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Original Study

A new application of Taylor expansion for approximate solution of systems of Fredholm integral equations

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Abstract

Systems of Fredholm integral equations are essential in engineering fields for modeling complex systems involving dynamic behavior, material properties, and interactions between different forces or fluids. In this research, the Taylor expansion approach is used to solve the systems of linear Fredholm integral equations (SLFIEs) of the second kind in a novel manner. By combining the m th-order Taylor polynomial of unknown functions at an arbitrary point and employing the repeated integration method; the given system of linear Fredholm integral equations is transformed into a new system of linear equations of unknown functions and their derivatives. Eventually, this new system is solved to obtain demanded m th-order approximate solutions. An error analysis is given as well as several numerical examples to demonstrate the efficiency, accuracy, and ability of the proposed method. Furthermore, this method always leads to the exact solution if the exact solution is a polynomial of degree less than or equal to m .

Keywords: Systems of linear Fredholm integral equations, the second kind Fredholm integral equations, error analysis, Taylor expansion.

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1 Introduction

Integral equations arise in many fields of applied mathematics and engineering problems [1]. Since Fredholm integral equations and systems of equations involving them appear in a lot of scientific applications and mathematical descriptions of various problems of engineering interest includes them, many significant improvements have been made in this field [1–4]. Systems of Fredholm integral equations offer a powerful mathematical framework for modeling a wide range of phenomena across diverse disciplines. Their ability to represent continuous interactions and complex relationships makes them a valuable tool in physics, engineering, biology, economics, and computer science. Continued research in developing more efficient and robust numerical methods for solving these equations will further enhance their applications and contribute to advancements in various fields. Fredholm integral equations are commonly encountered in transforming differential equations or boundary value problems modeled in different disciplines into integral equations [5–25].

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With due attention to the significance of numerical and approximate approaches for solving scientific problems, applying these methods is an essential and valuable work in scientific researches. Several numerical and approximate methods for solving systems of Fredholm integral equations of the second kind are available in literature and we thus briefly mention some of them. Babolian et al. [26] applied the Adomian decomposition method for systems of Fredholm integral equations of the second kind. The Haar wavelet method and the rationalized Haar function method have been applied to solve systems of linear Fredholm integral equations [27, 28]. A numerical solution of SLFIEs by Sinc function has been proposed in [29]. Maleknejad and his co-authors proposed Block-Pulse functions and the Taylor-series expansion method for solving the system of linear Fredholm integral equations of the second kind [30, 31]. Moreover, in recent years, various methods have been introduced which can be applied for solving systems of Fredholm integral equations, like the homotopy perturbation method [32], the modified homotopy perturbation method [33], the multi-parametric homotopy method [34], the homotopy analysis method [35], the modified Taylor expansion method [36], the triangular functions method [37], the delta basis functions [38], the feed-back neural network method [39], the B-spline wavelet collocation method [40], the Bernstein polynomials method and the hybrid Bernstein Block-Pulse functions method [41], the Hermite collocation method [42, 43], and Haar wavelet collocation method [44, 45].

Xian-Fang Li [46] suggested the Taylor expansion technique in a new way in order to approximate the solution of linear ordinary differential equations with variable coefficients. Next, Li and his co-authors expanded the above-mentioned method for solving Abel integral equation [47, 48], Riccati equation [49], an integral equation with fixed singularity for a cruciform crack [50], a class of linear integro-differential equations [51], and fractional integro-differential equations [52]. Vahidi and Didgar improved the Taylor expansion method presented in [49] to solve Riccati equations [53]. The method proposed in [46] was expanded by Didgar and Ahmadi [54] to determine the solutions of systems of linear ordinary and fractional differential equations. Furthermore, Maleknejad and Damercheli [55] presented a method to solve a linear system of Volterra integral equations of the second kind. Recently, Didgar et al. applied the method to solve systems of singular Volterra integral equations [56], Fredholm integral equations of the first kind [57], and systems of fractional integro-differential equations [58].

In this work, our study is focused on solving systems of linear Fredholm integral equations of the second kind based on applying the Taylor expansion in a novel manner [46–58]. By expanding unknown functions to be determined as an m th-order Taylor polynomial and employing repeated integration, the SLFIEs can be transformed into a system of linear equations of unknown functions and their derivatives. By solving the resultant system, the intended approximate solutions can be determined according to a standard method. The results of the obtained numerical approximations of the suggested method are compared with the results reported by applying different approaches. In the present investigation, the main powerful advantage of this approximate method besides reliability and applicability is that m th-order approximate solutions are exact if the exact solutions are polynomial functions of maximum degree m .

This paper is constructed as follows. In Section 2, we introduce our method for the SLFIEs with second kind. In Section 3, we give an error analysis. In Section 4, we investigate several numerical examples, which demonstrate the effectiveness of our technique. Finally, the paper is concluded in Section 5.

2 The SLFIEs with second kind

A SLFIEs with second kind can be considered as follows,

$$\lambda_i \psi_i(x) + \int_a^b \sum_{j=1}^n k_{ij}(x,t) \psi_j(t) dt = f_i(x), \quad i = 1, \dots, n, \quad (1)$$

where $f_i(x)$ and $k_{ij}(x, t)$ are known functions, λ_i are real constants and $\psi_j(x)$ are the unknown functions for $i, j = 1, \dots, n$, with $f_i(x), k_{ij}(x, t) \in C(I)$, in which I is considered as interest interval.

To estimate the solutions of Eq.(1), following the method used in [42–54], we reduce it to a linear equations system in terms of unknown functions and their derivatives. It is supposed that the solutions $\psi_j(t)$ are $m + 1$ times continuously differentiable on the interval I , that is to say, $\psi_j \in C^{m+1}(I)$. Therefore, for $\psi_j \in C^{m+1}(I)$, we can express unknown functions $\psi_j(t)$ in terms of the m th-order Taylor expansion at an arbitrary point $x \in I$ as follows,

$$\psi_j(t) = \psi_j(x) + \psi_j'(x)(t-x) + \dots + \frac{1}{m!} \psi_j^{(m)}(x)(t-x)^m + E_{j,m}(t, x), \quad (2)$$

in which $E_{j,m}(t, x)$ is the bound of Lagrange error defined as the following for some point ξ_j between x and t .

$$E_{j,m}(t, x) = \frac{\psi_j^{(m+1)}(\xi_j)}{(m+1)!} (t-x)^{m+1}. \quad (3)$$

In general, the Lagrange error bound $E_{j,m}(t, x)$ gets sufficiently small as m grows enough provided that $\psi_j^{(m+1)}(x)$ is a uniformly bounded function. It is important to note that the Lagrange error bound becomes zero for a polynomial function of maximum degree m , thus the above m th-order Taylor expansion is equal to the exact solution. With due attention to the aforesaid hypothesis, by omitting the last bound of Lagrange error, we approximately expand $\psi_j(t)$ as

$$\psi_j(t) \approx \sum_{k=0}^m \psi_j^{(k)}(x) \frac{(t-x)^k}{k!}. \quad (4)$$

Inserting the approximate relation (4), for unknown function $\psi_j(t)$, into Eq.(1) leads to

$$\lambda_i \psi_i(x) + \sum_{j=1}^n \sum_{k=0}^m \frac{(-1)^k}{k!} \psi_j^{(k)}(x) \int_a^b k_{ij}(x, t)(x-t)^k dt = f_i(x), \quad i = 1, \dots, n, \quad (5)$$

that can be simplified as

$$\sum_{j=1}^n \sum_{k=0}^m v_{jk}^{i0}(x) \psi_j^{(k)}(x) = f_i(x), \quad i = 1, \dots, n, \quad (6)$$

where

$$v_{jk}^{i0}(x) = \lambda_i \delta_{ij} + \frac{(-1)^k}{k!} \int_a^b k_{ij}(x, t)(x-t)^k dt, \quad (7)$$

δ is the Kronecker delta function.

