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Original Study

Further characteristics for certain newly formed solutions for two significant mathematical models by utilization of an efficient semi-analytic method

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Abstract

This study analyses the details of two mathematical models: the (2+1)-dimensional nonlinear time-fractional Hirota-Maccari (HM) model, which includes a novel fractional operator called the β -derivative, and the (3+1)-dimensional Korteweg-de Vries-type equation. Two different forms of the wave transformations are utilized to convert the studied models into nonlinear differential equations. The improved version of the generalized Kudryashov method (IGKhM), known for its efficiency, is applied for the first time to investigate studied models. Various computational programs provide essential visual representations, making them invaluable in academic and practical applications. The construction of traveling waves and singular solitons is achieved using exponential and hyperbolic trigonometric functions, which exhibit characteristics of darkness, brightness, and mixed states. In two- and three-dimensional representations, numerical simulations of the outcomes are incorporated to better express and understand the physical significance of the results. Every solution obtained is new and distinctive compared to previous studies. Additionally, the results are individually substituted into the corresponding equations and satisfy the conditions.

Keywords: The (3+1)-dimensional Korteweg-de Vries-type equation, Time-fractional Hirota-Maccari system, Beta derivative, soliton solutions, improved generalized Kudryashov method.

AMS 2020 codes: 45Kxx; 65Mxx; 35C07; 35Dxx.

1 Introduction

Partial differential equations (PDEs) play a central role in the modeling and simulation of diverse physical systems. Recent advancements in nonlinear science have led to the development of multidimensional and fractional-order PDEs to address more complex phenomena. Two such significant models are the (3+1)-dimensional Korteweg-de Vries (KdV)-type equation and the (2+1)-dimensional time-fractional HM system incorporating a beta derivative. These equations have seen growing attention due to their relevance in fluid dynamics, nonlinear optics, and plasma physics. Nonlinear dynamical structures have a significant role in mathematical physics. Recent developments in physics, including astronomy, computation science, fluid mechanics, statistical physics, physiological physics, and medical physics, often describe the underlying

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principles investigated by other scientific disciplines and propose new routes of inquiry in academic fields such as mathematics. Environment, scientific difficulties, and a majority of life and complex processes are clarified and developed by fractional wave models. Hirota in 1973 presented Hirota's equation, and he got several forms of solitaire solutions [1]. Maccari then enhanced the Hirota equation towards two dimensions of space. The Kadomtsev-Petviashvili equation's Lax pairing was examined using the elimination approach. By demonstrating the corresponding Lax pair, the integrability characteristic of the equation that followed was established. This newly generated equation, known as HM system, was introduced by Maccari in 1998. The presence of the associated Lax pair confirms that the HM framework is an integrable PDE [2]. The complex time-fractional HM system with β -derivative in (2+1)-dimensions is given by

$$\begin{aligned} i \frac{\partial u^\beta(x,y,t)}{\partial t} + i \frac{\partial^3 u(x,y,t)}{\partial x^3} + \frac{\partial^2 u(x,y,t)}{\partial x \partial y} - i |u(x,y,t)|^2 \frac{\partial u(x,y,t)}{\partial x} + u(x,y,t)v(x,y,t) &= 0, \\ 3 \frac{\partial v(x,y,t)}{\partial x} + \frac{\partial |u(x,y,t)|^2}{\partial y} &= 0, \text{ where } i = \sqrt{-1}. \end{aligned} \quad (1)$$

This setup (1) illustrates the dynamic characteristics of the ultra-fast soliton pulses in fibers with a single mode. The dependent variables are represented by the functions $u = u(x,y,t)$ and $v = v(x,y,t)$, where u and v are complex and real coordinate fields. The independent location's variables consist of the variables x and y . The temporal variable is represented by the symbol t . When the value of y is equal to x , and $\beta = 1$, the time fractional HM system (1) transforms and becomes the (1+1)-dimensional Hirota equation:

$$i \frac{\partial u(x,t)}{\partial t} + i \frac{\partial^3 u(x,t)}{\partial x^3} + \frac{\partial^2 u(x,t)}{\partial x \partial x} - \frac{1}{3} |u(x,t)|^2 u(x,t) - i |u(x,t)|^2 \frac{\partial u(x,t)}{\partial x} = 0.$$

Researching the HM system seeking to achieve accurate and analytic solutions became intriguing work for many scholars all over the globe. Many various sorts of strategies have been developed to show the concepts linked to the trigonometric and hyperbolic trigonometric functions [3], the trial equation approach has been implemented to generate solitary waves [4], the Kadomtsev-Petviashvili-hierarchy reduction method [5], the $\sin - \cos$, He's semi-inverse, and Riccati-Bernoulli sub-ODE approaches [6], the method of planar dynamics system [7], the improved tanh scheme, and auxiliary equation approach [8], the Jacobi elliptic function expansion approach [9, 10], the new version of the modified Kudryashov's, and the auxiliary equation techniques [11], the simple ansatz technique [12], the Nucci's reduction method applied to the (2+1)-dimensional Korteweg-de Vries equation [13], the Bilinear form and N -soliton solutions with higher-order breather solutions [14], the improved sub-equation method [15], the sine-Gordon expansion method [16], the (G'/G^2) -expansion method and the generalized (G'/G) -expansion method similar in reasoning to the M-fractional sub-equation methods applied to the same HM model [17], the Bernoulli sub-equation function method applied to the Caudrey-Dodd-Gibbon-Sawada-Kotera model [18], and so many existing references therein. Korteweg-de Vries-type equation in (3+1)-dimensions has the following structure:

$$\frac{\partial}{\partial x} \left(\beta_1 \frac{\partial u}{\partial x} + \beta_2 \frac{\partial u}{\partial y} + \beta_3 \frac{\partial u}{\partial z} \right) + \frac{\partial}{\partial x} \left(\frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} \right) + \beta_4 \frac{\partial^2 u}{\partial t^2} = 0. \quad (2)$$

The above statements clarify the central function of the analytic and semi-analytic methods in solving mathematics, physical, and engineering models. So after different methods with many improvement forms have been devoted to mentioning, the $(\frac{G'}{G'+G+A})$ technique, and the modified rational $\sin - \cos$ approach [19], the $\exp(-\Phi(\Xi))$ expansion method and addendum to Kudryashov's method [20], the improved exponential rational function method, the improved version of Bernoulli and the modified arrangements of the hyperbolic trigonometric function methods [21], the Homotopy analysis method and symbolic computation [22], the modified Sardar subequation function approach [23], the well-known asymptotic perturbation method [24], the improved version of the Bernoulli sub-equation function method [25], the natural transform decomposition technique [26], the space-time generalized finite difference scheme [27], the energy balance approach with new mapping method [28], the Lie symmetry analysis [29], the modified semi-separation of variables methods [30],

the use of the residual power series approach [31], the methods that used the hyperbolic trigonometric functions [32].

The $u(\xi) = u(x, y, z, t)$ is a scalar field representing wave motion, and β_i for $i = 1, \dots, 4$ are parameters accounting for directional dispersion and nonlinear effects. The studied model is used to investigate the evolution of nonlinear long waves in channels and coastal regions. Additionally, it describes ion-acoustic waves in magnetized plasma and is pertinent in fiber optics, where multidimensional effects cannot be overlooked. Different forms of equation (2) have been studied with some semi-analytical methods such as the Hirota bi-linear transformation and Riemann theta function [33], the Lie symmetry reductions Kudryashov's method based on ansatz, Noether's theorem relating symmetries to conservation [34], the Hirota method, through the binary Bell polynomial technique, has been utilized to explore N-soliton, breather, and periodic-wave solutions, along with their asymptotic properties [35]. The development of exact and approximate solution methods for nonlinear evolution equations and fractional differential equations has been one of the most active areas in applied mathematics and theoretical physics over the past two decades. In the realm of fractal calculus, use the method of separation of variables for finding the analytical solution, and the Grünwald-Letnikov approximation to dynamic characteristics of the fractional optimal control problem is defined in terms of the left and right Riemann-Liouville fractional derivatives [36]. A heat transfer problem defined by the Caputo-Fabrizio derivative [37] has been addressed in an axially symmetric cylindrical region where Laplace and finite Hankel integral transforms were applied, Özdemir [38] uses the generalized projective Riccati equations method (GPREM) to extract dark soliton and trigonometric traveling wave solutions of the fractional (4+1)-dimensional Fokas equation with the M-truncated derivative. Complementing these analytical approaches, a foundational result in the mathematical theory of physical systems is presented in [39], where exact structural properties were investigated using established analytical methods within the framework of mathematical physics. More recently, Wang et al. [40] introduced the novel fractional sub-equation neural network method, merging the expressive power of deep learning with fractional calculus to derive exact solutions of space-time fractional PDEs, marking a significant step toward data-driven mathematical physics. At the interface of machine learning and exact solution theory, Zhang and Bilige [41] proposed the bilinear neural network method-utilizing neural network tensor formulas after Hirota bilinearization to obtain a broad class of exact solutions for nonlinear PDEs including fractal soliton waves, while authors in [42] investigated the third-order nonlinear (2+1)-dimensional Novikov-Veselov system with constant coefficients using an appropriate traveling-wave transformation. They employed two reliable and powerful methods, that is, the extended rational sin-cos technique and the modified exponential rational function method, using the Generalized-Kudryashov-Auxiliary-Jacobian method [43]. Finally, the results reported in [44] contribute additional exact analytical solutions in contemporary nonlinear physics, further enriching the landscape of solution methods available for physically relevant equations.

