq j \leq n_1)$ has more than 3 pebbles we can move one pebble to both the targets by using 2 vertices with exactly one pebble forming a transmitting subgraph.

If not, there is a contradiction because there are $n_1 + n_2 \ldots + n_r$ pebbles in total. If every vertex of V_i $(2 \leq i \leq r)$ has zero pebbles then we place all the $n_1 + n_2 \ldots + n_r + 1$ pebbles in $n_1 - 2$ vertices of $V_1 - \{x_{11}, x_{12}\}$. We will have a vertex with at least 8 pebbles or there are at least two vertices in $V_1 - \{x_{11}, x_{12}\}$ with at least 4 pebbles each.

If not, the maximum count of placed pebbles would be $n_1 + 4$, conflicting with the total count of pebbles being $n_1 + n_2 + \ldots + n_r$ where $r \geq 3$.

CASE 2:

If x_{ij} and x_{kj}, $(1 \leq i, k \leq r)$, $i \neq k$, $1 \leq j \leq n_i)$ i.e., vertices from different partitions be the target vertices.

Suppose x_{11} and x_{21} be the target vertices. If there are two vertices with at least two pebbles on two different partitions, we can move a pebble to the targets x_{11} and x_{21}.

If there is a vertex of $V_1 - \{x_{11}\}$ or $V_2 - \{x_{21}\}$ with at least 6 pebbles we can move a pebble to the target.

If there is a vertex of $V_i, i \neq 1,2$ with at least 4 pebbles we can put 1 pebble on both the targets.

If there are two vertices x_{ij}, x_{pq} $(3 \leq i, p \leq r, 1 \leq j, q \leq n_i), i \neq p$ that has 1 pebble each then in the distribution of $n_1 + n_2 \ldots + n_r + 1$ over the vertices other than x_{11} and x_{21} at least two x_{ij} $(2 \leq j \leq n_1)$ and x_{pq} $(2 \leq q \leq n_2)$ has more than one pebble we can move a pebble to both the targets.

COROLLARY 6.8. From the result of the theorem 6.7 if all the other vertex sets have zero pebbles except V_1 and V_2 where $n_1, n_2 \geq 1$, they form a bipartite graph whose 2-target pebbling number is $n_1 + n_2 + 1$

THEOREM 6.9. Let $K_{1,n}$ be an n-star. Then $\phi(K_{1,n}, u, v) = n + 5$ if $n \geq 1$. PROOF. Let $V(K_{1,n}) = A \cup B$ where $A = a_1$ and $B = b_1, b_2, \ldots, b_n$.

CASE 1: Let any of b_i and b_j be the target vertices in $K_{(1,n)}$ where $n \geq 3$. Without loss of generality, let b_1 and b_2 be the target vertices. Place 7 pebbles at b_3 and one pebble at each of the vertices $b_4, b_5, \ldots, b_n$. Now we have placed $n + 4$ pebbles. In this positioning, no pebble can be moved to either b_1 or b_2. Therefore $\phi(b_1, b_2, K_{1,n}) > n + 5$. Let us prove that $\phi(b_1, b_2, K_{1,n}) \leq n + 5$.

If a_1 has four pebbles, then we are done. If a_1 has only one pebble, then the other $n + 4$ pebbles are placed in the $n - 2$ vertices of $B - \{b_1, b_2\}$ in $K_{1,n}$ except the target vertices one pebble could be shifted to both targets.

If a_1 has no pebble in the distribution, and if there is a vertex in $B - \{b_1, b_2\}$ having eight pebbles then we are done. Otherwise, if there are two vertices of $B - \{b_1, b_2\}$ having at least four pebbles. This is because if a vertex has 4 pebbles and the other vertex has only 3 pebbles and all the vertices have one pebble each, the total count comes to only $n + 3$ which contradicts the fact that we have $n + 5$ pebbles. So, if two vertices have four pebbles each, then four pebbles can be moved to a_1 and so one can be moved to b_1 and b_2. If four vertices have 2 pebbles each we could move 4 pebbles to a_1 so one pebble can be moved to both the targets.

CASE 2: Let a_1 and b_i where $i = 1$ to n be the target vertices. Without loss of generality, let a_1 and b_1 are the target vertices. If a vertex in B other than b_i has at least 6 pebbles then one pebble can be moved to a_1, b_1. If there exist two vertices in B other than b_i, $j, k \neq i$ that have at least 2 and 4 pebbles each then one pebble can be moved to a_1, b_1. If there exist 3 vertices in B other than b_i, $j, k \neq i$ that have at least 2 pebbles each then one pebble can be moved to a_1, b_1. So $\phi(b_1, b_2, K_{1,n}) \leq n + 5$.

Hence $\phi(b_1, b_2, K_{1,n}) = n + 5$. Therefore, $\phi(K_{1,n}) = n + 5$ since b_1 and b_2 is arbitrary.

THEOREM 6.10. A fan graph consists of a path P_{n-1} with vertices $(v_1, v_2, \ldots, v_{n-1}$ with an additional vertex v_0 connected to all the vertices of the path. The 2-target pebbling number of a fan graph F_n is $\phi(F_n, u, v) = n+2$. PROOF. Let v_1 and v_2 be the target vertices. $p(v_1) = 0$ and $p(v_2) = 0$. If $p(v_0) = 0$, $p(v_k) = 5, k \in P_{n-1}, k \neq 1, 2$ and all the other $n-3$ vertices have exactly one pebble each, we will not be able to move a pebble to v_1 or v_2. Therefore, with $n+1$ pebbles we cannot move a pebble to the target. $\phi(F_n) \geq n+2$ Now, let us prove $\phi(F_n) \leq n+2$

CASE 1: Let v_0 and v_i, $1 \leq i \leq n-1$ be the target vertices. Suppose v_0 and v_1 be the target vertices $p(v_0) = 0$ and $p(v_1) = 0$.

