Paradigm International Journal of Mathematics and Computer in Engineering 4(1) (2026) 107–126

International Journal of Mathematics and Computer in Engineering

https://reference-global.com/journal/IJMCE

Original Study

On the soliton solutions of the generalized stochastic nonlinear Schrödinger equation with Kerr effect and higher order nonlinearity via two analytical methods

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Communicated by Nikolai A. Kudryashov; Received: 26.06.2025; Accepted: 17.08.2025; Online: 00.00.2026

Abstract

In this study, we investigate the generalized stochastic nonlinear Schrödinger equation, which models the propagation of ultrashort optical pulses in nonlinear and dispersive media, incorporating both Kerr effect and higher-order nonlinearities. To construct exact analytical solutions, we employ the $\tan\left(\frac{\varpi(\xi)}{2}\right)$-expansion method and the $(G'/G, 1/G)$-expansion method. These methods yield a variety of exact solutions, including dark, singular, and singular periodic soliton solutions, each representing different physical wave behaviors. We further perform a stability analysis to determine the robustness of these solutions under perturbations and examine their temporal evolution to better understand their propagation dynamics. Graphical illustrations of selected solutions are provided to visualize their dynamics and to demonstrate how the passage of time influences the structure and stability of the resulting wave forms.

Keywords: The $\tan\left(\frac{\varpi(\xi)}{2}\right)$-expansion method, stability analysis, $(G'/G, 1/G)$-expansion method, traveling wave solution.
AMS 2020 codes: 35C07; 35D05; 45K05; 65M12; 65M70.

1 Introduction

Nonlinear partial differential equations (NLPDEs) are extensively utilized across multiple fields, including mathematical biology, chemistry, engineering, plasma physics, quantum mechanics, and fluid dynamics [1]. In order to explore and explain nonlinear effects, researchers in mathematics and physics focus on obtaining exact solutions of the partial differential equations that govern such systems [2, 3]. The study of complex nonlinear partial differential equations has garnered significant academic interest, driven by the inherent challenges of solving them and their importance in understanding intricate natural phenomena [4]. Significant advancements have been observed in the development of effective methods for obtaining accurate solutions to NLPDEs in recent years, including Hirota direct method [5, 6], the new extended auxiliary equation method [7], the new extended direct algebraic method [8], the sin-Gordon expansion method [9], modified simple equation method [10], the sub-ODE method [11], F-expansion method [12], the $(m+G'/G)$ expansion method [13], the improved $\tan(\phi(\xi)/2)$-expansion method [14], the Bernoulli sub-equation function method [15], Jacobi elliptic function method [16, 17], the unified Riccati equation expansion method [18, 19], the modified Sardar sub-equation approach [20, 21], and many more.

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Paradigm ISSN 2956-7068 doi:10.2478/ijmce-2026-0007 Paradigm

A stochastic process mathematically represents the evolution of a random phenomenon over time. Stochastic differential equations (SDEs) extend this idea by integrating random fluctuations into system dynamics. Similarly, stochastic partial differential equations (SPDEs) incorporate random variables or noise functions, providing suitable mathematical representations for complex systems influenced by uncertainty. In nonlinear optics, optical solitons, widely used in high-speed data transmission, are subject to stochastic disturbances from external influences. This necessitates the formulation of differential equations with stochastic elements for accurate modeling. The stochastic nonlinear Schrödinger equation (SNLSE) provides a mathematical framework for analyzing such systems, especially in quantum mechanics and nonlinear optics [22]. It describes the evolution of a quantum field that incorporates both nonlinear interactions and stochastic effects. Beyond optics, stochastic networked linear systems are also employed to capture random influences across diverse fields such as chemistry, physics, and electrical engineering. Increasingly, the SNLSE has been applied in areas including physics modeling, climate research, and information technology, where it serves as a powerful tool for constructing mathematical models of complex phenomena.