This work is organized as follows: Section 1 is designed specifically for summarizing the literature relevant to the techniques and the examined model through a brief outline. In Section 2, some fundamental concepts connected to the beta-derivative operator are reflected. The methodology of the offered technique is detailed in Section 3. The formalism of the proposed approach for finding specific semi-analytic solutions to Eq.(1) is explained in Section 4. Section 5 bridges the gap between the pure mathematics of the solutions and their real-world physical interpretation where outcomes are discussed. Finally, in Section 6, the results of the research are reported.

2 Preliminaries

The recently presented derivatives possess several attributes that were previously regarded as limits of fractional derivatives and are used to represent certain physical phenomena. Although these derivatives may not be commonly recognized as fractional derivatives, they may be seen as an appropriate extension of the traditional derivative. The β -derivative is formally defined as:

Definition 1. Let $H : [\delta, \infty) \rightarrow \mathbb{R}$ be a known function, then, the β -operator is represented as follows [45]:

$${}_0^A D_x^\beta (H) = \lim_{\Xi \rightarrow 0} \frac{H\left(x + \Xi \left(x + \frac{1}{\Gamma[\beta]}\right)^{1-\beta}\right) - H(x)}{\Xi}, \quad \forall x > \delta, \beta \in (0, 1], \quad (3)$$

if the limit of the above exists, it is said to be β -differentiable.

Here are some impressive belongings for the above characterization where assume $K(x) \neq 0$.

(i)

$${}_0^A D_x^\beta (H \circ K(x)) = K(x) {}_0^A D_x^\beta H(K(x)).$$

(ii)

$${}_0^A D_x^\beta (\gamma H(x) + \alpha K(x)) = \gamma {}_0^A D_x^\beta H(x) + \alpha {}_0^A D_x^\beta K(x), \text{ for any constants } \gamma, \alpha.$$

(iii)

$${}_0^A D_x^\beta (\sigma) = 0, \text{ for any constant } \sigma.$$

(iv)

$${}_0^A D_x^\beta (H(x)K(x)) = H(x) {}_0^A D_x^\beta (K(x)) + K(x) {}_0^A D_x^\beta (H(x)).$$

(v)

$${}_0^A D_x^\beta \left(\frac{H(x)}{K(x)} \right) = \frac{H(x) {}_0^A D_x^\beta (K(x)) - K(x) {}_0^A D_x^\beta (H(x))}{K^2(x)}.$$

(vi)

$${}_0^A D_x^\beta (H(x)) = \frac{dH(x)}{dx} \left(x + \frac{1}{\Gamma[\beta]} \right)^{1-\beta}.$$

3 General properties of the scheme

Here is the structure of the implemented method where the first operation is considered as follows:

Step 1. We are examining the following general forms of nonlinear β -differentiable temporal-fractional:

$$P\left({}_0^A D_t^\beta E, E, E_x, E_y, E_t, E_{xt}, E_{xx}, E_{yt}, Q_x, Q_y, Q_t, \dots\right) = 0, \quad (4)$$

where both of E, Q , defined as follows:

$$\begin{aligned} E(x, y, t) &= U(\xi) \exp(i\kappa\xi), \quad Q(x, y, t) = V(\xi), \\ \xi &= \gamma_1 x + \gamma_2 y - \frac{\gamma_3}{\beta} \left(t + \frac{1}{\Gamma[\beta]} \right)^\beta, \end{aligned} \quad (5)$$

and the next PDE:

$$P(E, E_x, E_y, E_z, E_t, E_{xt}, E_{xx}, E_{yt}, \dots) = 0, \quad (6)$$

for $E = E(x, y, z, t)$

$$\xi = \gamma_1 x + \gamma_2 y + \gamma_3 z - \gamma_4 t, \quad (7)$$

where $\gamma_1, \gamma_2, \gamma_3$, and γ_4 , are arbitrary non-zero parameters. Upon substituting equations (5) - (7) into equation (4) and (6) respectively, the resulting expression is as follows:

$$N(V, V', V'', U, U', U'', \dots) = 0, \quad (8)$$

and

$$N(V, V', V'', \dots) = 0, \quad (9)$$

wherein

$$V = V(L), V' = \frac{dV}{dL}, V'' = \frac{d^2V}{dL^2}, \dots,$$

and

$$U = U(L), U' = \frac{dU}{dL}, U'' = \frac{d^2U}{dL^2}, \dots.$$

Step 2. Let us consider that the solution of (8) or (9) might be communicated with the following arrangement:

$$U(L) = \frac{\omega_0 + \omega_1 T + \omega_2 T^2 + \dots + \omega_n T^n}{v_0 + v_1 T + v_2 T^2 + \dots + v_m T^m}. \quad (10)$$

Here, must be both $\omega_m, v_n \neq 0$, and they should be considered later. Also, $T(L)$ satisfies:

$$T' = T^2 - T. \quad (11)$$

By combining equation (10) with equation (11) and inserting in equation (8), the following result is obtained:

$$\Phi(T(L)) = \varepsilon_s T(L)^s + \dots + \varepsilon_1 T(L) + \varepsilon_0 = 0. \quad (12)$$

Utilizing the concepts of balance, one can come up with a formula for n , and M by comparing the highest-order derivatives with the terms of the highest degree in equation (8).

Step 3. When the parameters of $\Phi(T(L))$ were set to zero, the obtained algebraic equations was produced the next system:

$$\varepsilon_i = 0, \quad i = 0, \dots, s. \quad (13)$$

The values of $\alpha, \varpi, \omega_0, \dots, \omega_m, v_0, \dots, v_n$, and so solutions of (8) are realized if one learns how to decode (11). These matters are obtained by solving the system (13).

Step 4. The commanded solutions for equation (11) have the following manners:

$$T(L) = \frac{1}{\delta e^L + 1}, \quad 0 \neq \delta \in \mathbb{R}. \quad (14)$$

4 Outcomes of the specified method

Let us propose that equation (1) indeed has a solution that can be expressed in the following manner:

$$\left. \begin{aligned} u &= U(\xi) \exp(i\kappa\xi), \\ v &= V(\xi), \\ \xi &= \gamma_1 x + \gamma_2 y - \frac{\gamma_3}{\beta} \left(t + \frac{1}{\Gamma[\beta]} \right)^\beta. \end{aligned} \right\} \quad (15)$$

By directly applying the wave transformation formula (15) to equation (1), one can immediately obtain the following outcomes:

$$\begin{aligned} i\gamma_1^3 U^{(3)}(\xi) - 3\gamma_1^3 \kappa U''(\xi) + \gamma_2 \gamma_1 U''(\xi) - 3\gamma_1^3 \kappa^2 U'(\xi) + 2i\gamma_2 \gamma_1 \kappa U'(\xi) - i\gamma_1 U(\xi)^2 U'(\xi) \\ - i\gamma_3 U'(\xi) + \gamma_1^3 \kappa^3 U(\xi) - \gamma_2 \gamma_1 \kappa^2 U(\xi) + \gamma_1 \kappa U(\xi)^3 + \gamma_3 \kappa U(\xi) + U(\xi)V(\xi) = 0, \\ 2\gamma_2 U(\xi)U'(\xi) + 3\gamma_1 V'(\xi) = 0. \end{aligned} \quad (16)$$

The second expression in equation (16) leads us to a compelling conclusion:

$$V(\xi) = -\frac{\gamma_2}{3\gamma_1} U^2(\xi), \quad (17)$$

by substituting equation (17) into the first part of equation (16) and then simplifying mathematically, the imaginary and real components can be separated as follows:

$$\left(\gamma_1 \kappa - \frac{\gamma_2}{3\gamma_1}\right) U^3 + (\gamma_1 \gamma_2 - 3\gamma_1^3 \kappa) U'' + (\gamma_1^3 \kappa^3 - \gamma_2 \gamma_1 \kappa^2 + \gamma_3 \kappa) U = 0, \quad (18)$$

and

$$\gamma_1^3 U^{(3)} + (-3\gamma_1^3 \kappa^2 + 2\gamma_2 \gamma_1 \kappa - \gamma_3) U' - \gamma_1 U^2 U' = 0. \quad (19)$$

Take the integration of (19), the result is

$$\gamma_1^3 U'' + (-3\gamma_1^3 \kappa^2 + 2\gamma_2 \gamma_1 \kappa - \gamma_3) U - \frac{1}{3} \gamma_1 U^3 = 0. \quad (20)$$

From (20) it is possible to find out U^3 , after turning it back into (18), the following restricted equation will be obtained.

$$(2\gamma_1^2 \kappa - \gamma_2) (4\gamma_1^3 \kappa^2 - 2\gamma_2 \gamma_1 \kappa + \gamma_3) = 0. \quad (21)$$

Remark 1. The restricted condition (21) is a fixed equation that must be included in the set of algebraic equations for every case.

