Numerical analysis of stochastic PDEs in traffic flow: Investigating density-flow relations

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Abstract

Understanding traffic flow dynamics is crucial for modeling real-world scenarios, where stochastic factors often introduce variability and unpredictability. This study explores the impact of stochastic influences on traffic flow, focusing on the relationship between flow and density in both deterministic and stochastic models. Using the Lighthill-Whitham-Richards (LWR) framework, the study examines a stochastic partial differential equation (SPDE) to simulate traffic behavior under varying conditions. The lognormal random numbers are incorporated for the Brownian motion for stochasticity after analysing its normalitynusing various tests. Numerical solutions were obtained through Godunov's scheme, incorporating boundary conditions to capture stochastic effects. The findings show that the stochastic approach enhances predictive accuracy by capturing real-world traffic uncertainties.

Mathematics Subject Classification 2000: 60H15, 60H35, 65C30
Keywords: Stochastic PDEs, Traffic Flow, Numerical Methods, Density-Flow Relations.

1. INTRODUCTION

Modeling a physical problem in a mathematical or scientific way is an art that describes the physical scenario. Traffic flow modeling starts when Lighthill, Whitham, and Richards (1955) present the scenario of traffic. Traffic flow can be described in different ways. When considering the effect of individual drivers on vehicle disturbances, it is referred to as microscopic modeling of traffic. This approach, also known as the Follow-the-Leader model, assumes that each vehicle reacts to the behavior of the vehicle ahead. It can be expressed as: response = sensitivity × stimulus where the stimulus refers to speed, and sensitivity refers to the distance between two successive vehicles. If the effect of individual drivers is not considered, the modeling is referred to as mesoscopic traffic flow modeling. Alternatively, when focusing on the collective behavior of vehicles, it is known as macroscopic or car-following traffic flow modeling. Since the occurrence of vehicle flow on roads is random and human driving behavior causes vehicles to arrive and depart at different times and locations, traffic flow models should account for this stochasticity. By considering the randomness in vehicle behavior, models can better represent real-world traffic dynamics. Many researchers have suggested that

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macroscopic traffic flow models are well-suited for capturing the overall flow of traffic under such conditions. The parameters of a vehicle flow model—such as flow/spacing (q), the number of vehicles represented as density/gap/headway, and vehicle velocity—can be used to describe the fundamental relationship: flow (q) = density × velocity, represented by the equation