The governing model under consideration is the stochastic generalized nonlinear Schrödinger equation (SGNLSE), as described in [23]:

$$iq_t + q_{xx} + |q|^2 q + i\sigma\left(q_{xxx} + \gamma_1 |q|^2 q_x + \gamma_2 q^2 q_x^*\right) + \sigma \frac{\partial W}{\partial t} q = 0. \tag{1}$$

Here, Eq.(1) describes the propagation of light in a medium by incorporating the Kerr effect, which causes intensity-dependent changes in the refractive index. This nonlinearity affects both the phase and intensity of the light, thereby altering the energy distribution of the wave. The generalized nonlinear Schrödinger equation (GNLSE) is often used to predict energy changes, such as gain or loss, that arise from these nonlinear interactions. As a wave travels through a medium, it may experience energy gain similar to amplification in electronic systems or energy loss due to mechanisms such as absorption, scattering, and transmission. A filter is a device that selectively permits certain frequency components of a wave to pass while blocking others, thereby manipulating the energy distribution of the wave and often causing a net change in its energy. In this framework, $q = q(x,t)$ denotes the soliton pulse profile, with q_t representing temporal evolution, q_{xx} corresponding to group velocity dispersion (GVD), and q_{xxx} accounting for third-order dispersion. The term $|q|^2 q$ captures the Kerr effect, while q with an asterisk (q^*) denotes the complex conjugate of q. The expressions $|q|^2 q_x$ and $q^2 q_x^*$ represent higher-order nonlinear terms, with γ_1 and γ_2 as real-valued physical parameters. The term $W(t)$ denotes a one-dimensional Wiener process, which introduces stochasticity into the system and is formally defined through the Itô integral as [24]:

$$W(t) = \int_0^{t} q(\tau) dW(\tau), \text{ where } \tau < t.$$

The aim of this paper is to employ the $\tan\left(\frac{\varpi(\xi)}{2}\right)$ expansion and the ($G'/G, 1/G$)-expansion technique to Eq. (1) to find exact solutions and apply novel analytical methods to obtain exact soliton solutions. The $\tan\left(\frac{\varpi(\xi)}{2}\right)$-expansion technique, presented as an alternative and effective analytical method, has demonstrated its efficacy as a powerful instrument for deriving accurate solutions to NLPDEs. This method employs the hyperbolic tangent function to convert nonlinear equations into algebraic equations. For instance, the modified Zakharov-Kuznetsov equation [25] and Schrödinger type nonlinear evolution equations [26]. The two-variable ($G'/G, 1/G$)-expansion method is a recently developed analytical technique for constructing exact solutions of nonlinear evolution equations (NLEEs). This method involves introducing two independent auxiliary functions based on the ratio of derivatives and reciprocals of a function G, which satisfies a second-order linear ordinary differential equation. By converting complex NLPDEs into algebraic forms, the method facilitates the derivation of exact traveling wave solutions. It was first applied by Li et al. [27] to the Zakharov equations and has since been employed by several researchers to solve a wide range of NLEEs, including higher-order nonlinear Schrödinger and quantum Zakharov-Kuznetsov equations [28, 29].

In addition to classical expansion approaches, several modern analytical techniques have been developed for constructing exact solutions of NLPDEs. Kudryashov proposed effective methods for obtaining exact solitary solutions to the generalized Kuramoto-Sivashinsky equation, providing an important basis for further analytical investigations [30, 31]. Later, Kudryashov et al. [32] applied Painlevé analysis and the first integral method to the traveling wave reductions of a nonlinear equation, namely, the Radhakrishnan-Kundu-Lakshmanan equation. Demina and Kudryashov [33] presented a general method for constructing explicit meromorphic solutions of autonomous nonlinear ordinary differential equations based on the analysis of Laurent series and singularity structures. In a related but more specific study, they applied this approach to derive exact meromorphic solutions of the Kawahara equation, demonstrating its effectiveness for higher-order dispersive wave equations [34]. In another contribution, Kudryashov presented a robust approach for finding exact solutions to nonlinear differential equations, emphasizing its generality and efficiency [35]. Our study incorporates the $\tan\left(\frac{\varpi(\xi)}{2}\right)$-expansion and $(G'/G, 1/G)$-expansion methods while exploring their potential for generating physically meaningful solutions for the generalized stochastic nonlinear Schrödinger equation with Kerr effect and higher-order nonlinearity.