Indeed, Eq.(1) has been transformed into a linear ordinary differential equations system in terms of $\psi_j(x)$ and its derivatives up to order m . In other words, n linear equations in (6) have been obtained in terms of $n \times (m+1)$ unknown functions $\psi_j^{(k)}(x)$, for $k = 0, \dots, m, j = 1, \dots, n$. By solving a system of linear equations, we are going to acquire $\psi_j(x), \dots, \psi_j^{(m)}(x)$. To reach this goal, other $n \times m$ independent linear equations with respect to $\psi_j(x), \dots, \psi_j^{(m)}(x)$ are required, which are derived by integrating both sides of Eq.(1) m times with respect to x from a to s and by changing the order of the integrations. Therefore, we get

$$\lambda_i \int_a^x (x-t)^{l-1} \psi_i(t) dt + \sum_{j=1}^n \int_a^b \int_a^x (x-s)^{l-1} k_{ij}(s, t) \psi_j(t) ds dt = f_i^{[l]}(x), \quad l = 1, \dots, m, \quad (8)$$

where

$$f_i^{[l]}(x) = \int_a^x (x-t)^{l-1} f_i(t) dt, \quad i = 1, \dots, n, \quad (9)$$

where the variable s was replaced by x , for the sake of easiness. Similarly, the Taylor expansion is applied again. After substituting Eq.(4) for $\psi_j(t)$ into Eq.(8), we obtain

$$\begin{aligned} & \lambda_i \sum_{k=0}^m \frac{(-1)^k}{k!} \psi_i^{(k)}(x) \int_a^x (x-t)^{l+k-1} dt \\ & + \sum_{j=1}^n \sum_{k=0}^m \frac{(-1)^k}{k!} \psi_j^{(k)}(x) \int_a^b \int_a^x (x-s)^{l-1} (x-t)^k k_{ij}(s,t) ds dt = f_i^{[l]}(x), \quad l = 1, \dots, m, \end{aligned} \quad (10)$$

or equivalently

$$\sum_{j=1}^n \sum_{k=0}^m v_{jk}^{il}(x) \psi_j^{(k)}(x) = f_i^{[l]}(x), \quad l = 1, \dots, m, \quad (11)$$

where

$$v_{jk}^{il}(x) = \frac{(-1)^k}{k!} \left[\lambda_i \frac{(x-a)^{l+k}}{l+k} + \int_a^b \int_a^x (x-s)^{l-1} (x-t)^k k_{ij}(s,t) ds dt \right]. \quad (12)$$

Hence, Eqs. (6) and (11) produce a new system of linear equations in terms of the unknown functions $\psi_j(x)$ and its derivatives up to order m . Now, we demonstrate this system as

$$V(x)\Psi(x) = F(x), \quad (13)$$

where

$$V(x) = \begin{bmatrix} v_{10}^{10}(x) & \cdots & v_{n0}^{10}(x) & \cdots & v_{1k}^{10}(x) & \cdots & v_{nk}^{10}(x) & \cdots & v_{1m}^{10}(x) & \cdots & v_{nm}^{10}(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ v_{10}^{n0}(x) & \cdots & v_{n0}^{n0}(x) & \cdots & v_{1k}^{n0}(x) & \cdots & v_{nk}^{n0}(x) & \cdots & v_{1m}^{n0}(x) & \cdots & v_{nm}^{n0}(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ v_{10}^{1l}(x) & \cdots & v_{n0}^{1l}(x) & \cdots & v_{1k}^{1l}(x) & \cdots & v_{nk}^{1l}(x) & \cdots & v_{1m}^{1l}(x) & \cdots & v_{nm}^{1l}(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ v_{10}^{nl}(x) & \cdots & v_{n0}^{nl}(x) & \cdots & v_{1k}^{nl}(x) & \cdots & v_{nk}^{nl}(x) & \cdots & v_{1m}^{nl}(x) & \cdots & v_{nm}^{nl}(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ v_{10}^{1m}(x) & \cdots & v_{n0}^{1m}(x) & \cdots & v_{1k}^{1m}(x) & \cdots & v_{nk}^{1m}(x) & \cdots & v_{1m}^{1m}(x) & \cdots & v_{nm}^{1m}(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ v_{10}^{nm}(x) & \cdots & v_{n0}^{nm}(x) & \cdots & v_{1k}^{nm}(x) & \cdots & v_{nk}^{nm}(x) & \cdots & v_{1m}^{nm}(x) & \cdots & v_{nm}^{nm}(x) \end{bmatrix}, \quad (14)$$

$$\Psi(x) = \left[\psi_1(x), \dots, \psi_n(x), \dots, \psi_1^{(k)}(x), \dots, \psi_n^{(k)}(x), \dots, \psi_1^{(m)}(x), \dots, \psi_n^{(m)}(x) \right]^T, \quad (15)$$

$$F(x) = \left[f_1(x), \dots, f_n(x), \dots, f_1^{(l)}(x), \dots, f_n^{(l)}(x), \dots, f_1^{(m)}(x), \dots, f_n^{(m)}(x) \right]^T. \quad (16)$$

In the sequel, the application of a standard rule to the resulting new system yields an m th-order approximate solution of Eq.(1) as $\psi_{j,m}(x)$. We note that not only $\psi_j(x)$ but also $\psi_j^{(k)}(x)$, for $k = 1, \dots, m$ are obtained by solving the resulting new system. But actually, $\psi_j(x)$ is what we are seeking.