$$Q = \rho v$$

The above model is deterministic, as density and velocity are considered as average values. To develop a stochastic vehicle flow model, the parameters of the deterministic model must be treated as random variables. Hence by considering the density or velocity as a random variable the model can be characterized as stochastic traffic flow model. Many researchers (Nagel and Schreckenberg, 1992, Takayasu and Takayasu, 1993, Benjamin, Johnson, and Hui, 1996, Jiang and Wu, 2002) have described the stochastic traffic flow model using speed-density mechanism stochasticity of the model, (Knospe et al., 2000) define brake-light effect as stochasticity in traffic model, (Kerner, Klenov, and Wolf, 2002) suggested stochasticity in acceleration rule, Memory effects in microscopic traffic models have been investigated in previous studies. (Treiber and Helbing, 2003) and described the variance of time as stochasticity, A stochastic dynamic traffic model was developed for traffic state surveillance. (Sumalee et al., 2011) has added the effect of noise as stochastic parameter, The impact of uncertainties on the fundamental diagram of the LWR model has been analyzed by Li et al. (2012), who used a stochastic flux function by treating the free-flow speed as a random variable. Treiber and Helbing, 2003) considered the variance in time gap as stochasticity factor, (Treiber, Kesting, and Helbing, 2006) consider the time headway which effect the velocity/speed of car in free-flow model as a random variable, (Sumalee et al., 2011, Thonhofer and Jakubek, 2018) describe the variation in stochastic parameters using three different ways, the stochasticity at boundaries, source term as an inflow and variation in speed flow or speed density. Zheng et al. (2020) discussed that the link flow model may not converge if the path becomes infinite due to cyclic flow. This situation can occur when a very small value is considered. The model is then solved using properties of a Markov process. The researchers in (Nishinari, Treiber, and Helbing, 2003) use data for model and data is collected using some experiments for equilibrium and non-equilibrium. This data is always affected by some external factors like weather condition, road condition, time and the experimental data used by considering equilibrium and non-equilibrium conditions. Distribution function is used to find velocity profile by considering different values of density. Akamatsu (1996) provided a detailed model for route choice, where travel time is considered a random variable influenced by driver decisions. At least 700 iterations are required for the convergence of the model for a Medium size model. A heuristic solution algorithm based on projection method is used to solve the model. Bonzani and Mussone (2002) proposed a model that captures the complete traffic scenario, including entry, exit, and exchange points. This model predicts both quantitative and qualitative aspects of traffic. A Markov chain process is developed and simulated using the kinetic Monte Carlo method to solve the model. In (Shao, Lam, and Tam, 2006) the authors suggested the stochastic model by using traffic density as random parameter of the model to check the random performance of vehicles. This model gives the distribution of traffic flow, using a normal distribution to represent it. This distribution is then used to predict real traffic flow corresponding to the stochastic model. The proposed model is used for real life traffic data and is numerically simulated by the Godunov's technique. The Monte –Carlo simulation is used to generate the number of paths for flow of vehicles as density. (Alperovich and Sopasakis, 2008) proposed the stochastic model with LWR model which is extended using stochastic fundamental diagram by considering capacity as a random variable. The model shows the first order non-linear traffic phenomena using scatter fundamental diagram and is not capable for the higher order, the Harten-Lax-van Leer-Einfeldt (or HLLE) numerical scheme and is used to solve the model. (Ho, Leung, and Polak, 2010) suggested the stochastic traffic model with 18 paths of the section of road for collecting the real life data for the vehicles and used to capture the random behavior of the traffic flow and the model used Brownian motion for random behavior of the model with normal distribution of the collected data. (Tight et al., 2011) proposed model is able to develop traffic flow scenario with heterogeneity effect in traffic steam. A Poisson process for homogeneous and non-homogeneous traffic flow is considered for the model. Lim et al. (2012) proposed a model for the movement of a following vehicle based on the speed of the vehicle ahead. The model is designed to capture traffic flow dynamics. The model is solved using the probability density function of the beta distribution. Calvert, S. (Jabari and Liu, 2012) gives the model which is presented for stochastic behavior and interaction of vehicles and the authors used Upwind scheme for model.Calvert, S. (Jabari and Liu, 2012) presented a model that captures the stochastic behavior and interaction of vehicles. The authors used the Upwind scheme to solve the model. Zamith et al. (2015) provided a data-driven model developed without assumptions, which does not fully satisfy the fundamental diagram of stochastic traffic flow. The model further incorporates the Laplace distribution. Calvert et al. (2015) proposed a model that employs the Unscented Kalman Filter to estimate the speed and density of selected vehicles using real traffic data. In the case of stop-and-go traffic, this method also looks into predicting the traffic congestion density. The prototype considered the speed variable as a stochastic factor and used Unscented Kalman filter (UKF) and unscented transform for the model. Kendziorra, Wagner, and Toledo (2016) used Lagrangian coordinates to describe a proposed stochastic model for predicting stochastic traffic flow. The driver's speed varies in free-flow conditions and decreases as the distance between cars becomes smaller. Gaussian Approximate and Kalman Filter methods is used to solve the proposed model. Chu, Saigal, and Saitou (2016) proposed a stochastic model that incorporates the fundamental relationships of deterministic parameters as sources of stochasticity in traffic flow. The model was applied to real traffic data to describe the relationship between flow and density under stochastic conditions. A Beta distribution was used to model the data. (Jabari et al. 2018) proposed the stochastic model to forecast the location of vehicles in the movement of vehicles in traffic flow and used Newell's car-following model. Probability density function with mean and standard deviation is suggested to forecast the traffic flow in congested traffic. (Lin et al., 2018) suggest the model which is developed to estimate the behavior of stochastic car-following model for multilane. Convolutional Neural Network (CNN) is used to estimate the Lane changing model, Genetic algorithm for maximum likelihood estimation (MLE) is used, which is one of widely used methods for estimating the parameters of a stochastic traffic flow model. Xu and Laval (2019) developed a stochastic Lighthill–Whitham–Richards (LWR) model by considering mixed traffic flow, which accounts for vehicle dependency and jam density. Monte Carlo simulation was used to analyze the stochastic behavior of traffic flow. (Bhutto et al. 2023) analyzed the effect of uniform and exponential streams on magnetohydrodynamic (MHD) flows of viscous fluids. The study investigated flow behavior under different stream conditions, applying mathematical modeling and numerical simulations to understand MHD fluid dynamics. Lee, Ngoduy, and Keyvan-Ekbatani (2019) proposed a model that uses a stochastic speed–density relationship to characterize the randomness in macroscopic traffic flow. Simulation with Optimization Toolbox of