4.1 Implementations of IGKhM to Hirota-Maccari model

Considering the balancing principles of the nonlinear terms in equation (18) we can obtain a formula for N and M as follows:

$$N = M + 1. \quad (22)$$

The equation (22) generates consequential illustrations for the positive integers that already exist.

Case 1. In equation (22), if $M = 1$ and $N = 2$, so equation (10) becomes:

$$U(\eta) = \frac{\omega_0 + \omega_1 T + \omega_2 T^2}{v_0 + v_1 T}, \quad (23)$$

where equation (11) is considered, hence,

$$T' = (T^2 - T) \left(\frac{2\omega_2 T + \omega_1}{v_1 T + v_0} - \frac{v_1 (\omega_2 T^2 + \omega_1 T + \omega_0)}{(v_1 T + v_0)^2} \right) = \frac{\Psi(T)}{\Omega(T)}, \quad (24)$$

notice that there should be at least $\omega_2 \neq 0, v_1 \neq 0$, one can reach this:

$$T'' = \frac{\Omega(T)\Psi'(T) - \Psi(T)\Omega'(T)}{(\Omega(T))^2}. \quad (25)$$

By substituting equations (23)-(25) into equation (18), we obtain the following algebraic system of equations by setting all coefficients of the same powers of T^i to zero for $i = 0, 1, 2, \dots, 7$.

$$eq1 = 0, \dots, eq8 = 0. \quad (26)$$

The following sub-cases are decisively established through the solution of the system outlined in equation (26).

Case 1.1. Here is a list of the parameters that were obtained:

$$\gamma_2 = \frac{\sqrt{\omega_2} \sqrt{\omega_2^3 + 12\sqrt{6}v_1^3\gamma_3}}{3\sqrt{2}v_1^2}, \omega_1 = -2\omega_0 - \frac{\omega_2}{2}, \gamma_1 = \frac{\omega_2}{\sqrt{6}v_1}, \kappa = \frac{\sqrt{\frac{\omega_2^3}{2} + 6\sqrt{6}v_1^3\gamma_3}}{\omega_2^{3/2}}, v_0 = -\frac{2\omega_0 v_1}{\omega_2}. \quad (27)$$

From the equation (27), the next solution is attained:

$$u_{1,1} = \frac{\omega_2 \mathcal{A} \exp\left(\frac{i\sqrt{\frac{\omega_2^3}{2} + 6\sqrt{6}v_1^3\gamma_3}}{6\omega_2^{3/2}}\right)}{2v_1} \left(\frac{2}{\delta \exp\left(\frac{\sqrt{6}\omega_2 v_1 x + \sqrt{2}\sqrt{\omega_2} y \sqrt{\omega_2^3 + 12\sqrt{6}v_1^3\gamma_3}}{6v_1^2} - \frac{\gamma_3 \left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta}\right) + 1} - 1 \right), \quad (28)$$

where

$$\mathcal{A} = \frac{\sqrt{6}\omega_2 v_1 x + \sqrt{2}\sqrt{\omega_2} y \sqrt{\omega_2^3 + 12\sqrt{6}v_1^3\gamma_3}}{v_1^2} - \frac{6\gamma_3 \left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta}.$$

Profile of (28) where $\delta = \frac{3}{2}, \omega_1 = \frac{4}{5}, \beta = \frac{2}{3}, v_2 = \frac{5}{3}, \gamma_3 = \frac{3}{4}, t = \frac{5}{2}$, and $-5 \leq x \leq 5, -5 \leq y \leq 5$ are graphed below :

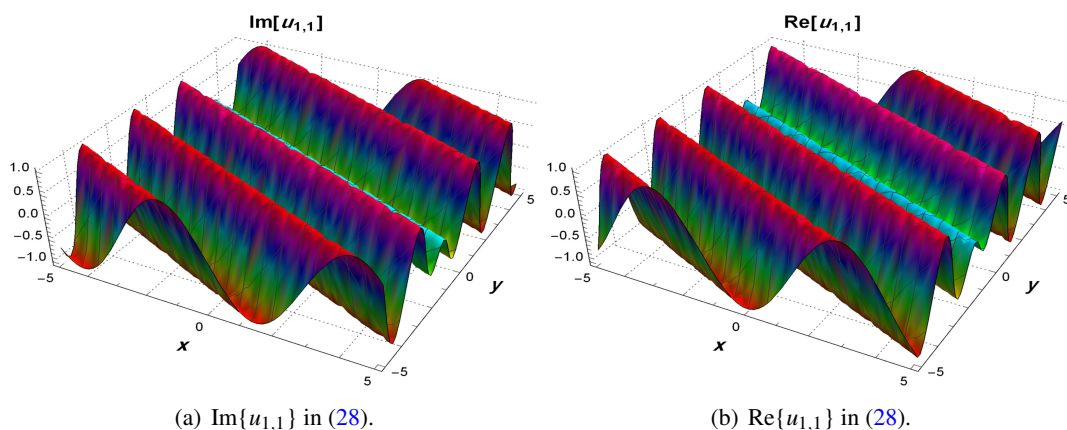


Fig. 1 Profile of (28) in three-dimensional figures.

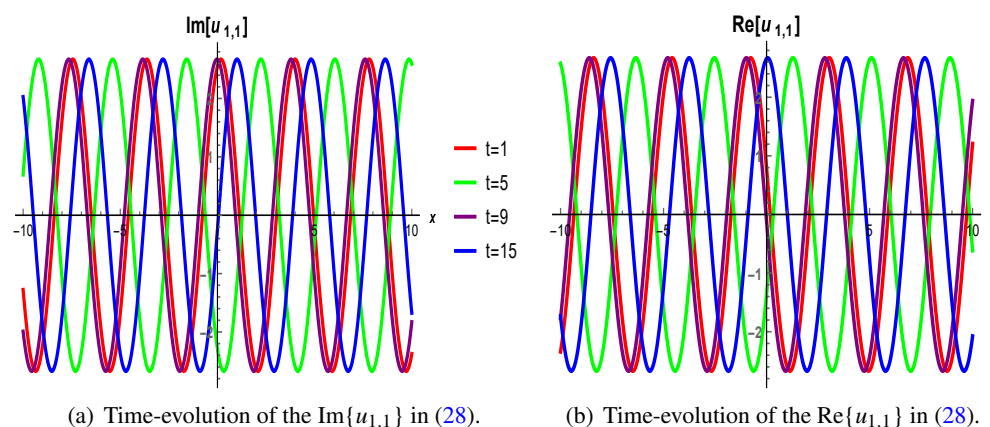


Fig. 2 Profile of (28) in two-dimensional time-evolution graphs where values of t are presented in the legend.