3 Error analysis

In this section, we expand the error analysis proposed in [48] for the derived m th-order approximate solution of Eq.(1) to obtain theoretical characteristics about the convergence of the proposed approach. The exact solutions $\psi_j(t)$ are supposed to be infinitely differentiable on the interval I ; thus we can expand $\psi_j(t)$ as a uniformly convergent Taylor series in I as the following.

$$\psi_j(t) = \sum_{k=0}^{\infty} \psi_j^{(k)}(x) \frac{(t-x)^k}{k!}. \quad (17)$$

Applying the suggested method presented in Section 2, we can transform SLFIEs into an equivalent system of linear equations in terms of unknown functions $\psi_i^{(k)}(x)$, $k = 0, 1, \dots$ as

$$\mathbf{V}\Psi = \mathbf{F}, \quad (18)$$

where

$$\mathbf{V} = \lim_{n \rightarrow \infty} \mathbf{V}_{nn}^{nn}, \quad \Psi = \lim_{n \rightarrow \infty} \Psi_n, \quad \mathbf{F} = \lim_{n \rightarrow \infty} \mathbf{F}_n, \quad (19)$$

in which $\mathbf{V}_{nn}^{nn}$, Ψ_n , and $\mathbf{F}_n$, as indicated in the former section, are defined as the following.

$$\mathbf{V}_{nn}^{nn} = \left[v_{ij}^{pq}(x) \right]_{n(m+1) \times n(m+1)}, \quad \Psi_n = \left[\psi_i^{(k)}(x) \right]_{n(m+1) \times 1}, \quad \mathbf{F}_n = \left[f_i^{(l)}(x) \right]_{n(m+1) \times 1}. \quad (20)$$

Therefore, putting $\mathbf{B} = \mathbf{V}^{-1}$, the solution of system (18) is uniquely indicated as

$$\Psi = \mathbf{B}\mathbf{F}. \quad (21)$$

The relation (21) is rewritten in an alternative matrix form as follows.

$$\begin{bmatrix} \Psi_n \\ \Psi_{\infty} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{nn}^{nn} & \mathbf{B}_{n\infty}^{n\infty} \\ \mathbf{B}_{\infty n}^{\infty n} & \mathbf{B}_{\infty\infty}^{\infty\infty} \end{bmatrix} \begin{bmatrix} \mathbf{F}_n \\ \mathbf{F}_{\infty} \end{bmatrix}. \quad (22)$$

Therefore, we realize that the vector Ψ_n must satisfy the relation below.

$$\Psi_n = \mathbf{B}_{nn}^{nn} \mathbf{F}_n + \mathbf{B}_{n\infty}^{n\infty} \mathbf{F}_{\infty}. \quad (23)$$

According to the suggested method, the unique solution of SLFIEs (1) can be represented as

$$\tilde{\Psi}_n = \mathbf{V}_{nn}^{nn-1} \mathbf{F}_n, \quad (24)$$

where $\tilde{\Psi}_n$ is a notation for approximate solution which is replaced by Ψ_n for convenience.

Subtracting (24) from (23) leads to

$$\Psi_n - \tilde{\Psi}_n = \mathbf{D}_{nn}^{nn} \mathbf{F}_n + \mathbf{B}_{n\infty}^{n\infty} \mathbf{F}_{\infty}, \quad (25)$$

where $\mathbf{D}_{nn}^{nn} = \mathbf{B}_{nn}^{nn} - \mathbf{V}_{nn}^{nn-1}$.

Now, the right-hand side of (25) is expanded. We can express the first n elements of the vector at the left-hand side of (25) as follows.

$$\psi^n(x) - \tilde{\psi}^n(x) = \sum_{j=0}^m \sum_{i=1}^n d_{ij}^{p0}(x) f_i^{(j)}(x) + \sum_{j=m+1}^{\infty} \sum_{i=1}^n b_{ij}^{p0}(x) f_i^{(j)}(x), \quad p = 1, \dots, n, \quad (26)$$

where

$$\psi^n(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_n(x) \end{bmatrix}, \quad \tilde{\psi}^n(x) = \begin{bmatrix} \tilde{\psi}_1(x) \\ \tilde{\psi}_2(x) \\ \vdots \\ \tilde{\psi}_n(x) \end{bmatrix}, \quad (27)$$

and $d_{ij}^{p0}(x)$, $b_{ij}^{p0}(x)$ are the elements of $\mathbf{D}_{nn}^{nn}$ and $\mathbf{B}_{n\infty}^{n\infty}$, respectively. Hence, based on the Cauchy-Schwarz inequality we have

$$|\psi^n(x) - \tilde{\psi}^n(x)| \leq \left(\sum_{j=0}^m \sum_{i=1}^n |d_{ij}^{p0}(x)|^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^m \sum_{i=1}^n |f_i^{(j)}(x)|^2 \right)^{\frac{1}{2}} + \left(\sum_{j=m+1}^{\infty} \sum_{i=1}^n |b_{ij}^{p0}(x)|^2 \right)^{\frac{1}{2}} \left(\sum_{j=m+1}^{\infty} \sum_{i=1}^n |f_i^{(j)}(x)|^2 \right)^{\frac{1}{2}}. \quad (28)$$

It must be noted that as $\lim_{n \rightarrow \infty} \mathbf{D}_{nn}^{nn} = 0$ and $\lim_{n \rightarrow \infty} \mathbf{B}_{n\infty}^{n\infty} = 0$, we have $\lim_{n \rightarrow \infty} |\psi^n(x) - \tilde{\psi}^n(x)| = 0$.

4 Applications

Here, to illustrate the efficiency and the accuracy of the method presented in this paper, several test applications are provided. The results are compared with referenced results to find the accuracy of the method. For convenience, absolute errors between m th-order approximate values $\psi_{i,m}(x)$ and the corresponding exact values $\psi_i(x)$ as $|\psi_{i,m}(x) - \psi_i(x)|$ are determined. All computations have been accomplished by applying Mathematica 11 in a computer with hardware configuration: Intel Core i5 CPU 1.33 GHz, 4 GB of RAM and 64-bit Operating System.

4.1 Application 1

We consider the system of Fredholm integral equations as follows [31, 40, 41]

$$\begin{cases} \psi_1(x) - \int_0^1 (x-t)^3 \psi_1(t) dt - \int_0^1 (x-t)^2 \psi_2(t) dt = \frac{1}{20} - \frac{11}{30}x + \frac{5}{3}x^2 - \frac{1}{3}x^3, \\ \psi_2(x) - \int_0^1 (x-t)^4 \psi_1(t) dt - \int_0^1 (x-t)^3 \psi_2(t) dt = -\frac{1}{30} - \frac{41}{60}x + \frac{3}{20}x^2 + \frac{23}{12}x^3 - \frac{1}{3}x^4, \end{cases} \quad (29)$$

with the exact solutions $\psi_1(x) = x^2$ and $\psi_2(x) = -x + x^2 + x^3$. We apply the process discussed in Section 2 to get the approximating approximate solutions by putting $m = 1, 2, 3$. It is worth mentioning that the Mathematica command 'LinearSolve' is employed to solve the resulting system after transforming the system (29) into a system of linear equations. The obtained absolute errors are listed in Table 1 and Table 2 together with the results given in [31]. It can be seen the third-order approximate solution gives the exact solution as expected since the m th-order approximation gives the exact solution if the exact solution is a polynomial function of maximum degree m . Figures 1, 2, and 3 show the approximate solutions of $\psi_1(x)$ and $\psi_2(x)$ for $m = 1, 2, 3$, respectively, in Taylor expansion in comparison with the exact solutions. In order to show the variation of the accuracy of approximations, the errors, between several approximate solutions and the exact values are plotted in Figures 4 and 5 for $\psi_1(x)$ and $\psi_2(x)$, respectively.