MATLAB using fitness function is used to implement the model. Wada, Martínez, and Jin (2020) developed a model for predicting traffic using experimental data and vehicle trajectories on a static road section. Graph Convolutional Networks are used for trajectories. Zheng et al. (2020) critically studied the impact of stochasticity on dynamic traffic flow models. Ni, Hsieh, and Jiang (2018) explored stochastic aspects using a traffic model that incorporates reservation-based systems, where each link can exist in one of four states, with a probability assigned to each state. The work is based on the data provided by a microscopic traffic flow model and shows that the link model estimates the state uncertainties quite well. (Khokhar et al. 2023) conducted a numerical analysis of Newtonian fluid flows, focusing on the effects of flow rates, porous media, and Reynolds numbers on fluid mixing and separation. The study employed computational techniques to examine fluid behavior under varying conditions, providing insights into flow dynamics in porous environments. The work of Luo, Li, and Zhang (2019) reviews the methodology used to construct the fundamental diagram and a stochastic traffic flow model. This was implemented on a real traffic model to regulate and establish the flow-density relationship using a stochastic model. Subsequent data collection and statistical analysis demonstrated that the fitted process was sufficient. Ostré et al. (Fan et al., 2023) formed a high level stochastic model of traffic flow using Lagrangian coordinates to capture disparate driving habits based on an individual driver's speed and distance relationship – which caused the action of traffic on the road to reflect noisy stochastic flow patterns. The method is suitable for real time circumstances in which feedback data are used. The model developed by (Cheng et al., 2024, Zheng et al., 2020) is used to illustrate the stochastic behavior of car-following traffic flow where the relationship between velocity and density is employed. The study explores the influence of traffic flow over stochastic noise, taking into consideration bottlenecks situated upstream and downstream. Zheng et al. (2020) suggest using stochastic speed-density relationships in macroscopic modeling to reproduce spatiotemporal patterns in stochastic processes. In addition, he has also developed a stochastic traffic model for mixed transport traffic types employing Lagrangian approach vehicles, which accounts for the diversity in human driver's activity together with the automated and human operated vehicles. Stochastic models describe the mean and covariance of intersection dynamics involving traffic streams composed of both randomly acting human drivers and autonomously driven vehicles. This model is in a position to look into the relationships between man driven cars and automated driven cars. (Martínez and Jin, 2020) The stochastic version of the Lighthill-Whitham Richards model, which incorporates heterogeneity and is solved using Lagrangian coordinates, is utilized to examine how driver diversity affects the macroscopic relationships between traffic flow parameters, both analytically and via simulations. The findings indicate that both static and dynamic macroscopic traffic flow characteristics, such as speed and flow rate at bottlenecks, align with the deterministic model, with jam density serving as the harmonic mean of the distribution. Additionally, the model examines the relationship between the flow rates of automated vehicles and dispersed vehicles, with the goal of maximizing the number of vehicles at bottlenecks while reducing the impact of capacity drop. In a traffic congested situation, a number of researchers have used microscopic traffic modeling, while others use a macroscopic model as suggested in this paper as a remedy for the congestion, because in reality randomness has to be factored into the situation, factors such as endless supply of cars on the roads traffic jam poor weather and changing road infrastructures. The prediction of traffic flow under congestion usually involves a myriad of random events and discontinuities that make the process complicated. The conservation of vehicles flow with macroscopic car-following model condition does not express the random manners of traffic parameter. The main aim of the article is to develop a stochastic traffic flow model that accounts for the randomness in vehicle behavior, such as speed variations, density fluctuations, and driver decision-making, current study seeks better prediction of real-world traffic dynamics under uncertain conditions.