Case 1.2. Then following coefficients are obtained:

$$\gamma_1 = -\frac{\sqrt{\gamma_2}}{\sqrt{2}\sqrt{\kappa}}, \gamma_3 = \frac{\gamma_2^{3/2}(1-2\kappa^2)}{4\sqrt{2}\kappa^{3/2}}, \omega_0 = \frac{\sqrt{3}v_0\sqrt{\gamma_2}}{2\sqrt{\kappa}}, \omega_1 = -\frac{\omega_2}{2} - \frac{\sqrt{3}v_0\sqrt{\gamma_2}}{\sqrt{\kappa}}, v_1 = -\frac{\omega_2\sqrt{\kappa}}{\sqrt{3}\sqrt{\gamma_2}}, \quad (29)$$

with the referenced values in equation (29), the following solution has been acquired:

$$u_{1,2} = \frac{\sqrt{3}\sqrt{\gamma_2} \exp\left(\frac{i\sqrt{\gamma_2}\left(\sqrt{2}\gamma_2(2\kappa^2-1)\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta - 4\sqrt{2}\beta\kappa x + 8\beta\sqrt{\gamma_2}\kappa^{3/2}y\right)}{8\beta\sqrt{\kappa}}\right) \left(1 - \frac{2}{\delta \exp(L)+1}\right)}{2\sqrt{\kappa}}, \quad (30)$$

where

$$L = \frac{\gamma_2^{3/2}(2\kappa^2-1)\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{4\sqrt{2}\beta\kappa^{3/2}} - \frac{\sqrt{\gamma_2}x}{\sqrt{2}\sqrt{\kappa}} + \gamma_2 y.$$

Case 1.3. We will get the following coefficients:

$$\gamma_1 = -\frac{\omega_2}{\sqrt{6}v_1}, \gamma_2 = -\frac{i\omega_2^2}{2v_1^2}, \gamma_3 = \frac{\omega_2^3}{3\sqrt{6}v_1^3}, \omega_0 = 0, \omega_1 = \frac{1}{2}\omega_2\left(\frac{2v_0}{v_1}-1\right), \kappa = -i, \quad (31)$$

utilizing (31), one derives the corresponding outcome.

$$u_{1,3} = -\frac{\omega_2}{2v_0\delta + 2(v_0 + v_1) \exp\left(\frac{\sqrt{6}\omega_2^3\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta + 3\omega_2\beta v_1(\sqrt{6}v_1x + 3i\omega_2y)}{18\beta v_1^3}\right)} + \frac{\omega_2}{v_1\left(\delta + \exp\left(\frac{\sqrt{6}\omega_2^3\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta + 3\omega_2\beta v_1(\sqrt{6}v_1x + 3i\omega_2y)}{18\beta v_1^3}\right)\right)}. \quad (32)$$

Graphs of (32) where $\delta = \frac{1}{5}$, $v_0 = \frac{1}{2}$, $v_1 = \frac{3}{4}$, $\beta = \frac{7}{8}$, $\omega_2 = \frac{4}{5}$, $\gamma_3 = \frac{7}{4}$, $t = \frac{5}{2}$, and $-20 \leq x \leq 20$, $-20 \leq y \leq 20$ are presented below

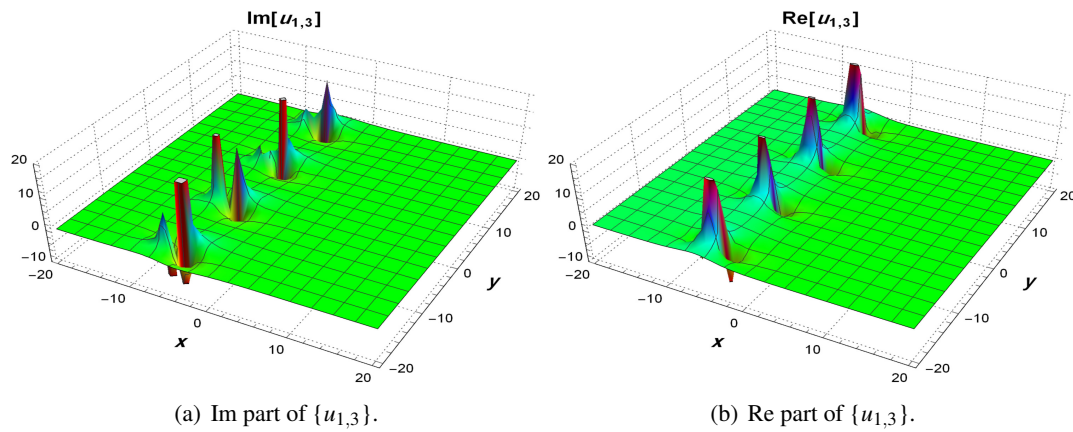


Fig. 3 Figures of (32) in three-dimensions.

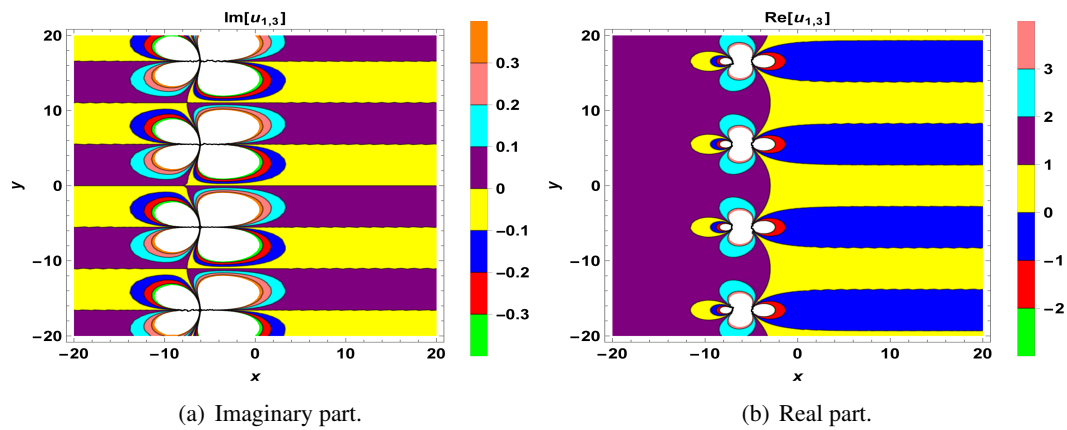


Fig. 4 Contour surfaces figures of (32).

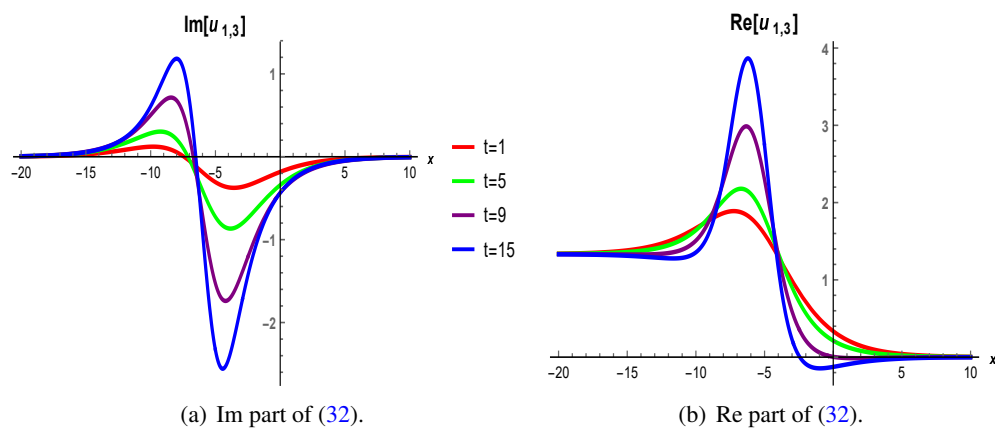


Fig. 5 Two-dimensional temporal-evolution plots (32) for given t values in the legend.

Case 2. If $M = 2$ in equation (22), then $N = 3$, so (10) is appearing with the given form:

$$U(\eta) = \frac{\omega_0 + \omega_1 T + \omega_2 T^2 + \omega_3 T^3}{v_0 + v_1 T + v_2 T^2}, \quad (33)$$

where equation (11) is considered, thereafter:

$$U' = (T^2 - T) \left(\frac{3\omega_3 T^2 + 2\omega_2 T + \omega_1}{v_2 T^2 + v_1 T + v_0} - \frac{(\omega_3 T^3 + \omega_2 T^2 + \omega_1 T + \omega_0)(2v_2 T + v_1)}{(v_2 T^2 + v_1 T + v_0)^2} \right), \quad (34)$$

there should be $\omega_3 \neq 0$, $v_2 \neq 0$, one can find this:

$$U'' = \frac{\Omega(T)\Psi'(T) - \Psi(T)\Omega'(T)}{(\Omega(T))^2}. \quad (35)$$

By appointing equations (33)-(35) into equation (18), we acquire an algebraic system of equations involving coefficients of equation (18). Henceforth, we will try to equate the coefficients of the same powers of T^i for $i = 0, 1, 2, \dots, 10$, to zero, aiming to obtain the following algebraic system:

$$eq1 = 0, \dots, eq11 = 0. \quad (36)$$

The following sub-cases have been generated using computer software programs to solve (36).