Table 1 Absolute errors of Application 1 for $\psi_1(x)$.

x	Method in [31] (m=10)	Suggested method	Suggested method	Suggested method
	$\psi_{Exact} - \psi_{Taylor}$	m=1	m=2	m=3
0.1	2.548×10^{-4}	2.89365×10^{-1}	8.76658×10^{-2}	0
0.2	1.055×10^{-3}	1.69063×10^{-1}	5.32303×10^{-2}	0
0.3	1.558×10^{-3}	9.81397×10^{-2}	2.99342×10^{-2}	0
0.4	1.513×10^{-3}	5.38480×10^{-2}	1.52024×10^{-2}	0
0.5	1.467×10^{-3}	2.82321×10^{-2}	6.67616×10^{-3}	0
0.6	2.634×10^{-3}	1.73289×10^{-2}	2.29812×10^{-3}	0
0.7	6.293×10^{-3}	1.93255×10^{-2}	3.73043×10^{-4}	0
0.8	1.248×10^{-2}	3.43349×10^{-2}	4.01271×10^{-4}	0
0.9	1.655×10^{-2}	6.45784×10^{-2}	9.36226×10^{-4}	0
1.0	4.620×10^{-3}	1.14521×10^{-1}	1.79048×10^{-3}	0

Table 2 Absolute errors of Application 1 for $\psi_2(x)$.

x	Method in [31] (m=10)	Suggested method	Suggested method	Suggested method
	$\psi_{Exact} - \psi_{Taylor}$	m=1	m=2	m=3
0.1	2.229×10^{-3}	2.17774×10^{-1}	6.77644×10^{-2}	0
0.2	2.271×10^{-2}	1.12737×10^{-1}	3.64407×10^{-2}	0
0.3	1.835×10^{-2}	5.73750×10^{-2}	1.80167×10^{-2}	0
0.4	7.450×10^{-4}	2.69276×10^{-2}	8.04810×10^{-3}	0
0.5	2.317×10^{-2}	1.03606×10^{-2}	3.20315×10^{-3}	0
0.6	3.458×10^{-2}	2.43533×10^{-4}	1.17962×10^{-3}	0
0.7	9.332×10^{-3}	9.73055×10^{-3}	5.69291×10^{-4}	0
0.8	1.111×10^{-1}	2.60431×10^{-2}	6.88201×10^{-4}	0
0.9	6.109×10^{-2}	5.64250×10^{-2}	1.39386×10^{-3}	0
1.0	1.101×10^{-1}	1.10788×10^{-1}	2.91265×10^{-3}	0

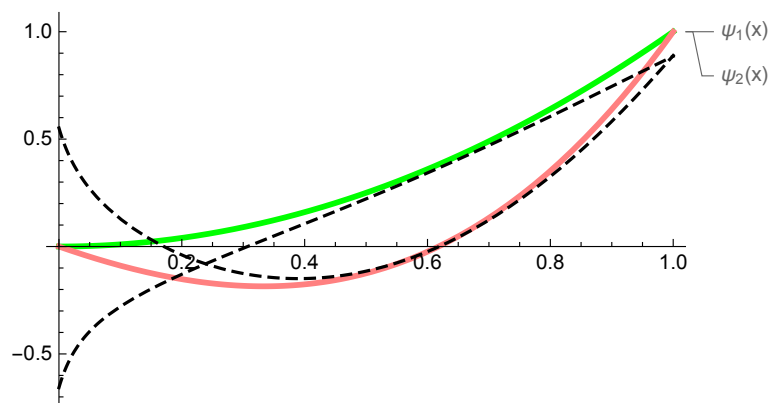
This application was used in [31] and has been solved by the Taylor expansion method. It seems that the current method is more rapidly convergent than the method given in [31]. Moreover, system (29) has been solved using the B-spline wavelet method in [40], the Bernstein polynomials method (BPM) and the hybrid Bernstein Block-Pulse functions method (HBBPFM) in [41]. We present the results given in [40] and [41] via Tables 3 and 4, respectively. From Tables 3 and 4, it can be observed that the absolute errors obtained by the present method are much better than those reported in [40, 41].

Table 3 Absolute errors of Application 1 by B-spline wavelet method in [40] for $(\psi_1(x), \psi_2(x))$.

x	M=2	M=4
0.1	$(1.05947E-4, 3.16487E-4)$	$(6.51740E-6, 5.47084E-6)$
0.2	$(1.14356E-3, 1.75687E-3)$	$(7.16057E-5, 1.16058E-4)$
0.3	$(1.14313E-3, 2.26496E-3)$	$(7.16040E-5, 1.34611E-4)$
0.4	$(1.07237E-4, 3.70402E-5)$	$(6.52242E-6, 1.72473E-5)$
0.5	$(2.60757E-3, 6.50994E-3)$	$(1.62774E-4, 4.06899E-4)$
0.6	$(1.07887E-4, 4.83014E-4)$	$(6.52497E-6, 1.53021E-5)$
0.7	$(1.14179E-3, 3.46452E-3)$	$(7.15988E-5, 2.23463E-4)$
0.8	$(1.14144E-3, 3.97184E-3)$	$(7.15974E-5, 2.42013E-4)$
0.9	$(1.08955E-4, 2.05924E-4)$	$(6.52915E-6, 2.70877E-5)$
1.0	$(2.60940E-3, 1.01976E-2)$	$(1.62781E-4, 6.47543E-4)$

Table 4 Absolute errors of Application 1 by BPM and HBBPFM in [41] for $(\psi_1(x), \psi_2(x))$.

x	BPM $n = 10$	HBBPFM $n=6, M=25$	HBBPFM $n=7, M=14$
0.1	$(3.47 \times 10^{-4}, 9.66 \times 10^{-4})$	$(1.34 \times 10^{-4}, 1.28 \times 10^{-4})$	$(9.88 \times 10^{-6}, 5.60 \times 10^{-6})$
0.2	$(4.78 \times 10^{-4}, 8.25 \times 10^{-4})$	$(2.77 \times 10^{-4}, 2.95 \times 10^{-4})$	$(7.63 \times 10^{-6}, 9.90 \times 10^{-6})$
0.3	$(5.28 \times 10^{-4}, 6.35 \times 10^{-4})$	$(4.13 \times 10^{-4}, 4.88 \times 10^{-4})$	$(3.65 \times 10^{-6}, 9.40 \times 10^{-6})$
0.4	$(7.92 \times 10^{-4}, 4.56 \times 10^{-4})$	$(6.75 \times 10^{-5}, 9.67 \times 10^{-5})$	$(1.33 \times 10^{-6}, 3.50 \times 10^{-6})$
0.5	$(3.68 \times 10^{-4}, 1.96 \times 10^{-4})$	$(6.03 \times 10^{-5}, 7.36 \times 10^{-5})$	$(6.94 \times 10^{-6}, 7.00 \times 10^{-7})$
0.6	$(4.73 \times 10^{-4}, 9.25 \times 10^{-4})$	$(2.04 \times 10^{-4}, 1.66 \times 10^{-4})$	$(3.77 \times 10^{-6}, 1.00 \times 10^{-6})$
0.7	$(3.11 \times 10^{-5}, 7.41 \times 10^{-4})$	$(9.25 \times 10^{-5}, 3.55 \times 10^{-4})$	$(6.46 \times 10^{-6}, 3.70 \times 10^{-6})$
0.8	$(9.43 \times 10^{-5}, 3.67 \times 10^{-4})$	$(6.04 \times 10^{-5}, 3.56 \times 10^{-4})$	$(8.80 \times 10^{-6}, 8.00 \times 10^{-7})$
0.9	$(7.36 \times 10^{-4}, 8.93 \times 10^{-4})$	$(3.82 \times 10^{-5}, 2.00 \times 10^{-4})$	$(6.14 \times 10^{-6}, 5.09 \times 10^{-2})$