The rest of this paper is organized as follows. In Section 2, the core procedures of the $\tan\left(\frac{\varpi(\xi)}{2}\right)$-expansion method and the $(G'/G, 1/G)$-expansion method are outlined. Section 3 focuses on the application of both techniques to Eq.(1), followed by a detailed analysis of the resulting exact analytical solutions. Section 4 presents the stability analysis of the obtained soliton solution. Section 5 includes 2D and 3D visualizations and examines how the solution evolves over time. The results are discussed in Section 6. Section 7 concludes the paper with a summary of the main findings.

2 Description of the methods

2.1 The $\tan\left(\frac{\varpi(\xi)}{2}\right)$-expansion method

Suppose a NPDE follows the following form:

$$P\left(q, q_x, q_t, q_{tx}, \cdots\right) = 0, \tag{2}$$

where $q = q(x,t)$ and it can be transformed into an ordinary differential equation (ODE):

$$O\left(Q, Q', Q', Q'', \cdots\right) = 0. \tag{3}$$

By using the wave transformation $q = Q(\xi)$ and $\xi = x - vt$. The following expression is proposed as a traveling wave solution to Eq.(3):

$$Q(\xi) = W(\varpi) = \sum_{k=0}^{n} a_k \left(\frac{\tan(\varpi(\xi))}{2}\right)^k, 0 \le k \le n, \tag{4}$$

with constants $a_k \neq 0$, which will be evaluated subsequently and must satisfy the governing ODE.

$$\varpi'(\xi) = A\sin(\varpi(\xi)) + B\cos(\varpi(\xi)) + C. \tag{5}$$

Here are a class of solutions to Eq.(5):

Family 1: When $A^2 + B^2 - C^2 < 0$ and $B - C \neq 0$, the corresponding solution is given by

$$\varpi(\xi) = 2\tan^{-1}\left(\frac{A}{B-C} - \left(\frac{\sqrt{-A^2 - B^2 + C^2}}{B-C}\right)\tan\left(\frac{\sqrt{-A^2 - B^2 + C^2}}{2}\xi\right)\right).$$

Family 2: In case $A^2 + B^2 - C^2 > 0$ and $B - C \neq 0$, the corresponding solution is given by

$$\varpi(\xi) = 2\tan^{-1}\left(\frac{A}{B-C} + \left(\frac{\sqrt{A^2 + B^2 - C^2}}{B-C}\right)\tanh\left(\frac{\sqrt{A^2 + B^2 - C^2}}{2}\xi\right)\right).$$

Family 3: In case $A^2+B^2-C^2>0, B\neq 0$ and $C=0$, the corresponding solution is given by

$$\varpi(\xi)=2\tan^{-1}\left(\frac{A}{B-C}+\frac{\sqrt{B^2-A^2}}{B}\tanh\left(\frac{\sqrt{B^2-A^2}}{2}\xi\right)\right).$$

Family 4: In case $A^2+B^2-C^2<0, C\neq 0$ and $B=0$, the corresponding solution is given by

$$\varpi(\xi)=2\tan^{-1}\left(-\frac{A}{C}+\frac{\sqrt{C^2-A^2}}{C}\tan\left(\frac{\sqrt{C^2-A^2}}{2}\xi\right)\right).$$

Family 5: In case $A^2+B^2-C^2>0, B-C\neq 0$ and $A=0$, the corresponding solution is given by

$$\varpi(\xi)=2\tan^{-1}\left(\sqrt{\frac{B+C}{B-C}}\tanh\left(\frac{\sqrt{B^2-C^2}}{2}\xi\right)\right).$$

