

Analytic computation of digamma function using some new identities

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Abstract

Motivated by the substantial development in the theory of digamma function, we derive some new identities for the digamma function. These new identities enable us to compute the values of the digamma function for fractional orders in an analogous manner. Also, we mention two errata, found in Jensen's article (An elementary exposition of the theory of the Gamma function, 1916), and present their correct forms.

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1. INTRODUCTION AND PRELIMINARIES

A natural property of digamma (Psi)function is as application in the theory of beta distributions-probability models for the domain $[0,1]$. It is used mainly in the theory of special functions in wide range of applications. Digamma functions are directly connected with many special functions such as Riemann's zeta function and Clausen's function etc.

Many authors have contributed to develop the theory of polygamma function with respect to properties [25; 9; 13; 14; 16], inequalities [2; 3; 6], monotonicity [21; 22; 23; 24], series [5; 7; 15; 27; 10; 12], and fractional calculus [1; 19; 20].

The Gamma function, $\Gamma(x)$, was introduced by Leonard Euler as a generalization of the factorial function on the sets, $\mathbb{R}$ of all real numbers, and $\mathbb{C}$ of all complex numbers. It (or, Euler's integral of second kind) is defined by

$$\begin{aligned}\Gamma(z) &= \int_0^{\infty} \exp(-t)t^{z-1}dt, & \Re(z) > 0 \\ &= \lim_{n \rightarrow \infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}dt.\end{aligned}\tag{1.1}$$

In 1856, Karl Weierstrass gave a novel definition of gamma function

$$\frac{1}{\Gamma(z)} = z \exp(\gamma z) \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \right], \quad (1.2)$$

where $\gamma = 0.577215664901532860606512090082402431042\dots$, is called Euler-Mascheroni constant, and $\frac{1}{\Gamma(z)}$ is an entire function of z , and

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln(n) \right).$$

The function

$$\psi(z) = \frac{d}{dz} \{ \ln \Gamma(z) \} = \frac{\Gamma'(z)}{\Gamma(z)}, \quad (1.3)$$

or, equivalently

$$\ln \Gamma(z) = \int_1^z \psi(\zeta) d\zeta, \quad (1.4)$$

is the logarithmic derivative of the gamma function or digamma function.

$\psi^{(i)}(z)$ for $i \in \mathbb{N}$ are called the polygamma functions, and ψ has the presentation as

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} dt \quad (\gamma = \text{Euler's constant}). \quad (1.5)$$

The Psi function has following series representation

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)}, \quad z \neq -1, -2, -3, \dots \quad (1.6)$$

In 1813, Gauss [9] (see also, Jensen [13, p.146, eq.(32)]; [8, p.19, (1.7.3) eq.(29)], Böhmer [4, p.77]) discovered an interesting formula for digamma (Psi) function as follows

$$\psi(p/q) = -\gamma - \ln(q) - \frac{\pi}{2} \cot\left(\frac{\pi p}{q}\right) + \sum_{j=1}^{\left[\frac{q}{2}\right]'} \left\{ \cos\left(\frac{2\pi j p}{q}\right) \ln\left(2 - 2 \cos \frac{2\pi j}{q}\right) \right\}, \quad (1.7)$$

where $1 \leq p < q$ and p, q are positive integers, and accent(prime) to right of the summation sign indicates the term corresponding to (last term) $j = \frac{q}{2}$ (when q is positive even integer) should be divided by 2.

A different form of Gauss formula is also given in N. Nielsen [18, p. 22, an equation between equations (7) and (8)] as follows

$$\psi(p/q) = -\gamma - \ln(q) - \frac{\pi}{2} \cot\left(\frac{\pi p}{q}\right) + \sum_{j=1}^{q-1} \left\{ \cos\left(\frac{2\pi p j}{q}\right) \ln\left(2 \sin\left(\frac{\pi j}{q}\right)\right) \right\}, \quad (1.8)$$

where $1 \leq p < q$ and p, q are positive integers.

Afterwards, in 2007, an attempt was made by Murty and Saradha [17, p. 300, after eq.(4)] (see also, Lehmer [14, p. 135, after eq.(20)]) for the simplification of the above Gauss formula (1.7) as follows

$$\psi(p/q) = -\gamma - \ln(2q) - \frac{\pi}{2} \cot\left(\frac{\pi p}{q}\right) + 2 \sum_{j=1}^{\left[\frac{q}{2}\right]} \left\{ \cos\left(\frac{2\pi p j}{q}\right) \ln \sin\left(\frac{\pi j}{q}\right) \right\}, \quad (1.9)$$

where $p = 1, 2, 3, \dots, (q-1)$, $q = 2, 3, 4, \dots$; $(p, q) = 1$.

Also, we have verified the results (1.7), (1.8) and (1.9) by taking different values of p and q .