**Fig. 1** The exact solutions (solid) versus the approximate solutions (dashed) for $m = 1$.

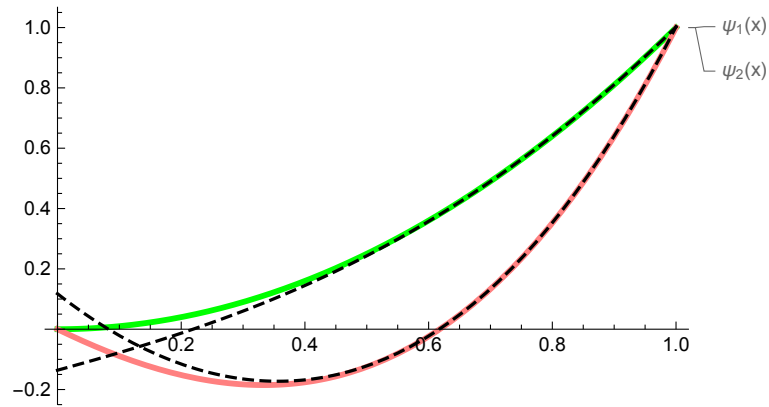


Fig. 2 The exact solutions (solid) versus the approximate solutions (dashed) for $m = 2$.

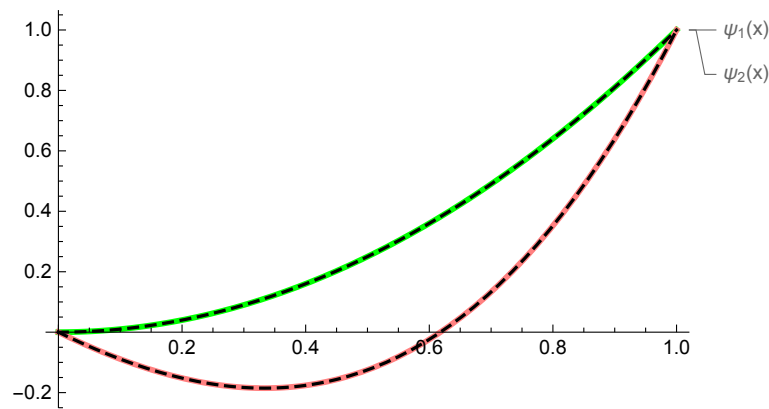


Fig. 3 The exact solutions (solid) versus the approximate solutions (dashed) for $m = 3$.

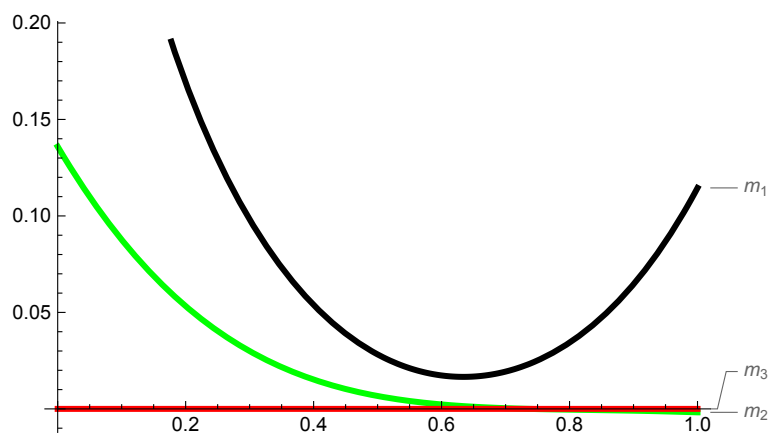


Fig. 4 Variations of the errors between several approximations of $\psi_1(x)$ and the corresponding exact value.

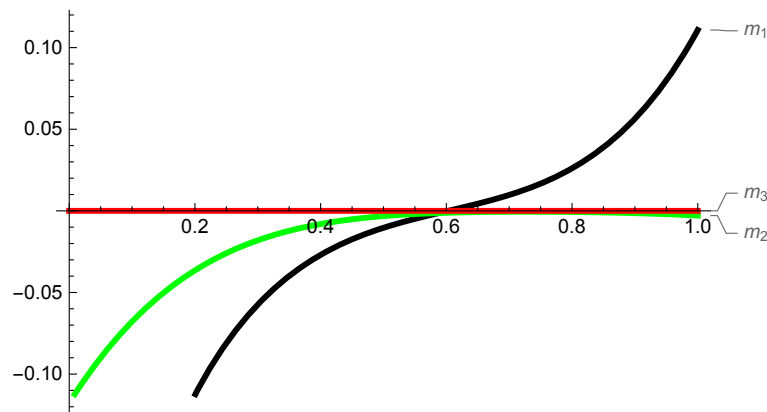


Fig. 5 Variations of the errors between several approximations of $\psi_2(x)$ and the corresponding exact value.

4.2 Application 2

We consider the following system of Fredholm integral equations [26,40]

$$\begin{cases} \psi_1(x) - \int_0^1 \frac{x+t}{3} (\psi_1(t) + \psi_2(t)) dt = \frac{x}{18} + \frac{17}{36}, \\ \psi_2(x) - \int_0^1 xt (\psi_1(t) + \psi_2(t)) dt = x^2 - \frac{19}{12}x + 1, \end{cases} \quad (30)$$

with the exact solutions $\psi_1(x) = 1 + x$ and $\psi_2(x) = 1 + x^2$.

We apply the procedure described in this paper to obtain the first and second-order approximate solutions. The absolute errors between the exact solution and approximate solutions and the results given in [26] are tabulated in Tables 5 and 6. Quite satisfactory accuracy of results is observed from these tables. Also, the second-order approximate solution yields the exact solution as anticipated. This application has been solved by the well-known Adomian decomposition method with eleven terms. Moreover, system (30) has been solved using the B-spline wavelet method in [40] and we present the results obtained in [40] via Table 7. From Table 7, it can be observed that the approximate results obtained by the present method are more accurate than those reported in [40].