2. METHODOLOGY

2.1. **Area Selection and Data collection**:

To collect data on congestion influenced by stochastic factors, a specific road section was selected, as shown in Figure 1. This section includes both inflow and outflow of traffic, with a total of 18 paths where vehicles enter and exit. The inflow from ramps is disregarded, and only the inflow and outflow at designated path locations are considered. Data was manually recorded for all paths over a one-week period from December 23, 2024 to December 29,2024, between 7:00 AM and 7:00 PM, with measurements taken at one-hour intervals.

The methodology follows these steps: The deterministic macroscopic LWR model is used as a foundation to develop the stochastic model. A homogeneous congested road section is selected for collecting random traffic data. The recorded data is analyzed to examine vehicle flow patterns. A normality test is conducted on the collected data, followed by fitting an appropriate probability distribution to capture stochastic variations. Finally, the Godunov numerical scheme is applied to develop an efficient algorithm in MATLAB for predicting traffic flow based on the proposed model.

Fig. 1: Satellite view of the section of road, 23rd December, 2024

3. DATA COLLECTION AND ANALYSIS

Table 3.1 shows the number of cars entering in the S1 of road section from Monday to Sunday; from 07:00 hr to 19:00 hr for each day. The hour wise statistics, day-wise statistics and week-wise statistics can be find usually. It can be summarized that on Monday the total cars from 07:00 to 19:00 hr were recoded as 3937 with average 328.08 cars per hour having standard deviation and range as 174.76 and 514 respectively; and the data have positive skewness of 0.60 from its average. Also, the total number of cars on 7:00 am from Monday to Sunday were recorded as 1138 with average 162.57 cars per day at 7:00 am having standard deviation and range as 27 and 76 respectively; and the data have positive skewness of 1.37 from its average. Finally, the total number of cars on whole week from 7:00 to 18:00 hr. and from Monday to Sunday were recorded as 25391 with average 302 cars per hour per day having standard deviation and range as 40 and 203 respectively; and the data have positive skewness of 0.81 from its average. Similarly, the data is also recorded for all other days of the week from 7:00 to 19:00 and statistical analysis is presented to show the state of the considered section of road on the paths S1. Similarly for all the considered paths. Table I and Table II show the number of cars entering the S1 road section from Monday to Sunday, recorded from 07:00 to 19:00 each day. Hourly statistics are given in the top-right panel of the table, and day-wise statistics are in the bottom panel.

Table I, Table II, and Table III show the number of cars entering the S1 road section from Monday to Sunday, recorded from 07:00 to 19:00 each day.

Table I: Traffic Data for Path S1 (Part 1)

Day/Time	07:00-08:00	08:00-09:00	09:00-10:00	10:00-11:00
Monday	138	525	430	312
Tuesday	142	412	370	285
Wednesday	214	387	390	310
Thursday	182	318	291	317
Friday	162	416	380	280
Saturday	148	456	430	318
Sunday	152	373	375	287

Table II: Traffic Data for Path S1 (Part 2)

Day/Time	11:00-12:00	12:00-13:00	13:00-14:00	14:00-15:00
Monday	218	178	210	116
Tuesday	206	156	318	230
Wednesday	183	202	323	211
Thursday	157	216	347	170
Friday	116	176	287	203
Saturday	233	220	273	167
Sunday	290	218	316	216

Table III: Traffic Data for Path S1 (Part 3)

Day/Time	15:00-16:00	16:00-17:00	17:00-18:00	18:00-19:00
Monday	318	282	630	580
Tuesday	287	169	418	466
Wednesday	278	216	495	610
Thursday	192	223	412	417
Friday	323	267	378	530
Saturday	297	390	298	518
Sunday	270	365	310	496

4. NORMALITY TEST

Table 4.1 summarizes the statistical results for path S1 using three tests: Kolmogorov-Smirnov, Anderson-Darling, and Chi-Square. Table 4.1 shows the Normal distribution was rejected across all tests, indicating a poor fit. The Lognormal distribution, however, was the only one that consistently passed all three tests with acceptable p-values (Kolmogorov-Smirnov: 0.66604, Anderson-Darling: 0.28583, Chi-Square: 0.0719), suggesting that the data for path S1 follows a lognormal distribution. Other distributions, such as Weibull, Exponential, and Gamma, were also rejected due to low p-values in at least one test, signifying their inadequacy in modeling the dataset. These results confirm that the lognormal distribution provides the best fit for the data among the tested alternatives. The findings emphasize the importance of conducting multiple tests to validate the statistical alignment of a dataset with a specific distribution model, ensuring robust and reliable interpretation. Table IV summarizes the statistical results for path S1 using three tests: Kolmogorov-Smirnov, Anderson-Darling, and Chi-Square.