Gradshteyn and Ryzhik [11, p. 904, eq 8.363(6)] recorded an erroneous formula for digamma function such that

$$\psi(p/q) \doteq -\gamma - \ln(2q) - \frac{\pi}{2} \cot\left(\frac{\pi p}{q}\right) + 2 \sum_{j=1}^{\left[\frac{q+1}{2}\right]-1} \left\{ \cos\left(\frac{2\pi p j}{q}\right) \ln \sin\left(\frac{\pi j}{q}\right) \right\}, \quad (1.10)$$

where $p = 1, 2, 3, \dots, (q-1)$, $q = 2, 3, 4, \dots$; $(p, q) = 1$ and the symbol $\doteq$ exhibits the fact that equation (1.10) does not hold true as stated.

Some important facts, which appreciate us to work in this direction, are as follows

- We cannot compute the value of digamma function when $p > q$ or (and) $\frac{p}{q}$ is negative fraction using Gauss formula [9].
- We cannot compute the value of digamma function when $p > q$ using Jensen formula [13].
- We cannot compute the value of digamma function when $\frac{p}{q}$ is negative using Jensen [13].
- Murty and Saradha [17, p. 300] corrected a formula of Lehmer [14, p. 135] for $\psi\left(\frac{p}{q}\right)$
- Some specific values of digamma function were proved transcendental by Murty and Saradha [17].

2. SOME NEW IDENTITIES FOR DIGAMMA FUNCTION

Some functional relations for digamma function, that are easily derivable from the properties of the gamma function, are recalled here. Indeed, from the formula

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \neq 0, \pm 1, \pm 2, \pm 3, \dots \quad (2.1)$$

taking ℓn both sides and differentiating the above equation with respect to z , we derive the some basic identities for digamma function as follows

$$\psi(z+1) = \psi(z) + \frac{1}{z}, \quad \psi(1-z) = \psi(z) + \pi \cot(\pi z), \quad z \neq 0, \pm 1, \pm 2, \pm 3, \dots \quad (2.2)$$

$$\psi(z+n) = \frac{1}{z} + \frac{1}{z+1} + \dots + \frac{1}{z+n-1} + \psi(z). \quad (2.3)$$

On setting $z = (1-z)$ in equation (2.2), we get

$$\psi(-z) = \frac{1}{z} + \psi(1-z). \quad (2.4)$$

On comparing the values of $\psi(1-z)$ from the equations (2.2) and (2.4), we get a new identity

$$\psi(z) + \pi \cot(\pi z) = \psi(-z) - \frac{1}{z}. \quad (2.5)$$

By setting $z = \frac{p}{q}$, $1 \leq p < q$ in equations (2.2) and (2.4), we get more identities. These identities, enable us to derive our main identities, are as follows

$$\psi\left(\frac{p+q}{q}\right) = \frac{q}{p} + \psi\left(\frac{p}{q}\right), \text{ and } \psi\left(\frac{-p}{q}\right) = \frac{q}{p} + \psi\left(\frac{q-p}{q}\right), \quad 1 \leq p < q. \quad (2.6)$$

For the sake of convenient computation of digamma function, we derive some more identities, which are simple but more applicable in the computation of digamma function for $\frac{p}{q} > 1$. For this concern, we connect the Murty and Saradha's formula for digamma function (1.11) with our above identity (2.6) and get the result as follows

$$\psi\left(\frac{q-p}{q}\right) = -\gamma - \ell n(2q) + \frac{\pi}{2} \cot\left(\frac{\pi p}{q}\right) + 2 \sum_{j=1}^{\left[\frac{q}{2}\right]} \left\{ \cos\left(\frac{2\pi p j}{q}\right) \ell n \sin\left(\frac{\pi j}{q}\right) \right\}, \quad (2.7)$$

$(p, q) = 1$; $1 \leq p < q$. Now, we derive the identity for computation of the digamma function for negative fractions $\left(-\frac{p}{q}\right)$. For this motive, we derive the identity in the similar manner as used in the above identity and get the result as follows

$$\psi\left(\frac{-p}{q}\right) = \frac{q}{p} - \gamma - \ell n(2q) - \frac{\pi}{2} \cot\left(\frac{\pi(q-p)}{q}\right) + 2 \sum_{j=1}^{\left[\frac{q}{2}\right]} \left\{ \cos\left(\frac{2\pi(q-p)j}{q}\right) \ell n \sin\left(\frac{\pi j}{q}\right) \right\}, \quad (2.8)$$

$$1 \leq p < q.$$

3. NUMERIC COMPUTATIONS OF DIGAMMA FUNCTION

Table I. ψ - Function(Fractional Valued, $p > q$)