Case 2.1. We can get the following parameters:

$$\begin{aligned} \gamma_1 &= -\frac{\omega_3}{\sqrt{6}v_2}, \gamma_2 = \frac{\omega_3^2 \kappa}{3v_2^2}, \gamma_3 = \frac{\omega_3^3 (1 - 2\kappa^2)}{12\sqrt{6}v_2^3}, \omega_0 = -\frac{\omega_3 v_0}{2v_2}, \\ \omega_1 &= \frac{1}{4} \left(\omega_3 \left(\frac{4v_0}{v_2} - 1 \right) - 2\omega_2 \right), v_1 = \frac{1}{2} \left(\frac{2\omega_2}{\omega_3} + 1 \right) v_2, \end{aligned} \quad (37)$$

from (37), one generates the following result:

$$u_{2,1} = \frac{\omega_3 \exp\left(\frac{i\omega_3 \kappa}{72v_2^3} \mathcal{B}\right) \left(\frac{2}{\delta \exp\left(\frac{\omega_3}{72v_2^3} \mathcal{B}\right) + 1} - 1 \right)}{2v_2}, \quad (38)$$

where

$$\mathcal{B} = 24\omega_3 v_2 \kappa y + \frac{\sqrt{6}\omega_3^2 (2\kappa^2 - 1) \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} - 12\sqrt{6}v_2^2 x.$$

Case 2.2. One obtain the subsequent coefficients:

$$\gamma_1 = \frac{\omega_3}{\sqrt{6}v_2}, \gamma_2 = -\frac{i\omega_3^2}{2v_2^2}, \gamma_3 = -\frac{\omega_3^3}{3\sqrt{6}v_2^3}, \omega_0 = 0; \omega_2 = \frac{1}{2} \omega_3 \left(\frac{2v_1}{v_2} - 1 \right), v_0 = \frac{1}{4} (-2v_1 - v_2), \kappa = -i, \quad (39)$$

The solution is as follows when the parameters stated in the equation (39) are implemented.

$$u_{2,2} = \frac{e^{\mathcal{B}} \left(\frac{a_3 \left(\frac{2b_1}{b_2} - 1 \right)}{2(e^{\mathcal{B}} \delta + 1)^2} + \frac{a_1}{e^{\mathcal{B}} \delta + 1} + \frac{a_3}{(e^{\mathcal{B}} \delta + 1)^3} \right)}{\frac{b_1}{e^{\mathcal{B}} \delta + 1} + \frac{b_2}{(e^{\mathcal{B}} \delta + 1)^2} + \frac{1}{4} (-2b_1 - b_2)}, \quad (40)$$

where

$$\mathcal{B} = \frac{a_3^3 \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{3\sqrt{6}\beta b_2^3} + \frac{a_3 x}{\sqrt{6}b_2} - \frac{ia_3^2 y}{2b_2^2}.$$

Dynamical behavior of (40) where $\delta = \frac{1}{4}$, $v_1 = \frac{35}{2}$, $\beta = \frac{3}{4}$, $\omega_2 = \frac{6}{7}$, $\gamma_3 = \frac{3}{4}$, $t = \frac{1}{2}$, and $-10 \leq x \leq 10$, $-10 \leq y \leq 10$ are graphed below:

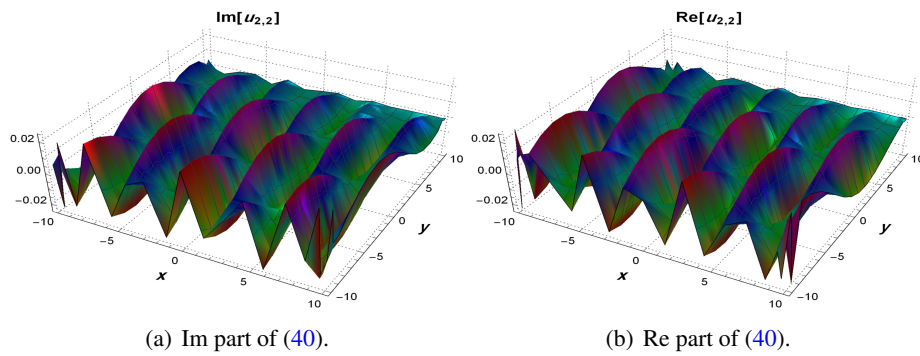


Fig. 6 Profile of (40) in 3-D .

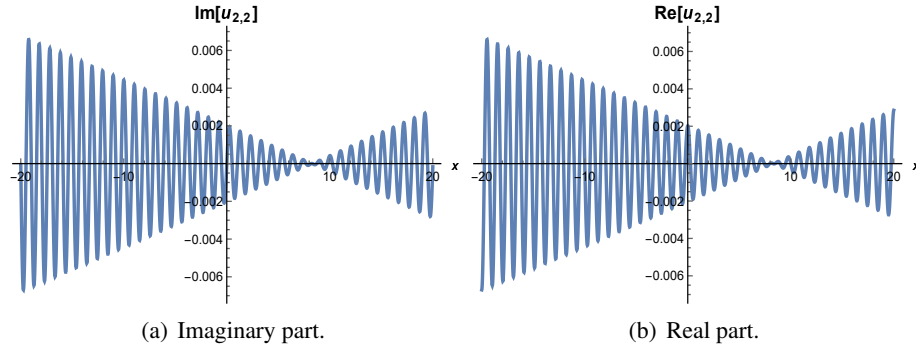


Fig. 7 Two-Dimensional graphs of (40) where $-20 \leq x \leq 20$, $y = 10$.

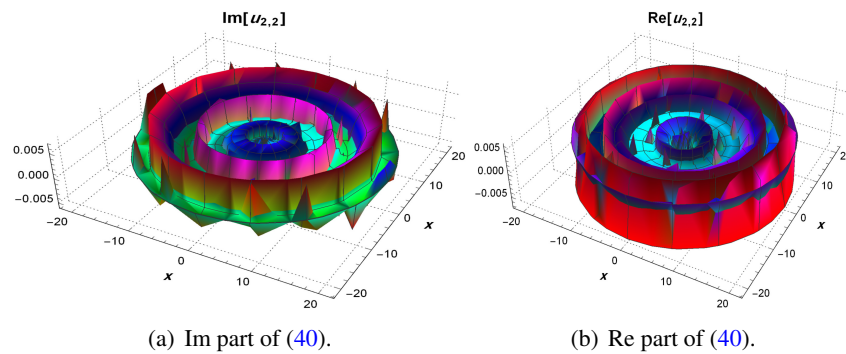


Fig. 8 Revolution plots in 3-dimensions of (40) where $y = 10$, $-20 \leq x \leq 20$.

Case 2.3. One gets the following coefficients:

$$\gamma_1 = \frac{\omega_3}{\sqrt{6}v_2}, \gamma_2 = \frac{\omega_3^2 \kappa}{2v_2^2}, \gamma_3 = \frac{\omega_3^3 \kappa^2}{3\sqrt{6}v_2^3}, v_1 = \frac{(2\omega_2 + \omega_3)v_2}{2\omega_3}, \quad (41)$$

the solution that follows will be derived from the parameters that have been identified in equation (41):

$$u_{2,3} = \frac{2\omega_3 \mathcal{L} \exp \left(\frac{i\omega_3 \kappa \left(3\beta v_2 (3\omega_3 \kappa y + \sqrt{6}v_2 x) - \sqrt{6}\omega_3^2 \kappa^2 \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta \right)}{18\beta v_2^3} \right)}{v_2 \left((2\omega_2 + \omega_3) \delta \exp \left(\frac{3\omega_3 \beta v_2 (3\omega_3 \kappa y + \sqrt{6}v_2 x) - \sqrt{6}\omega_3^3 \kappa^2 \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{18\beta v_2^3} \right) + 2\omega_2 + 3\omega_3 \right)} + 2\omega_3 v_0, \quad (42)$$

$$\left(\delta \exp \left(\frac{3\omega_3 \beta v_2 (3\omega_3 \kappa y + \sqrt{6}v_2 x) - \sqrt{6}\omega_3^3 \kappa^2 \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{18\beta v_2^3} \right) + 1 \right)^2$$

where

$$\mathcal{L} = \omega_0 + \frac{\omega_1 \left(\delta \exp \left(\frac{3\omega_3 \beta v_2 (3\omega_3 \kappa y + \sqrt{6}v_2 x) - \sqrt{6}\omega_3^3 \kappa^2 \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{18\beta v_2^3} \right) + 1 \right)^2 + \omega_3}{\left(\delta \exp \left(\frac{3\omega_3 \beta v_2 (3\omega_3 \kappa y + \sqrt{6}v_2 x) - \sqrt{6}\omega_3^3 \kappa^2 \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{18\beta v_2^3} \right) + 1 \right)^3} + \frac{\omega_2 + \left(\delta \exp \left(\frac{3\omega_3 \beta v_2 (3\omega_3 \kappa y + \sqrt{6}v_2 x) - \sqrt{6}\omega_3^3 \kappa^2 \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{18\beta v_2^3} \right) + 1 \right) + \omega_3}{\left(\delta \exp \left(\frac{3\omega_3 \beta v_2 (3\omega_3 \kappa y + \sqrt{6}v_2 x) - \sqrt{6}\omega_3^3 \kappa^2 \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{18\beta v_2^3} \right) + 1 \right)^3}.$$