Table IV: Normality Test for Path S1

Distribution	Kolmogorov-Smirnov	Anderson-Darling	Chi-Square
Normal	0.19174	0.00517	0.0020 (Reject)
Lognormal	0.66604	0.28583	0.0719 (Accept)
Weibull	>0.1	0.03037	0.0044 (Reject)
Exponential	<0.0001	<0.0001	2.93e-18 (Reject)
Gamma	0.58589	≥0.25	0.0165 (Reject)

5. DETERMINISTIC LWR TRAFFIC FLOW MODEL AND EXACT AND NUMERICAL SOLUTION

5.1. Analytical solution

The fundamental deterministic LWR model equation with exact solution using car-following fundamental relation between the traffic parameters is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{1}$$

The key relationships between velocity, density, and flux are:

$$v(x,t) = V_{\max}\left(1 - \frac{\rho(x,t)}{\rho_{\max}}\right), \tag{2}$$

$$q(x,t) = \rho(x,t)v(x,t), \tag{3}$$

$$q(x,t) = \rho(x,t)V_{\max}\left(1 - \frac{\rho(x,t)}{\rho_{\max}}\right). \tag{4}$$

The initial and boundary conditions are:

$$\rho(x,0) = \rho_0(x), \quad q(0) = 0, \quad V(0) = V_{\max}. \tag{5}$$

From equation (1), we derive:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \implies \frac{\partial q}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{6}$$

Using the velocity equation, we obtain:

$$\frac{dx}{dt} = V_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)^2. \tag{7}$$

Thus, the exact solution of the deterministic model is:

$$\rho(x,t) = \rho_{\max}\left(1 - \frac{x - x_0}{3V_{\max}t + c}\right)^2. \tag{8}$$

where, c is a constant as $c = \rho(x_0, 0)$, depending upon the initial density.

5.2. Numerical Solution:

Using the forward difference approximation in time and backward difference in space, the explicit numerical scheme is:

$$\rho_i^{j+1} = \rho_i^j - \frac{\delta t}{\delta x}\left(q_i^j - q_{i-1}^j\right), \tag{9}$$

where

$$q_i^j = \rho_i^j V_{\max}\left(1 - \frac{\rho_i^j}{\rho_{\max}}\right)^2. \tag{10}$$

6. STOCHASTIC LWR TRAFFIC FLOW MODEL

The stochastic model modifies the LWR equation by adding a noise term:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = u_e(x,t,\rho), \tag{11}$$

where

$$u_e(x,t,\rho) = \rho(x,t) + Lq(x,t) + W(x,t). \tag{12}$$

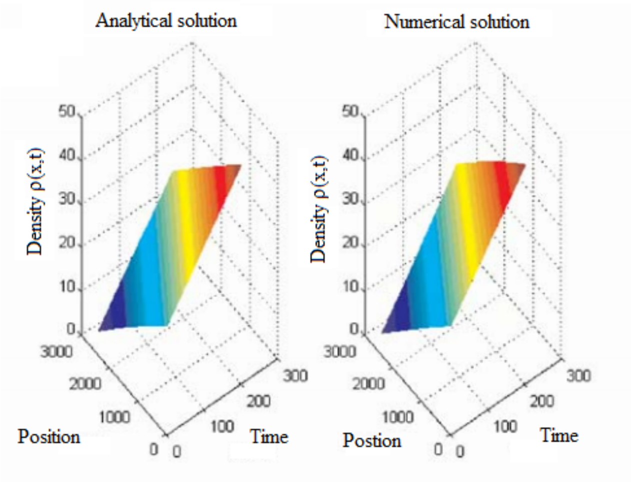

Fig. 2: Analyticla and Numerical solution

Thus, the **stochastic partial differential equation (SPDE)** is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = \rho(x,t) + Lq(x,t) + W(x,t), \tag{13}$$

where:

—L is the road section length ($L = 4$ km).

—$W(x,t)$ represents Brownian motion.