Table 5 Absolute errors of Application 2 for $\psi_1(x)$.

x	Method in [26]	Suggested method	Suggested method
	$ \psi_{Exact} - \psi_{ADM} $	m=1	m=2
0.1	1.33×10^{-2}	2.40124×10^{-1}	0
0.2	1.52×10^{-2}	2.18895×10^{-1}	0
0.3	1.71×10^{-2}	1.90581×10^{-1}	0
0.4	1.89×10^{-2}	1.57067×10^{-1}	0
0.5	2.08×10^{-2}	1.20238×10^{-1}	0
0.6	2.26×10^{-2}	8.19810×10^{-2}	0
0.7	2.45×10^{-2}	4.41810×10^{-2}	0
0.8	2.64×10^{-2}	8.72381×10^{-3}	0
0.9	2.82×10^{-2}	2.25048×10^{-2}	0
1.0	3.02×10^{-1}	4.76190×10^{-2}	0

Table 6 Absolute errors of Application 2 for $\psi_2(x)$.

x	Method in [26]	Suggested method	Suggested method
	$ \psi_{Exact} - \psi_{ADM} $	m=1	m=2
0.1	3.45×10^{-3}	6.25429×10^{-2}	0
0.2	6.90×10^{-3}	1.00914×10^{-1}	0
0.3	1.03×10^{-2}	1.18371×10^{-1}	0
0.4	1.38×10^{-2}	1.18171×10^{-1}	0
0.5	1.72×10^{-2}	1.03571×10^{-1}	0
0.6	2.07×10^{-2}	7.78286×10^{-2}	0
0.7	2.41×10^{-2}	4.42000×10^{-2}	0
0.8	2.76×10^{-2}	5.94286×10^{-3}	0
0.9	3.10×10^{-2}	3.36857×10^{-2}	0
1.0	3.45×10^{-2}	7.14286×10^{-2}	0

Table 7 Absolute errors of Application 2 by B-spline wavelet method in [40] for $(\psi_1(x), \psi_2(x))$.

x	M=2	M=4
0.1	(2.22045E-16, 1.04167E-4)	(4.44089E-16, 6.51042E-6)
0.2	(0.000000000000, 1.14583E-3)	(2.22045E-16, 7.16146E-5)
0.3	(2.22045E-16, 1.14583E-3)	(2.22045E-16, 7.16146E-5)
0.4	(2.22045E-16, 1.04167E-4)	(4.44089E-16, 6.51042E-6)
0.5	(4.44089E-16, 2.60417E-3)	(1.33227E-16, 1.62760E-4)
0.6	(2.22045E-16, 1.04167E-4)	(6.66134E-16, 6.51042E-6)
0.7	(6.66134E-16, 1.14583E-3)	(4.44089E-16, 7.16146E-5)
0.8	(4.44089E-16, 1.14583E-3)	(4.44089E-16, 7.16146E-5)
0.9	(4.44089E-16, 1.04167E-4)	(6.66134E-16, 6.51042E-6)
1.0	(6.66134E-16, 2.60417E-3)	(8.88178E-16, 1.62760E-4)

4.3 Application 3

We consider the following system of Fredholm integral equations [30, 40]

$$\begin{cases} \psi_1(x) + \int_0^1 (x+t)\psi_1(t)dt + \int_0^1 (x+2t^2)\psi_2(t)dt = \frac{11}{6}x + \frac{11}{15}, \\ \psi_2(x) + \int_0^1 xt^2\psi_1(t)dt + \int_0^1 x^2t\psi_2(t)dt = \frac{5}{4}x^2 + \frac{1}{4}x, \end{cases} \quad (31)$$

with the exact solutions $\psi_1(x) = x$ and $\psi_2(x) = x^2$. Using the proposed method, the first-order and the second-order approximate solutions for $\psi_1(x)$ and $\psi_2(x)$ are computed. The obtained approximate solutions and the results given in [30] are tabulated in Tables 8 and 9. From Tables 8 and 9, it is observed that the second-order approximating solution gives the exact solution.

This application was used in [30] and has been solved by Block-Pulse functions. The results show that the current method is more rapidly convergent than the method in [30]. Moreover, system (31) has been solved using the B-spline wavelet method in [40] and we present the results obtained in [40] via Table 10. From Table 10, it can be observed that the Taylor expansion method presented in this work provides better accuracy than those reported in [40].

Table 8 Absolute errors of Application 3 for $\psi_1(x)$.

x	Method in [30] m=16	Method in [30] m=32	Suggested method m=1	Suggested method m=2
0.1	8.698×10^{-2}	1.980×10^{-3}	2.38200×10^{-1}	0
0.2	1.898×10^{-2}	3.450×10^{-3}	1.91404×10^{-1}	0
0.3	1.855×10^{-2}	8.540×10^{-3}	1.49116×10^{-1}	0
0.4	5.420×10^{-3}	1.210×10^{-2}	1.11397×10^{-1}	0
0.5	3.123×10^{-2}	1.386×10^{-2}	7.83878×10^{-2}	0
0.6	8.538×10^{-2}	6.410×10^{-3}	5.03014×10^{-2}	0
0.7	1.876×10^{-2}	9.140×10^{-3}	2.74219×10^{-2}	0
0.8	1.877×10^{-2}	1.314×10^{-2}	1.01035×10^{-2}	0
0.9	6.150×10^{-3}	1.512×10^{-2}	1.22547×10^{-3}	0
1.0	3.137×10^{-2}	6.860×10^{-3}	6.05633×10^{-3}	0

Table 9 Absolute errors of Application 3 for $\psi_2(x)$.

x	Method in [30] m=16	Method in [30] m=32	Suggested method m=1	Suggested method m=2
0.1	5.600×10^{-4}	2.900×10^{-4}	6.62637×10^{-3}	0
0.2	2.810×10^{-4}	1.720×10^{-3}	6.70369×10^{-3}	0
0.3	8.230×10^{-3}	1.340×10^{-3}	3.58039×10^{-3}	0
0.4	1.123×10^{-2}	8.120×10^{-3}	3.55018×10^{-4}	0
0.5	3.136×10^{-2}	1.409×10^{-2}	3.43149×10^{-3}	0
0.6	3.171×10^{-2}	2.405×10^{-2}	4.53355×10^{-3}	0
0.7	2.701×10^{-2}	9.000×10^{-5}	2.99261×10^{-3}	0
0.8	2.666×10^{-2}	1.795×10^{-2}	1.48519×10^{-3}	0
0.9	9.855×10^{-3}	4.490×10^{-3}	8.86505×10^{-3}	0
1.0	2.900×10^{-2}	2.900×10^{-4}	1.88099×10^{-2}	0

Table 10 Absolute errors of Application 3 by B-spline wavelet method in [40] for $(\psi_1(x), \psi_2(x))$.

x	M=2	M=4
0.1	$(2.07979E - 6, 1.04205E - 4)$	$(8.12417E - 9, 6.51057E - 6)$
0.2	$(1.94288E - 6, 1.14576E - 3)$	$(7.58937E - 9, 7.16143E - 5)$
0.3	$(1.80597E - 6, 1.14572E - 3)$	$(7.05456E - 9, 7.16142E - 5)$
0.4	$(1.66906E - 6, 1.04308E - 4)$	$(6.51976E - 9, 6.51097E - 6)$
0.5	$(1.53215E - 6, 2.60434E - 3)$	$(5.98496E - 9, 1.62761E - 4)$
0.6	$(1.39524E - 6, 1.04367E - 4)$	$(5.45016E - 9, 6.51120E - 6)$
0.7	$(1.25833E - 6, 1.14561E - 3)$	$(4.91536E - 9, 7.16137E - 5)$
0.8	$(1.12142E - 6, 1.14558E - 3)$	$(4.38055E - 9, 7.16136E - 5)$
0.9	$(9.84513E - 7, 1.04438E - 4)$	$(3.84575E - 9, 6.51148E - 6)$
1.0	$(8.47603E - 7, 2.60446E - 3)$	$(3.31095E - 9, 1.62762E - 4)$

4.4 Application 4

We consider the following system of Fredholm integral equations [28,30]

$$\begin{cases} \psi_1(x) + \int_0^1 e^{x-t} \psi_1(t) + e^{(x+2)t} \psi_2(t) dt = 2e^x + \frac{e^{x+1} - 1}{x+1}, \\ \psi_2(x) + \int_0^1 e^{xt} \psi_1(t) + e^{x+t} \psi_2(t) dt = e^x + e^{-x} + \frac{e^{x+1} - 1}{x+1}, \end{cases}, 0 \leq x \leq 1, \quad (32)$$

with the exact solutions $\psi_1(x) = e^x$ and $\psi_2(x) = e^{-x}$. Applying the method described in Section 2, the approximating solutions are obtained by putting $m = 1, \dots, 4$. The absolute errors between the exact solution and approximate solutions for $\psi_1(x)$ and $\psi_2(x)$ are listed in Tables 11 and 12, respectively. This application was used in [28] and [30] and has been solved by rationalized Haar functions and Block-Pulse functions methods, respectively. We present the results obtained in [28,30] via Tables 13 and 14. It is worth mentioning that the proposed method provides better results.