4.2 Implementations of IGKhM to (3+1)-dimensional KdV-type model

Consider equation (2) has a solution in the form of

$$u(x, y, z, t) = U(\xi), \xi = \gamma_1 x + \gamma_2 y + \gamma_3 z - \gamma_4 t, \quad (43)$$

The following produces are straight away obtained by taking the wave transformation (43) into equation (2):

$$\gamma_1^4 U^{(4)} + (\beta_1 \gamma_1^2 + \beta_2 \gamma_2 \gamma_1 + \beta_3 \gamma_3 \gamma_1 + \beta_4 \gamma_4^2 - \gamma_4 \gamma_1) U'' + 6\gamma_1^2 (UU')' = 0. \quad (44)$$

The last expression in (44) after twice integration with zero constants, concludes up with:

$$\gamma_1^4 U'' + 3\gamma_1^2 U^2 + U (\beta_1 \gamma_1^2 + \beta_2 \gamma_2 \gamma_1 + \beta_3 \gamma_3 \gamma_1 + \beta_4 \gamma_4^2 - \gamma_4 \gamma_1) = 0. \quad (45)$$

Considering the balancing principles of the nonlinear terms in equation (45) we can obtain a formula for N and M as follows:

$$N = M + 2. \quad (46)$$

The equation (46) generates consequential illustrations for the positive integers that already exist.

Family 1. In (46), if $M = 1$ and $N = 2$, so (10) becomes:

$$U(\eta) = \frac{\omega_0 + \omega_1 T + \omega_2 T^2 + \omega_3 T^3}{v_0 + v_1 T}, \quad (47)$$

where equation (11) is considered, hence:

$$T' = (\phi^2 - \phi) \left(\frac{3\omega_3 \phi^2 + 2\omega_2 \phi + \omega_1}{v_1 \phi + v_0} - \frac{v_1 (\omega_3 \phi^3 + \omega_2 \phi^2 + \omega_1 \phi + \omega_0)}{(v_1 \phi + v_0)^2} \right) = \frac{\Psi(T)}{\Omega(T)}, \quad (48)$$

notice that there should be at least $\omega_3 \neq 0, v_1 \neq 0$, one can reach this:

$$T'' = \frac{\Omega(T)\Psi'(T) - \Psi(T)\Omega'(T)}{(\Omega(T))^2}. \quad (49)$$

By substituting (47)-(49) into (18), we acquire the following algebraic system of equations by correlating to zero all the coefficients of the same powers of T^i for $i = 0, 1, 2, \dots, 8$.

$$eq1 = 0, \dots, eq9 = 0. \quad (50)$$

The following sub-cases are constructed from the solving the system (50).

Family 1.1. Here is a set of the obtained parameters:

$$\begin{aligned} \gamma_1 &= -\frac{i\sqrt{\omega_3}}{\sqrt{2}\sqrt{v_1}}, \omega_1 = \frac{1}{6}\omega_3 \left(1 - \frac{6v_0}{v_1}\right), \omega_2 = \omega_3 \left(\frac{v_0}{v_1} - 1\right), \omega_0 = \frac{\omega_3 v_0}{6v_1}, \\ \beta_4 &= \frac{2i\sqrt{2}\sqrt{\omega_3}v_1^{3/2}(\beta_2\gamma_2 + \beta_3\gamma_3 - \gamma_4) + 2\omega_3\beta_1v_1 + \omega_3^2}{4v_1^2\gamma_4^2}. \end{aligned} \quad (51)$$

From (51), the subsequent solution is attained:

$$F_{1,1} = \frac{\omega_3}{6v_1} \left(1 - \frac{6\delta e^{\frac{i\sqrt{\omega_3}x}{\sqrt{2}\sqrt{v_1}} + \gamma_4 t + \gamma_2 y + \gamma_3 z}}{\left(\delta e^{\gamma_2 y + \gamma_3 z} + e^{\gamma_4 t + \frac{i\sqrt{\omega_3}x}{\sqrt{2}\sqrt{v_1}}} \right)^2} \right). \quad (52)$$

Characteristic of (52) where $\gamma_2 = \frac{3}{2}, \omega_3 = \frac{5}{2}, \gamma_3 = \frac{3}{4}, \gamma_4 = \frac{5}{4}, v_1 = \frac{1}{8}, \delta = \frac{3}{2}, y = \frac{2}{3}, z = \frac{1}{3}$, and $-10 \leq x \leq 10, -10 \leq y \leq 10$ are presented below:

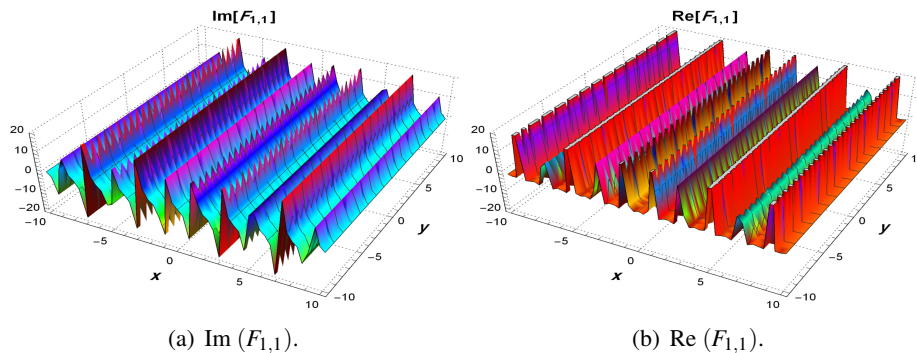


Fig. 9 Figures of (52) in 3D.

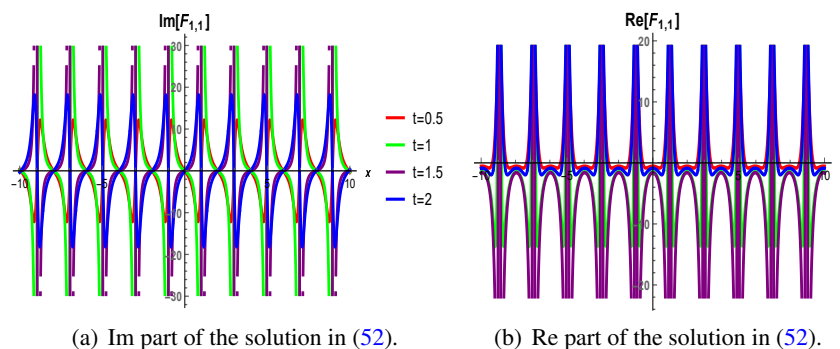


Fig. 10 Two-dimensional time-development plots of equation (52) where values of t are indicated in the legend.

Family 1.2. This family of the coefficients are reached:

$$\gamma_1 = -\frac{i\sqrt{\omega_3}}{\sqrt{2}\sqrt{v_1}}, \gamma_3 = \frac{i\sqrt{2}(2\omega_3 v_1 \beta_1 + \omega_3^2 - 4v_1^2 \beta_4 \gamma_4^2) - 4\beta_2 \gamma_2 + 4\gamma_4}{\sqrt{\omega_3} v_1^{3/2} 4\beta_3}, \quad (53)$$

$$\omega_0 = \frac{\omega_3 v_0}{6v_1}, \omega_1 = \frac{1}{6}\omega_3 \left(1 - \frac{6v_0}{v_1}\right), \omega_2 = \omega_3 \left(\frac{v_0}{v_1} - 1\right),$$

with the referenced values in (53), the following solution is acquired:

$$F_{1,2} = \frac{\omega_3}{6v_1} \left(1 - \frac{6\delta \exp \left(\frac{i\sqrt{\omega_3}x}{\sqrt{2}\sqrt{v_1}} + \frac{z \left(\frac{i\sqrt{2}(2\omega_3 v_1 \beta_1 + \omega_3^2 - 4v_1^2 \beta_4 \gamma_4^2) - 4\beta_2 \gamma_2 + 4\gamma_4}{\sqrt{\omega_3} v_1^{3/2}} \right) - 4\beta_2 \gamma_2 + 4\gamma_4}{4\beta_3} \right) + \gamma_4 t + \gamma_2 y}{\left(\delta \exp \left(\gamma_2 y + \frac{z \left(\frac{i\sqrt{2}(2\omega_3 v_1 \beta_1 + \omega_3^2 - 4v_1^2 \beta_4 \gamma_4^2) - 4\beta_2 \gamma_2 + 4\gamma_4}{\sqrt{\omega_3} v_1^{3/2}} \right) - 4\beta_2 \gamma_2 + 4\gamma_4}{4\beta_3} \right) + e^{\gamma_4 t + \frac{i\sqrt{\omega_3}x}{\sqrt{2}\sqrt{v_1}}} \right)^2} \right), \quad (54)$$