7. NUMERICAL SOLUTION OF THE SPDE

Using the central difference technique:

$$\frac{\rho(x+\Delta x,t) - \rho(x-\Delta x,t)}{2\Delta x} + \frac{Q(x+\Delta x,t) - Q(x-\Delta x,t)}{2\Delta x} + W(x,t) = 0. \tag{14}$$

Rewriting:

$$\frac{\rho(x+\Delta x,t) + \rho(x-\Delta x,t)}{2} = \frac{Q(x+\Delta x,t) - Q(x-\Delta x,t)}{\Delta x} + W(x,t). \tag{15}$$

Applying Godunov's numerical scheme:

$$\rho_i^{j+1} = \rho_i^j + \Delta t \left[Q_i^{j+0.5} - Q_{i-1}^{j+0.5} \right] + W(x,t). \tag{16}$$

where, $Q_{i-0.5}^j$ and $Q_{i+0.5}^j$ are determined using the Riemann solver. The solution of the stochastic partial differential equation (SPDE) is computed using Godunov's numerical scheme, incorporating randomness by generating normally distributed random numbers. The root mean square error is then estimated between the numerical and analytical solutions of the SPDE over multiple iterations. An error margin of 10% is considered acceptable between the deterministic and stochastic partial differential equations. The key parameters for the deterministic partial differential equation and stochastic partial differential equations are density, velocity and flow. Figure 2 shows the density profile for both the deterministic and stochastic models. The profile is generated for a 4 km section of road, with density observed over a two-hour period. The maximum density reaches 800 vehicles/km. The results indicate that density peaks between the 1 km and 2 km positions, after which the vehicle flow decreases and transitions into normal traffic conditions. Similarly, the right-hand figure in Figure 1 shows the numerically calculated density using a stochastic partial differential equation, which exhibits similar behavior and closely approximates the deterministic density profile. The Figure 3 shows the deterministic velocity (left) and stochastic velocity (right) and shows the stochastic velocity profile approximated to the deterministic velocity profile with random effect of the model, Figure 4 shows the deterministic (left) traffic flow profile and stochastic (right) traffic flow profile and shows the stochastic traffic flow profile is most approximate to the deterministic traffic flow model. Figure 5 shows the root mean square error (RMSE)

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{x}_i)^2} \tag{17}$$

where:

—x_i represents the actual (observed) traffic flow values (e.g., from the deterministic model or real-world data).

—$\hat{x}_i$ represents the predicted traffic flow values (e.g., from the stochastic model).

—N is the total number of observations.

and number of iterations on all the paths as on path S-1 the root mean square error between the deterministic and stochastic partial differential equation is 11 while applying the 52 number of iterations and by increasing or decreasing the number of iterations the error may be increase or may be decrease because of random generation of stochasticity into the stochastic traffic flow mode and its proposed solution, similarly the Figure 4 may be referred for the root mean square and number of iterations for all the paths.

8. RESULTS AND DISCUSSION

The results of this study provide a comprehensive understanding of traffic dynamics under both deterministic and stochastic frameworks. Numerical simulations using the Godunov scheme reveal that while the stochastic model closely follows deterministic trends, it also captures real-world uncertainties such as irregular driver behavior and unpredictable road conditions. The comparison of density and velocity profiles shows strong correlations between the two approaches, with stochastic variations introducing realistic fluctuations. These findings confirm that the stochastic model successfully preserves the fundamental traffic dynamics while enhancing adaptability to real-world conditions.

Furthermore, the analysis of traffic flow patterns and root mean square error (RMSE) evaluation highlights the model's reliability. Both deterministic and stochastic flow models exhibit similar trajectories, with the stochastic model incorporating minor fluctuations that account for random traffic variations. RMSE analysis indicates an error of approximately 11 after 52 iterations, emphasizing the importance of optimizing iteration counts to balance accuracy and computational efficiency. These results demonstrate the practical applicability of the stochastic model in traffic prediction, making it a valuable tool for real-world traffic flow analysis.