Table 11 Absolute errors of Application 4 for $\psi_1(x)$.

x	m=1	m=2	m=3	m=4
0.1	1.42576×10^{-1}	1.87892×10^{-1}	1.18936×10^{-2}	1.07546×10^{-2}
0.2	1.02909×10^{-1}	1.50166×10^{-1}	8.40862×10^{-3}	4.51723×10^{-3}
0.3	7.28585×10^{-2}	1.16840×10^{-1}	5.75061×10^{-3}	3.13013×10^{-3}
0.4	5.11274×10^{-2}	8.77018×10^{-2}	3.77465×10^{-3}	2.06896×10^{-3}
0.5	3.66628×10^{-2}	6.25978×10^{-2}	2.34919×10^{-3}	1.28032×10^{-3}
0.6	2.87143×10^{-2}	4.14305×10^{-2}	1.35764×10^{-3}	7.21594×10^{-4}
0.7	2.69156×10^{-2}	2.41633×10^{-2}	6.98557×10^{-4}	3.52211×10^{-4}
0.8	3.13962×10^{-2}	1.08267×10^{-2}	2.86319×10^{-4}	1.33281×10^{-4}
0.9	4.29419×10^{-2}	1.52918×10^{-3}	5.19763×10^{-5}	2.75368×10^{-5}
1.0	6.32297×10^{-2}	3.52550×10^{-3}	5.55572×10^{-5}	6.06983×10^{-7}

Table 12 Absolute errors of Application 4 for $\psi_2(x)$.

x	m=1	m=2	m=3	m=4
0.1	5.08674×10^{-2}	9.01505×10^{-2}	1.27633×10^{-3}	6.45456×10^{-3}
0.2	5.50850×10^{-2}	6.80369×10^{-2}	1.68664×10^{-4}	2.04838×10^{-3}
0.3	5.35456×10^{-2}	4.97035×10^{-2}	4.25853×10^{-4}	1.32564×10^{-3}
0.4	4.75126×10^{-2}	3.47864×10^{-2}	6.56463×10^{-4}	8.12332×10^{-4}
0.5	3.80803×10^{-2}	2.29547×10^{-2}	6.50718×10^{-4}	4.62022×10^{-4}
0.6	2.61641×10^{-2}	1.39002×10^{-2}	5.13929×10^{-4}	2.36744×10^{-4}
0.7	1.24855×10^{-2}	7.32388×10^{-3}	3.29481×10^{-4}	1.03630×10^{-4}
0.8	2.45176×10^{-3}	2.92136×10^{-3}	1.59256×10^{-4}	3.45574×10^{-5}
0.9	1.84003×10^{-2}	3.64138×10^{-4}	4.44145×10^{-5}	6.15417×10^{-6}
1.0	3.54318×10^{-2}	7.23693×10^{-4}	6.92641×10^{-6}	9.48343×10^{-8}

Table 13 Absolute errors of Application 4 by rationalized Haar functions method in [28] for $(\psi_1(x), \psi_2(x))$.

x	M=16	M=32
0.1	$(7.970 \times 10^{-3}, 5.950 \times 10^{-3})$	$(1.014 \times 10^{-2}, 8.370 \times 10^{-3})$
0.2	$(2.203 \times 10^{-2}, 1.490 \times 10^{-2})$	$(3.550 \times 10^{-3}, 2.480 \times 10^{-3})$
0.3	$(2.615 \times 10^{-2}, 1.436 \times 10^{-2})$	$(4.480 \times 10^{-3}, 2.410 \times 10^{-3})$
0.4	$(8.310 \times 10^{-3}, 3.790 \times 10^{-3})$	$(1.418 \times 10^{-2}, 6.410 \times 10^{-3})$
0.5	$(5.175 \times 10^{-2}, 1.968 \times 10^{-2})$	$(2.582 \times 10^{-2}, 9.660 \times 10^{-3})$
0.6	$(1.231 \times 10^{-2}, 3.950 \times 10^{-3})$	$(1.693 \times 10^{-2}, 4.990 \times 10^{-3})$
0.7	$(3.724 \times 10^{-2}, 8.610 \times 10^{-3})$	$(6.070 \times 10^{-3}, 1.410 \times 10^{-3})$
0.8	$(4.214 \times 10^{-2}, 9.200 \times 10^{-2})$	$(7.140 \times 10^{-3}, 1.590 \times 10^{-3})$
0.9	$(1.478 \times 10^{-3}, 1.690 \times 10^{-3})$	$(2.492 \times 10^{-2}, 4.030 \times 10^{-3})$
1.0	$(8.418 \times 10^{-2}, 1.261 \times 10^{-2})$	$(4.227 \times 10^{-2}, 6.030 \times 10^{-3})$

Table 14 Absolute errors of Application 4 by Block-Pulse functions method in [30] for $(\psi_1(x), \psi_2(x))$.

x	M=16	M=32
0.1	$(7.570 \times 10^{-3}, 5.960 \times 10^{-3})$	$(1.124 \times 10^{-2}, 8.260 \times 10^{-3})$
0.2	$(2.212 \times 10^{-2}, 1.481 \times 10^{-2})$	$(3.560 \times 10^{-3}, 2.370 \times 10^{-3})$
0.3	$(2.606 \times 10^{-2}, 1.441 \times 10^{-2})$	$(4.390 \times 10^{-3}, 2.700 \times 10^{-3})$
0.4	$(8.320 \times 10^{-3}, 3.660 \times 10^{-3})$	$(1.406 \times 10^{-2}, 6.500 \times 10^{-3})$
0.5	$(5.185 \times 10^{-2}, 1.989 \times 10^{-2})$	$(2.572 \times 10^{-2}, 9.680 \times 10^{-3})$
0.6	$(1.221 \times 10^{-2}, 4.020 \times 10^{-3})$	$(1.699 \times 10^{-2}, 4.950 \times 10^{-3})$
0.7	$(3.744 \times 10^{-2}, 8.690 \times 10^{-3})$	$(6.060 \times 10^{-3}, 1.390 \times 10^{-3})$
0.8	$(4.174 \times 10^{-2}, 9.480 \times 10^{-3})$	$(6.440 \times 10^{-3}, 7.800 \times 10^{-4})$
0.9	$(1.499 \times 10^{-2}, 1.670 \times 10^{-3})$	$(2.309 \times 10^{-2}, 4.130 \times 10^{-3})$
1.0	$(8.308 \times 10^{-2}, 1.299 \times 10^{-2})$	$(4.217 \times 10^{-2}, 6.130 \times 10^{-3})$