Family 6: In case $A=0$ and $C=0$, the corresponding solution is given by

$$\varpi(\xi)=\tan^{-1}\left(\frac{2e^{B\xi}}{e^{2B\xi}+1}\right).$$

Family 7: In case $B=C=0$, the corresponding solution is given by

$$\varpi(\xi)=\tan^{-1}\left(\frac{2e^{A\xi}}{e^{2A\xi}+1}\right).$$

Family 8: In case $A^2+B^2=C^2$, the corresponding solution is given by

$$\varpi(\xi)=2\tan^{-1}\left(\frac{A\xi+2}{(B-C)\xi}\right).$$

Family 9: In case $A=B=C=kA$, the corresponding solution is given by

$$\varpi(\xi)=2\tan^{-1}\left(e^{kA\xi}-1\right).$$

Family 10: In case $A=C=kA$ and $B=-kA$, the corresponding solution is given by

$$\varpi(\xi)=-2\tan^{-1}\left(\frac{e^{kA\xi}}{-1+e^{kA\xi}}\right).$$

Family 11: In case $C=A$, the corresponding solution is given by

$$\varpi(\xi)=-2\tan^{-1}\left(\frac{(A+B)e^{B\xi}-1}{(A-B)e^{B\xi}-1}\right).$$

Family 12: In case $A=C$, the corresponding solution is given by

$$\varpi(\xi)=2\tan^{-1}\left(\frac{(B+A)e^{B\xi}+1}{(B-A)e^{B\xi}-1}\right).$$

Family 13: In case $C=-A$, the corresponding solution is given by

$$\varpi(\xi)=2\tan^{-1}\left(\frac{B-A+e^{B\xi}}{-B-A+e^{B\xi}}\right).$$

Family 14: In case $B=-C$, the corresponding solution is given by

$$\varpi(\xi)=2\tan^{-1}\left(\frac{Ae^{A\xi}}{1-Ce^{A\xi}}\right).$$

Family 15: In case $B=0$ and $A=C$, the corresponding solution is given by

$$\varpi(\xi)=-2\tan^{-1}\left(\frac{C\xi+2}{C\xi}\right).$$

Family 16: In case $A=0$ while $B=C$, the corresponding solution is given by

$$\varpi(\xi)=2\tan^{-1}(C\xi).$$

Family 17: In case $A=0$ while $B=-C$, the corresponding solution is given by

$$\varpi(\xi)=-2\tan^{-1}\left(\frac{1}{C\xi}\right).$$

Family 18: In case $A=B=0$, the corresponding solution is given by

$$\varpi(\xi)=C\xi+\varepsilon.$$

Family 19: In case $B=C$, the corresponding solution is given by

$$\varpi(\xi)=2\tan^{-1}\left(\frac{e^{A\xi}-C}{A}\right).$$

The value of n is determined using the balance principle. By substituting Eq.(4) into Eq.(3) and collecting the coefficients of $(\tan(\varpi/2))^n$ for $(n=1,2,3,\cdots)$, a system of equations is obtained by setting each coefficient to zero. This results in an over-determined set of equations for A,B,C, and a_k $(k=1,2,\cdots,n)$. Symbolic computations were carried out using the computational package program.

2.2 The $(G'/G,1/G)$ - expansion method

This part focuses on elucidating fundamental ideas of mention technique, which is employed to determine precise solutions of non-linear equations. Let us examine second order of linear ODE.

$$G''[\xi]+\delta G[\xi]=\rho, \tag{6}$$

set as,

$$\psi=1/G, \text{ and } \phi=G'/G. \tag{7}$$

It may be inferred from the aforementioned relationships,

$$\phi'=-\phi^2+\rho\psi-\delta, \text{ and } \psi'=-\phi\psi. \tag{8}$$

Three types of solutions may be discovered for various values of δ :
Case 1: When $\delta<0$, the general solution corresponding to Eq.(6) is given by the following expression:

$$G(\xi)=A_1\cosh(\sqrt{-\delta}\xi)+B_2\sinh(\sqrt{-\delta}\xi)+\frac{\rho}{\delta}. \tag{9}$$