Ser. No.	$z = \frac{p}{q}$	$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$
1	$\frac{7}{3}$	$-\gamma + \frac{15}{4} - \frac{\pi\sqrt{3}}{6} - \frac{3}{2}\ell n 3$
2	$\frac{3}{2}$	$-\gamma + 2 - 2\ell n 2$
3	$\frac{5}{2}$	$-\gamma + \frac{8}{3} - 2\ell n 2$

Table II. ψ - Function(Positive fractional Order)

Ser. No.	$z = \frac{p}{q}$	$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$
1	$\frac{1}{2}$	$-\gamma - 2\ell n 2$
2	$\frac{1}{3}$	$-\gamma - \frac{\sqrt{3}\pi}{6} - \frac{3}{2}\ell n 3$
3	$\frac{1}{4}$	$-\gamma - \frac{\pi}{2} - 3\ell n 2$
4	$\frac{1}{5}$	$-\gamma - \ell n 10 - \left(\frac{1+\sqrt{5}}{\sqrt{(10-2\sqrt{5})}} \right) \frac{\pi}{2} + \frac{1}{2} \{ \sqrt{5}\ell n \left(\frac{\sqrt{5}-1}{2} \right) - \ell n \frac{\sqrt{5}}{4} \}$
5	$\frac{1}{6}$	$-\gamma - \ell n 12 - \frac{\pi\sqrt{3}}{2} - \ell n \sqrt{3}$
6	$\frac{1}{8}$	$-\gamma - \frac{(1+\sqrt{2})\pi}{2} - 4\ell n 2 - \sqrt{2}\ell n (1+\sqrt{2})$
7	$\frac{1}{10}$	$-\gamma - \ell n 20 - \left(\frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1} \right) \frac{\pi}{2} + \frac{1}{2} \{ \sqrt{5}\ell n (\sqrt{5}-2) - \ell n \sqrt{5} \}$
8	$\frac{1}{12}$	$-\gamma - \ell n 24 - (2+\sqrt{3}) \frac{\pi}{2} + \{ \sqrt{3}\ell n (2-\sqrt{3}) - \ell n \sqrt{3} \}$
9	$\frac{2}{3}$	$-\gamma + \frac{\sqrt{3}\pi}{6} - \frac{3}{2}\ell n 3$
10	$\frac{2}{5}$	$-\gamma - \ell n 10 - \left(\frac{\sqrt{5}-1}{\sqrt{(10+2\sqrt{5})}} \right) \frac{\pi}{2} + \frac{1}{2} \{ \sqrt{5}\ell n \left(\frac{\sqrt{5}+1}{2} \right) - \ell n \frac{\sqrt{5}}{4} \}$
11	$\frac{3}{4}$	$-\gamma + \frac{\pi}{2} - 3\ell n 2$
12	$\frac{3}{5}$	$-\gamma - \ell n 10 + \left(\frac{\sqrt{5}-1}{\sqrt{(10+2\sqrt{5})}} \right) \frac{\pi}{2} + \frac{1}{2} \{ \sqrt{5}\ell n \left(\frac{\sqrt{5}+1}{2} \right) - \ell n \frac{\sqrt{5}}{4} \}$
13	$\frac{3}{8}$	$-\gamma - \frac{(\sqrt{2}-1)\pi}{2} - 4\ell n 2 + \sqrt{2}\ell n (1+\sqrt{2})$
14	$\frac{3}{10}$	$-\gamma - \ell n 20 - \left(\frac{\sqrt{(10-2\sqrt{5})}}{1+\sqrt{5}} \right) \frac{\pi}{2} + \frac{1}{2} \{ \sqrt{5}\ell n (2+\sqrt{5}) - \ell n \sqrt{5} \}$
15	$\frac{4}{5}$	$-\gamma - \ell n 10 + \left(\frac{1+\sqrt{5}}{\sqrt{(10-2\sqrt{5})}} \right) \frac{\pi}{2} + \frac{1}{2} \{ \sqrt{5}\ell n \left(\frac{\sqrt{5}-1}{2} \right) - \ell n \frac{\sqrt{5}}{4} \}$
16	$\frac{5}{6}$	$-\gamma - \ell n 12 + \frac{\pi\sqrt{3}}{2} - \ell n \sqrt{3}$
17	$\frac{5}{8}$	$-\gamma + \frac{(\sqrt{2}-1)\pi}{2} - 4\ell n 2 + \sqrt{2}\ell n (1+\sqrt{2})$
18	$\frac{5}{12}$	$-\gamma - \ell n 24 - (2-\sqrt{3}) \frac{\pi}{2} + \{ \sqrt{3}\ell n (2+\sqrt{3}) - \ell n \sqrt{3} \}$
19	$\frac{7}{8}$	$-\gamma + \frac{(1+\sqrt{2})\pi}{2} - 4\ell n 2 - \sqrt{2}\ell n (1+\sqrt{2})$
20	$\frac{7}{10}$	$-\gamma - \ell n 20 + \left(\frac{\sqrt{(10-2\sqrt{5})}}{1+\sqrt{5}} \right) \frac{\pi}{2} + \frac{1}{2} \{ \sqrt{5}\ell n (2+\sqrt{5}) - \ell n \sqrt{5} \}$
21	$\frac{7}{12}$	$-\gamma - \ell n 24 + (2-\sqrt{3}) \frac{\pi}{2} + \{ \sqrt{3}\ell n (2+\sqrt{3}) - \ell n \sqrt{3} \}$
22	$\frac{9}{10}$	$-\gamma - \ell n 20 + \left(\frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1} \right) \frac{\pi}{2} + \frac{1}{2} \{ \sqrt{5}\ell n (\sqrt{5}-2) - \ell n \sqrt{5} \}$
23	$\frac{11}{12}$	$-\gamma - \ell n 24 + (2+\sqrt{3}) \frac{\pi}{2} + \{ \sqrt{3}\ell n (2-\sqrt{3}) - \ell n \sqrt{3} \}$