Family 1.3. We obtain the following parameters:

$$\omega_0 = -\frac{1}{3}v_0\gamma_1^2, \omega_1 = \frac{\omega_3}{6} + 2v_0\gamma_1^2, \omega_2 = -\omega_3 - 2v_0\gamma_1^2, \quad (55)$$

$$\beta_2 = \frac{-\beta_1\gamma_1^2 + \gamma_1(\gamma_4 - \beta_3\gamma_3) - \beta_4\gamma_4^2 + \gamma_1^4}{\gamma_1\gamma_2}, v_1 = -\frac{\omega_3}{2\gamma_1^2},$$

by utilizing (55), we derive the corresponding solution.

$$F_{1,3} = \frac{1}{3}\gamma_1^2 \left(\frac{6\delta e^{\gamma_4 t + \gamma_1 x + \gamma_2 y + \gamma_3 z}}{(e^{\gamma_4 t} + \delta e^{\gamma_1 x + \gamma_2 y + \gamma_3 z})^2} - 1 \right). \quad (56)$$

Profile of the real solution (56) where $\gamma_1 = -\frac{5}{2}$, $\gamma_2 = \frac{4}{3}$, $\gamma_3 = \frac{5}{4}$, $\gamma_4 = \frac{4}{5}$, $\delta = \frac{3}{2}$, $t = \frac{5}{3}$, $z = \frac{4}{3}$, and $-10 \leq x \leq 10$, $-10 \leq y \leq 10$ are graphed below :

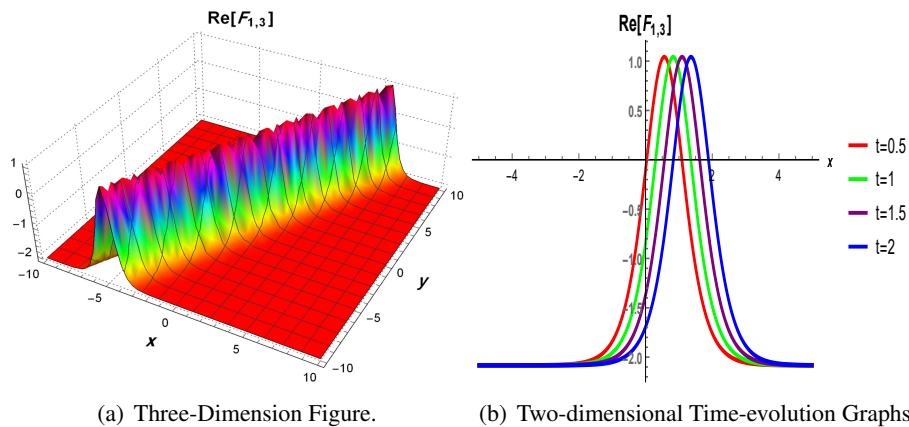


Fig. 11 Three-dimensional and two-dimensional (where t values are given) time-evolution figures of (56).

Family 2. From (46), if $M = 2$ then $N = 4$, so equation (10) is appearing in the following form:

$$U(\eta) = \frac{\omega_0 + \omega_1 T + \omega_2 T^2 + \omega_3 T^3 + \omega_4 T^4}{v_0 + v_1 T + v_2 T^2}, \quad (57)$$

where equation (11) is considered, thereafter:

$$U' = (\phi^2 - \phi) \left(\frac{4\omega_4 \phi^3 + 3\omega_3 \phi^2 + 2\omega_2 \phi + \omega_1}{v_2 \phi^2 + v_1 \phi + v_0} - \frac{(\omega_4 \phi^4 + \omega_3 \phi^3 + \omega_2 \phi^2 + \omega_1 \phi + \omega_0)(2v_2 \phi + v_1)}{(v_2 \phi^2 + v_1 \phi + v_0)^2} \right), \quad (58)$$

there should be $\omega_4 \neq 0$, $v_2 \neq 0$, one can find this:

$$U'' = \frac{\Omega(T)\Psi'(T) - \Psi(T)\Omega'(T)}{(\Omega(T))^2}. \quad (59)$$

By appointing (33)-(35) into (18), we acquire an algebraic system of equations involving coefficients of equation (18). Henceforth, we will try to equate the coefficients of the same powers of T^i for $i = 0, 1, 2, \dots, 10$, to zero, aiming to obtain the following algebraic system:

$$eq1 = 0, \dots, eq11 = 0. \quad (60)$$

The following sub-family are generated by using computer software programs to solve (60).

Family 2.1. We can get the following parameters:

$$\begin{aligned} \omega_0 &= \frac{1}{36} \left(\frac{\omega_4 v_1}{v_2} - 6\omega_1 \right), \omega_2 = \frac{1}{6} \omega_4 \left(1 - \frac{5v_1}{v_2} \right) - \omega_1, \omega_3 = \omega_4 \left(\frac{v_1}{v_2} - 1 \right), \gamma_1 = -\frac{i\sqrt{\omega_4}}{\sqrt{2}\sqrt{v_2}}, \\ v_0 &= \frac{v_1}{6} - \frac{\omega_1 v_2}{\omega_4}, \beta_2 = \frac{\frac{i\sqrt{2}(2\omega_4 v_2 \beta_1 + \omega_4^2 - 4v_2^2 \beta_4 \gamma_4^2)}{\sqrt{\omega_4 v_2^{3/2}}} - 4\beta_3 \gamma_3 + 4\gamma_4}{4\gamma_2}, \end{aligned} \quad (61)$$

from (61) one generates the following solution:

$$F_{2,1} = \frac{\omega_4}{6v_2} \left(1 - \frac{6\delta e^{\frac{i\sqrt{\omega_4}x}{\sqrt{2}\sqrt{v_2}} + \gamma_4 t + \gamma_2 y + \gamma_3 z}}{\left(\delta e^{\gamma_2 y + \gamma_3 z} + e^{\gamma_4 t + \frac{i\sqrt{\omega_4}x}{\sqrt{2}\sqrt{v_2}}} \right)^2} \right). \quad (62)$$

Family 2.2. One obtain the subsequent coefficients:

$$\begin{aligned} v_2 &= -\frac{\omega_4}{2\gamma_1^2}, \omega_0 = -\frac{1}{3}v_0\gamma_1^2, \omega_1 = 2v_0\gamma_1^2 - \frac{\omega_4}{6}, \omega_2 = \frac{7\omega_4}{6} - 2v_0\gamma_1^2, \\ \omega_3 &= -2\omega_4; v_1 = \frac{\omega_4}{2\gamma_1^2}, \beta_4 = \frac{\gamma_1(-\beta_1\gamma_1 - \beta_2\gamma_2 - \beta_3\gamma_3 + \gamma_1^3 + \gamma_4)}{\gamma_4^2}. \end{aligned} \quad (63)$$

The solution is as follows when the parameters stated in the equation (63) are implemented.

$$F_{2,2} = \frac{1}{3}\gamma_1^2 \left(\frac{6\delta e^{\gamma_4 t + \gamma_1 x + \gamma_2 y + \gamma_3 z}}{(e^{\gamma_4 t} + \delta e^{\gamma_1 x + \gamma_2 y + \gamma_3 z})^2} - 1 \right), \quad (64)$$

Family 2.3. One gets the following coefficients:

$$\begin{aligned} \gamma_1 &= -\frac{i\sqrt{\omega_4}}{\sqrt{2}\sqrt{v_2}}, \omega_0 = \frac{\omega_4 v_0}{6v_2}, \omega_1 = \frac{\omega_4(v_1 - 6v_0)}{6v_2}, \omega_2 = \frac{\omega_4(6v_0 - 6v_1 + v_2)}{6v_2}, \\ \omega_3 &= \omega_4 \left(\frac{v_1}{v_2} - 1 \right), \beta_4 = \frac{2i\sqrt{2}\sqrt{\omega_4}v_2^{3/2}(\beta_2\gamma_2 + \beta_3\gamma_3 - \gamma_4) + 2\omega_4\beta_1 v_2 + \omega_4^2}{4v_2^2\gamma_4^2}, \end{aligned} \quad (65)$$

the solution that follows will be derived from the parameters that have been identified in equation (65):

$$F_{2,3} = \frac{\omega_4}{6v_2} \left(1 - \frac{6\delta e^{\frac{i\sqrt{\omega_4}x}{\sqrt{2}\sqrt{v_2}} + \gamma_4 t + \gamma_2 y + \gamma_3 z}}{\left(\delta e^{\gamma_2 y + \gamma_3 z} + e^{\gamma_4 t + \frac{i\sqrt{\omega_4}x}{\sqrt{2}\sqrt{v_2}}} \right)^2} \right), \quad (66)$$