Therefore, we acquire

$$\psi^2=\frac{-\delta\left(\phi^2-2\rho\psi+\delta\right)}{\delta^2\left(A_1{}^2-A_2{}^2\right)+\rho^2}. \tag{10}$$

Here A_1, A_2 are constants.
Case 2: When $\delta > 0$, the general solution corresponding to Eq.(6) is given by the following expression:

$$G(\xi) = A_1 \cos(\sqrt{\delta}\xi) + A_2 \sin(\sqrt{\delta}\xi) + \frac{\rho}{\delta}, \tag{11}$$

and consequently, we obtain

$$\psi^2 = \frac{\delta\left(\phi^2 - 2\rho\psi + \delta\right)}{\delta^2\left({A_1}^2 + {A_2}^2\right) - \rho^2}. \tag{12}$$

Here A_1, A_2 are constants.
Case 3: When $\delta = 0$, the general solution corresponding to Eq.(6) is given by the following expression:

$$G(\xi) = \frac{\rho}{2}\xi^2 + A_1\xi + A_2, \tag{13}$$

and thus, we obtain

$$\psi^2 = \frac{\left(\phi^2 - 2\rho\psi\right)}{A_1^2 - 2\rho A_2}. \tag{14}$$

Here A_1, A_2 are constants.
The main steps of the ($G'/G, 1/G$) -expansion method are as follows:
Step 1. Apply the coordinate transformation $\xi = x - vt$ and assume $q(x,t) = Q(\xi)$. Under this transformation, Eq.(2) is reduced to ODE in terms of $Q(\xi)$, given by:

$$P\left(Q, -vQ', Q', v^2Q'', -vQ'', Q'', \cdots\right) = 0. \tag{15}$$

Step 2. Assume that the solution to ODE Eq.(15) can be represented as a polynomial involving the functions ϕ and ψ in the following form:

$$Q(\xi) = \sum_{i=0}^{n} a_i\phi^i + \sum_{i=1}^{n} B_i\phi^{i-1}\psi. \tag{16}$$

Here, the constants a_i $(i = 0, \cdots, n)$, B_i $(i = 1, \cdots, n)$, δ, and ρ are to be determined later. The positive integer n can be found by applying the homogeneous balance between the highest-order derivatives and the nonlinear terms present in ODE Eq. (15).
Step 3. By substituting Eq.(6) into Eq.(15) and applying Eq.(8) and Eq.(10), the left-hand side of Eq.(15) is transformed into a polynomial in ϕ and ψ. Setting each coefficient of this polynomial to zero produces a system of algebraic equations for the constants a_i $(i = 0, \cdots, n)$, b_i $(i = 1, \cdots, n)$, δ (with $\delta < 0$), ρ, A_1, and A_2.
Step 4. Using computational program, the system of algebraic equations obtained in Step 3 is solved. By substituting the computed values of a_i (for $i = 0, \cdots, n$), b_i (for $i = 1, \cdots, n$), together with the parameters δ, ρ, A_1, and A_2 into Eq.(16), the travelling wave solutions are derived. These solutions are expressed in terms of hyperbolic functions, corresponding to the form presented in Eq.(15).
Step 5. By applying a similar procedure as in Steps 3 and 4, Eq.(16) is substituted into Eq.(15). Then, by making use of equations (8) and (12) (or alternatively, equations (8) and (14)), the travelling wave solutions of equation (15) are obtained. These solutions are expressed either in terms of trigonometric functions or as rational functions, depending on the specific case.