If any one vertex v_i where $i \in (2, 3, \ldots, n-1)$, with $p(v_i) \geq 6$, we can shift a pebble to both the targets.

If any two vertices v_i and v_j where $i, j \in (2, 3, \ldots, n-1), i \neq j$ with $p(v_i) \geq 2$ and $p(v_j) \geq 4$ we can transfer one pebble to both the targets.

If any three vertices v_i, v_j and v_k where $i, j, k \in (2, 3, \ldots, n-1), i \neq j \neq k$ with $p(v_i), p(v_j), p(v_k) \geq 2$ we can transfer one pebble to both the targets.

CASE 2: Let v_i and v_j, $1 \leq i, j \leq n-1, i \neq j$ be the targets. Suppose v_1 and v_2 be the target vertices $p(v_1) = 0$ and $p(v_2) = 0$.

If any one vertex v_i where $i \in (3, 4, \ldots, n-1)$, with $p(v_i) \geq 8$, we can shift a pebble to both the targets.

If any two vertices v_i and v_j where $i, j \in (3, 4, \ldots, n-1), i \neq j$ with $p(v_i) \geq 4$ and $p(v_j) \geq 4$ we can transfer one pebble to both the targets.

If any four vertices v_i, v_j and v_j where $i, j, k, l \in (3, 4, \ldots, n-1), i \neq j \neq k \neq l$ with $p(v_i, v_j, v_k, v_l) \geq 2$ we can transfer one pebble to both the targets. Therefore, $\phi(F_n) \leq n+2$.

Hence, $\phi(F_n, u, v) = n+2$

THEOREM 6.11. The wheel graph W_n consists of $K_1 = a$ and $C_n = (b_1, b_2, \ldots, b_{n-1})$, cycle of length $n-1$. K_1 is connected with all the vertices of C_n. Then the 2-target pebbling number of the wheel graph W_n is $\phi(W_n) = n+1$.

PROOF. Let b_1 and b_2 be the target vertices. $p(b_1) = 0$ and $p(b_2) = 0$. If $p(b_k) = 3, k \in C_n, k \neq 1, 2$ and all the other $n-2$ vertices have exactly one pebble each, with the total of $n+1$ pebble, we will not be able to move a pebble to b_1 or b_2. Therefore, $\phi(u, v, W_n) \geq n+2$.

CASE 1: Let a and b_i, $1 \leq i \leq n-1$ be the targets.

Suppose a and b_1 be the target vertices $p(a) = 0$ and $p(b_1) = 0$. If vertex b_i where

$i \in (2,3,\ldots,n), i \neq 1$ with $p(b_i) \geq n$ has three pebbles and if all the other vertices have one pebble each, we cannot shift a pebble to both the targets with n pebbles. Therefore we need at least $n+1$ pebbles, to pebble both targets simultaneously.

If b_2 or b_n has four pebbles then we are done.

If b_2 and b_n each have two pebbles then both targets could be pebbled.

If any vertex of $\{b_3, b_4, \ldots, b_{n-1}\}$ has at least six pebbles then we put three pebbles on a and one pebble on b_1.

CASE 2: Let b_i and b_j, $1 \leq i, j \leq n-1, i \neq j$ be the targets. Suppose b_1 and b_2 are the target vertices. If a has four pebbles we can move a pebble to both targets.
If a has at most 3 pebbles then if any of the other b_i, $i \in (3,4,\ldots,n)$ has at least 2 pebbles we could move a pebble to both the targets at the same time.
If a has at most one pebble then if any has b_i where $i \in (3,4,\ldots,n-1), i \neq 1,2$ with $p(b_i) \geq 6$ then, we could move one pebble to the target.
Or else, If a has at most one pebble then if any have b_i and b_j where $i, j \in (3,4,\ldots,n-1), i, j \neq 1,2$ and $i \neq j$ with $p(b_i) \geq 4$ and $p(b_j) \geq 2$ then we could move one pebble to the target.
Also, the cases when the targets b_1 and b_2 can be pebbled with $p(a) = 0$ is shown as follows: If a has zero pebbles, b_3 and b_{n-1} have at most one pebble each, and no vertex of $\{b_4, b_5, \ldots, b_{n-3}\}$ has eight pebbles. There are at least two vertices of $\{b_4, b_5, \ldots, b_{n-2}\}$ with at least four pebbles each. There is a vertex in $\{b_4, b_5, \ldots, b_{n-2}\}$ with four pebbles and there is a pebble on every other vertex except on a, b_1 and b_2. If not, there is a contradiction since the total number of pebbles placed is at most n. We can also come around the cycle in both directions in order to put a pebble on b_1 and b_2. Hence, $\phi(b_1, b_2, W_n) \leq n+1$. Therefore, $\phi(W_n) = n+1$. since b_1 and b_2 are arbitrary.

CONCLUSION

Our results provide insights into the 2-target pebbling number for various standard graph families, highlighting patterns and establishing bounds that contribute to a deeper understanding of simultaneous reachability in graph pebbling. These findings lay the groundwork for further exploration of multi-target pebbling in more complex graph structures.

Declaration of Competing Interest:
The authors declare that there are no conflicts of interest regarding this work.

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