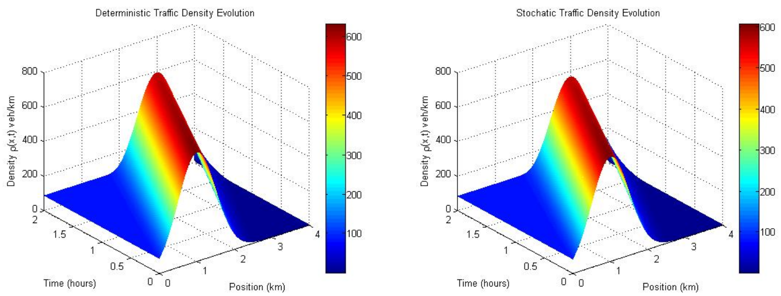

Fig. 3: Deterministic (left) and Stochastic (right) density profile

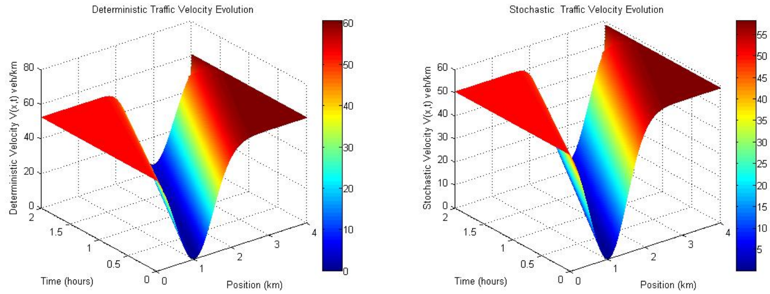

Fig. 4: Deterministic (left) and Stochastic (right) Velocity profile

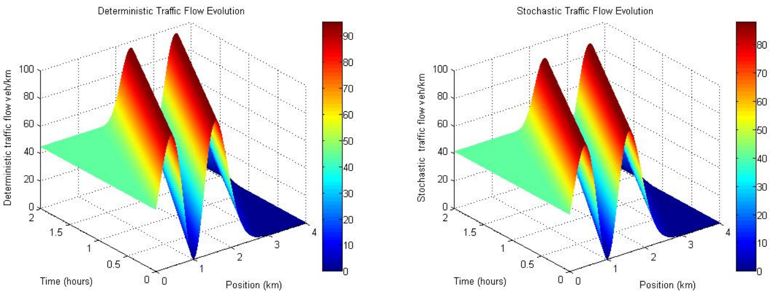

Fig. 5: Deterministic (left) and Stochastic (right) density profile

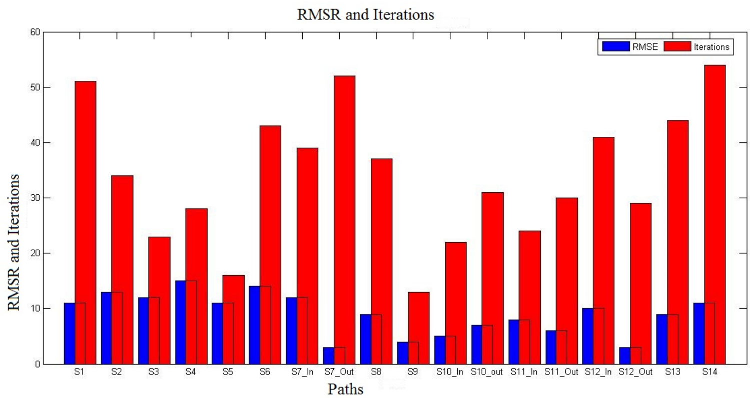

Fig. 6: Root mean square error (RMSE) and number of iteration

9. CONCLUSION

Traffic flow analysis remains an essential area of research due to its implications for urban planning and environmental sustainability. This study highlights the critical role of stochastic factors, particularly driver behavior and random fluctuations, in influencing traffic dynamics. By employing a macroscopic approach to model flow-density relations, both deterministic and stochastic scenarios were analyzed. The numerical implementation using Godunov's method effectively incorporated stochastic variability, demonstrating its applicability to real-world traffic conditions. The results emphasize that incorporating stochastic elements into traditional traffic models can enhance their accuracy and predictive capabilities, offering valuable insights for addressing congestion and optimizing traffic systems. The study's limitations include idealized assumptions, the exclusion of microscopic interactions, and computational challenges. Future work should address these by incorporating multi-lane and mixed-traffic conditions, considering external factors like weather and accidents, and utilizing machine learning for real-time traffic prediction. Experimental validation with large-scale data will further improve the model's applicability.

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