Physical structure of the solution (66) when $\omega_4 = \frac{5}{2}, v_2 = \frac{8}{3}, \gamma_2 = \frac{2}{3}, \gamma_3 = \frac{5}{4}, \gamma_4 = \frac{4}{3}, \delta = -\frac{5}{2}, y = \frac{1}{3}, z = \frac{4}{3}$, and $-20 \leq x \leq 20, -20 \leq y \leq 20$, are outlined in the following:

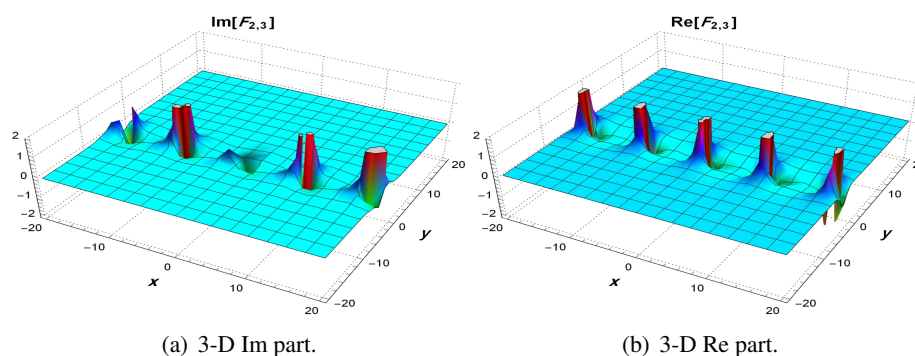


Fig. 12 Plots of (66) in three-dimensions.

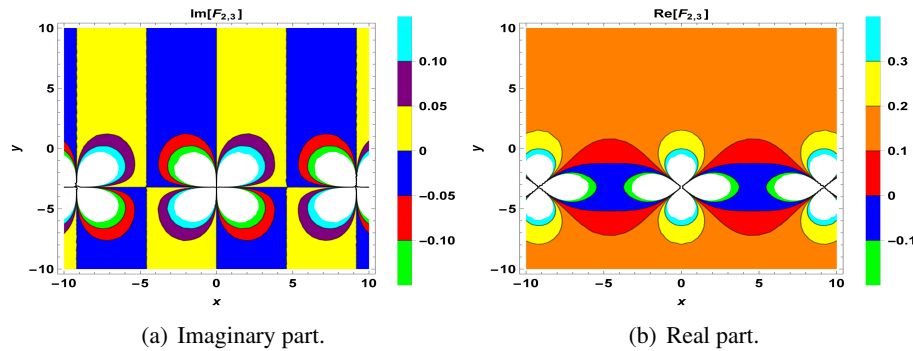


Fig. 13 Figures of (66), where $-10 \leq x \leq 10$, $-10 \leq y \leq 10$ (Contour surfaces).

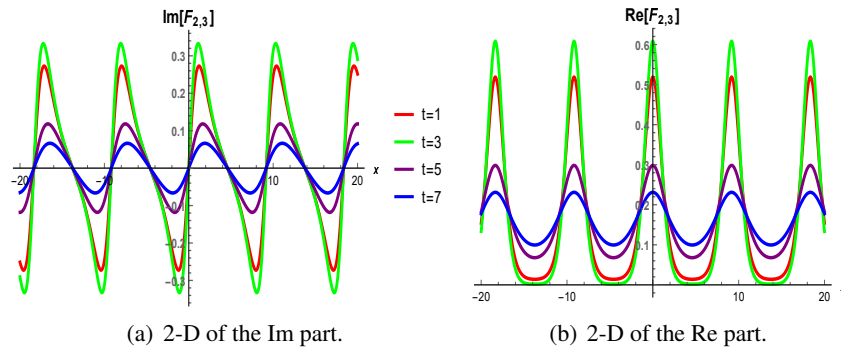


Fig. 14 Time-growth figures in two-dimensions represent (66) for t values in the legend, where $y = -\frac{3}{2}$, $-20 \leq x \leq 20$.

5 Results and discussion

The solutions obtained to the governing models exhibit a rich variety of physically meaningful wave structures such as bright solitons, dark solitons, singular solitons, dark-bright composites, and continuous periodic waves. For the Hirota–Maccari system, the variable $u(x, y, t)$ models the complex envelope of an ultra-fast optical pulse in a single-mode fiber. The solution $u_{1,1}$ characterized by Figures (1)–(2) represents a bright oscillatory multi-soliton whose real and imaginary components translate at constant speed without any change in shape, confirming stable energy transport along the fibre. The solution $u_{1,3}$ represented by Figures (3)–(5) is a dark singular soliton characterized by sharp, isolated spikes rising from a flat background, a wave type relevant to ultra-short pulse dynamics in nonlinear optical media. The mixed-mode solution $u_{2,2}$ represented by Figures (6)–(8) arises from the interplay between self-focusing and self-refocusing nonlinear, producing a dark-bright composite profile with a low-amplitude periodic tail, as seen in bi-modal fibers. In all solutions of HM, the fractional order β compresses the effective time axis: reducing β below unity slows the apparent wave speed, modeling anomalous dispersion in engineered photon media. For the KdV-type equation, solutions $F_{1,1}$ and $F_{2,3}$ symbolized by Figures (9)–(10), and (12)–(14) are singular solitons modeling ion-acoustic spikes in magnetized plasma, while solution $F_{1,3}$ characterized by Figure (11) is a smooth dark solitons representing rarefaction waves in stratified fluid channels, where nonlinear steepening precisely balances linear dispersion. Across all solution families, the free parameters γ_1 , γ_2 control the wave-front orientation in the spatial plane, ω/ν governs the soliton width and carrier frequency, and δ sets the initial position of the wave peak.

6 Conclusion

The existing research shows that the results presented here are new for the complex nonlinear (2+1)-dimensional time-fractional HM system and the (3+1)-dimensional Korteweg-de Vries-type equation. The improved generalized Kudryashov method, a semi-analytical technique, has been used for the first time on these models. The new “beta-derivative” meets certain characteristics called limits for fractional derivatives and is used to describe specific physical problems. In this study, numerous key solutions to the HM system were found, including multi-solitons, dark solitons, dark-bright solitons, dark singular solitons, bright singular solitons, M-shaped periodic waves, and singular soliton solutions, all using a specific approach. Some of these solutions have been visualized as multi-dimensional graphs. The models studied illustrate many nonlinear phenomena found in optical fibers, plasma, fluid mechanics, physics, engineering, and various scientific fields. The optical solitons generated may be important for understanding nonlinear processes in optical fiber interactions and signal processing. The free parameters in the solutions significantly influence the shape of the wave forms. Choosing these parameters is deliberate; they are not arbitrary or randomly selected. Additionally, the spatial and temporal intervals are crucial in shaping the solutions. The structure of these solutions demonstrates the reliability and effectiveness of the method used. The solutions are presented in combined forms, typically involving exponential functions coupled with either trigonometric or hyperbolic functions. We have also shown many two-dimensional time-evolution graphs that are new in literature for this model. These graphs illustrate how solutions change over time. Furthermore, three-dimensional plots, revolving plots around the x -axis, and surface contours are also depicted clearly. The dynamic behavior of the solutions can be observed in these visuals, which display various periodic traveling waves, M-shaped waves, continuous periodic waves, discontinuous waves, dark and bright solitons, and multi-periodic waves. This work can be expanded in several exciting ways. One idea is to replace the standard derivative with nonlocal operators like Caputo or Riemann-Liouville derivatives for the future direction.

7 Declarations

7.1 Competing interests:

The authors confirm that there is no conflict of interest related to the publishing of this work.

7.2 Funding:

No funding was offered to support the completion of this work.

7.3 Author’s contributions:

A.A.M.-Conceptualization, Writing Original Draft, Writing-Review and Editing, Methodology, Software. K.A.M.-Investigation, Methodology, Resources, Validation, Formal Analysis, Visualization. T.T.-Supervision, Formal Analysis, Investigation, Conceptualization, Methodology, Resources. All authors read and approved the final submitted version of this manuscript.

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7.5 Data availability statement:

The article contains all the data supporting the study’s outcomes.

7.6 Using of AI tools:

The authors affirm that Artificial Intelligence (AI) techniques were not used in creating this content.

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