4.5 Application 5

We consider the following system of linear Fredholm integral equations [28, 31, 59]

$$\begin{cases} \psi_1(x) + \int_0^1 t \cos(x) \psi_1(t) dt + \int_0^1 x \sin(t) \psi_2(t) dt = x + \frac{\cos x}{3} + \frac{x \sin^2 1}{2}, \\ \psi_2(x) + \int_0^1 e^{xt^2} \psi_1(t) dt + \int_0^1 (x+t) \psi_2(t) dt = \frac{e^x - 1}{2x} + \cos x + (x+1) \sin 1 + \cos 1 - 1, \end{cases} \quad , 0 \leq x \leq 1, \quad (33)$$

with the exact solutions $\psi_1(x) = x$ and $\psi_2(x) = \cos x$. We employ the procedure described in Section 2 to determine the first and second-order approximate solutions by setting $m = 1, 2$. The absolute errors between the exact solution and approximate solutions are tabulated in Table 15. This application was used in [28], [31], and [59] and has been solved by rationalized Haar functions, Taylor-series expansion, and collocation method based on q.i. splines, respectively. In the following, we present the results obtained in [28, 31, 59] via Tables 16, 17 and 18. From Tables 15, 16, 17, and 18, it can be seen that the accuracy of the results is fully satisfactory. Moreover, the absolute errors obtained by the proposed method in this paper are much better than those given in [28, 31, 59].

Table 15 Absolute errors of Application 5 for $(\psi_1(x), \psi_2(x))$.

x	$m=1$	$m=2$
0.1	$(6.39030 \times 10^{-3}, 4.89613 \times 10^{-2})$	$(5.03104 \times 10^{-4}, 3.35710 \times 10^{-3})$
0.2	$(9.88092 \times 10^{-4}, 4.28668 \times 10^{-2})$	$(1.21393 \times 10^{-4}, 3.29583 \times 10^{-3})$
0.3	$(2.53419 \times 10^{-3}, 3.56531 \times 10^{-2})$	$(1.91079 \times 10^{-4}, 3.02207 \times 10^{-3})$
0.4	$(4.35938 \times 10^{-3}, 2.79192 \times 10^{-2})$	$(3.97546 \times 10^{-4}, 2.57961 \times 10^{-3})$
0.5	$(4.69526 \times 10^{-3}, 2.02420 \times 10^{-2})$	$(4.84346 \times 10^{-4}, 2.02815 \times 10^{-3})$
0.6	$(3.76983 \times 10^{-3}, 1.31603 \times 10^{-2})$	$(4.59428 \times 10^{-4}, 1.43648 \times 10^{-3})$
0.7	$(1.82537 \times 10^{-3}, 7.15873 \times 10^{-3})$	$(3.50314 \times 10^{-4}, 8.75759 \times 10^{-4})$
0.8	$(8.88641 \times 10^{-4}, 2.65294 \times 10^{-3})$	$(2.01529 \times 10^{-4}, 4.12923 \times 10^{-4})$
0.9	$(4.12378 \times 10^{-3}, 2.38319 \times 10^{-5})$	$(7.16129 \times 10^{-5}, 1.04535 \times 10^{-4})$
1.0	$(7.64147 \times 10^{-3}, 6.32640 \times 10^{-4})$	$(2.97710 \times 10^{-5}, 8.93103 \times 10^{-6})$

Table 16 Absolute errors of Application 5 by rationalized Haar functions method in [28] for $(\psi_1(x), \psi_2(x))$.

x	$M=16$	$M=32$
0.1	$(5.98 \times 10^{-3}, 2.40 \times 10^{-4})$	$(9.44 \times 10^{-3}, 1.07 \times 10^{-3})$
0.2	$(1.89 \times 10^{-2}, 4.12 \times 10^{-3})$	$(3.18 \times 10^{-3}, 7.00 \times 10^{-4})$
0.3	$(1.85 \times 10^{-2}, 5.12 \times 10^{-3})$	$(3.08 \times 10^{-3}, 8.60 \times 10^{-4})$
0.4	$(6.41 \times 10^{-3}, 2.62 \times 10^{-3})$	$(9.34 \times 10^{-3}, 3.56 \times 10^{-3})$
0.5	$(3.11 \times 10^{-2}, 1.44 \times 10^{-2})$	$(1.56 \times 10^{-2}, 7.36 \times 10^{-3})$
0.6	$(6.18 \times 10^{-3}, 3.53 \times 10^{-3})$	$(9.39 \times 10^{-3}, 5.32 \times 10^{-3})$
0.7	$(1.87 \times 10^{-2}, 1.20 \times 10^{-2})$	$(3.13 \times 10^{-3}, 1.98 \times 10^{-3})$
0.8	$(1.87 \times 10^{-2}, 1.35 \times 10^{-2})$	$(3.13 \times 10^{-3}, 2.31 \times 10^{-3})$
0.9	$(6.17 \times 10^{-3}, 4.45 \times 10^{-3})$	$(9.40 \times 10^{-3}, 7.42 \times 10^{-3})$
1.0	$(3.13 \times 10^{-2}, 2.65 \times 10^{-2})$	$(1.56 \times 10^{-2}, 1.32 \times 10^{-2})$

Table 17 Absolute errors of Application 5 by Taylor-series expansion method in [31] with $m = 10$.

x	$\psi_1(x)$	$\psi_2(x)$
0.1	3.809×10^{-2}	6.406×10^{-3}
0.2	2.748×10^{-2}	4.259×10^{-3}
0.3	1.745×10^{-2}	4.240×10^{-4}
0.4	8.999×10^{-3}	2.835×10^{-3}
0.5	2.914×10^{-3}	3.644×10^{-3}
0.6	3.770×10^{-4}	5.120×10^{-4}
0.7	9.700×10^{-4}	7.755×10^{-3}
0.8	3.290×10^{-4}	2.216×10^{-2}
0.9	1.818×10^{-3}	4.367×10^{-2}
1.0	7.100×10^{-4}	7.329×10^{-2}

Table 18 The maximum of the absolute errors in [59] by collocation method.

x	$\cos x$
1.91×10^{-2}	1.02×10^{-2}

5 Conclusion

In this research, an efficient method based on Taylor expansion has been suggested to determine the approximate solutions of systems of linear Fredholm integral equations of the second kind. An interesting feature of the proposed method is to find the exact solutions if the system has the exact solutions as polynomial functions. The method can be easily extended to systems of Fredholm integro-differential equations. Numerical results from tables have illustrated the efficiency and the applicability of the approach compared to those of some given methods.

6 Declarations

6.1 Conflicts of interest:

Not applicable.

6.2 Funding:

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6.3 Author's contribution:

M.D.-Conceptualization, Methodology, Software, Writing-Original Draft. F.E.- Investigation, Writing, Review and Editing. All authors read and approved the final submitted version of this manuscript.

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6.5 Data availability statement:

All data that support the findings of this study are included within the article.

6.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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