Table III. ψ - Function(Negative fractional Valued)

Ser. No.	$z = \frac{p}{q}$	$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$
1	$-\frac{2}{3}$	$-\gamma + \frac{3}{2} - \frac{\pi\sqrt{3}}{6} - \frac{3\ln 3}{2}$
2	$-\frac{3}{4}$	$-\gamma + \frac{4}{3} - \frac{\pi}{2} - 3\ln 2$
3	$-\frac{1}{2}$	$-\gamma + 2 - 2\ln 2$
4	$-\frac{1}{3}$	$-\gamma + 3 + \frac{\sqrt{3}\pi}{6} - \frac{3}{2}\ln 3$
5	$-\frac{1}{4}$	$-\gamma + 4 + \frac{\pi}{2} - 3\ln 2$
6	$-\frac{5}{8}$	$-\gamma + \frac{8}{5} - \frac{(\sqrt{2}-1)\pi}{2} - 4\ln 2 + \sqrt{2}\ln(1+\sqrt{2})$
7	$-\frac{3}{8}$	$-\gamma + \frac{8}{3} + \frac{(\sqrt{2}-1)\pi}{2} - 4\ln 2 + \sqrt{2}\ln(1+\sqrt{2})$
8	$-\frac{1}{8}$	$-\gamma + 8 + \frac{(1+\sqrt{2})\pi}{2} - 4\ln 2 - \sqrt{2}\ln(1+\sqrt{2})$
9	$-\frac{5}{6}$	$-\gamma + \frac{6}{5} - \frac{\pi\sqrt{3}}{2} - \frac{3}{2}\ln 3 - 2\ln 2$
10	$-\frac{3}{2}$	$-\gamma + \frac{8}{3} - 2\ln 2$
11	$-\frac{7}{3}$	$-\gamma + \frac{117}{28} + \frac{\pi\sqrt{3}}{6} - \frac{3}{2}\ln 3$

The following errata are found in a paper of Jensen [13, p. 147] such that

$$\psi(3/5) \doteq -\gamma + \frac{\pi}{2} \sqrt{\left(1 - \frac{2}{\sqrt{5}}\right)} - \frac{5}{4}\ln 5 + \frac{\sqrt{5}}{4}\ln \left(\frac{3+2\sqrt{5}}{2}\right), \quad (3.1)$$

$$\psi(4/5) \doteq -\gamma + \frac{\pi}{2} \sqrt{\left(1 + \frac{2}{\sqrt{5}}\right)} - \frac{5}{4}\ln 5 - \frac{\sqrt{5}}{4}\ln \left(\frac{3+2\sqrt{5}}{2}\right), \quad (3.2)$$

where the symbol $\doteq$ exhibits the fact that each of the equations (3.1) and (3.2) does not hold true as stated.

The following are the corrected forms of above equations

$$\psi(3/5) = -\gamma + \frac{\pi}{2} \sqrt{\left(1 - \frac{2}{\sqrt{5}}\right)} - \frac{5}{4}\ln 5 + \frac{\sqrt{5}}{4}\ln \left(\frac{3+\sqrt{5}}{2}\right), \quad (3.3)$$

$$\psi(4/5) = -\gamma + \frac{\pi}{2} \sqrt{\left(1 + \frac{2}{\sqrt{5}}\right)} - \frac{5}{4}\ln 5 - \frac{\sqrt{5}}{4}\ln \left(\frac{3+\sqrt{5}}{2}\right). \quad (3.4)$$

4. CONCLUDING REMARK

We conclude our present investigation by observing that some new identities for digamma function have been deduced as the equations (2.6), (2.7) and (2.8) in an analogous manner. Using these new identities, we have calculated the values of digamma function for positive and negative fractional orders. Also, we have presented correct forms of two errata found in Jensen's article [13, p. 147].

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