3 Applications

3.1 Application of $\tan\left(\frac{\varpi(\xi)}{2}\right)$-expansion method

This part of the study employs the subsequent transformation to get analytical solutions for Eq.(1)

$$q(x,t) = Q(\xi)e^{i\left[-\kappa x + \omega t + \sigma W(t) - \sigma^2 t\right]}, \xi = x - vt, \tag{17}$$

where κ, ω, σ, and v are constants. By substituting Eq.(17) into Eq.(1), the real and imaginary parts are obtained as

$$(1+k\gamma_1\sigma-k\gamma_2\sigma)\,Q^3-\left(k^2+k^3\sigma-\sigma^2+\omega\right)Q+(1+3k\sigma)Q''=0, \tag{18}$$

and

$$\left(-2k-v-3k^2\sigma\right)Q'+(\gamma_1+\gamma_2)\,\sigma Q'Q^2+\sigma Q'''=0. \tag{19}$$

Integrating Eq.(19), and assuming the constant of integration $C_0=0$ for simplicity, we obtain

$$\left(-2k-v-3k^2\sigma\right)Q+\frac{1}{3}(\gamma_1+\gamma_2)\,\sigma Q^3+\sigma Q''=0. \tag{20}$$

Balancing Q^3 with Q'' obtaining $n=1$ is feasible. Utilizing the derived value of n, the solution for Eq.(18) and Eq.(20) may be expressed as

$$\mathrm{Q}(\xi)=a_0+a_1\tan\left(\frac{\varpi(\xi)}{2}\right), \tag{21}$$

a system of equations is generated by substituting Eq.(21) into equations (18) and (20). The solutions of this system yield the subsequent set of solutions as following

Set 1: When

$$a_0=\frac{A\sqrt{-1-6k\sigma}}{\sqrt{2+4k\gamma_1\sigma}},a_1=\frac{(-B+C)\sqrt{-1-6k\sigma}}{\sqrt{2+4k\gamma_1\sigma}},\gamma_2=\frac{3-\gamma_1}{1+6k\sigma},v=-2k-\frac{1}{2}\left(A^2+B^2-C^2+6k^2\right)\sigma,$$
$$\omega=\frac{1}{2}\left(C^2-2k^2+\sigma\left(3C^2k-2k^3+2\sigma\right)-A^2(1+3k\sigma)-B^2(1+3k\sigma)\right), \tag{22}$$

where $-1-6k\sigma>0, 2+4k\gamma_1\sigma>0$, the outcomes solutions are:
Solution 1: When $-A^2-B^2+C^2>0$, the singular periodic solution corresponding to Family 1 is obtained

$$q(x,t)=\left(\frac{\sqrt{(-1-6k\sigma)\left(-A^2-B^2+C^2\right)}}{\sqrt{2+4k\gamma_1\sigma}}\right)\tan\left(\frac{1}{4}\sqrt{-A^2-B^2+C^2}\left(4kt+2x+\left(A^2+B^2-C^2\right)t\sigma+6k^2t\sigma\right)\right)$$
$$\times\,\mathrm{e}^{-\frac{1}{2}\left(B^2t-C^2t+2k^2t+2kx+3B^2kt\sigma-3C^2kt\sigma+2k^3t\sigma+A^2(t+3kt\sigma)-2\sigma W(t)\right)}. \tag{23}$$

Solution 2: When $A^2+B^2-C^2>0$, the dark soliton solution corresponding to Family 2 is obtained

$$q(x,t)=\left(-\frac{\sqrt{(-1-6k\sigma)\left(A^2+B^2-C^2\right)}}{\sqrt{2+4k\gamma_1\sigma}}\right)\tanh\left(\frac{1}{4}\sqrt{A^2+B^2-C^2}\left(4kt+2x+\left(A^2+B^2-C^2\right)t\sigma+6k^2t\sigma\right)\right)$$
$$\times\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\left(-C^2t+2k^2t+2kx-3C^2kt\sigma+2k^3t\sigma-2W(t)\sigma+A^2(t+3kt\sigma)+B^2(t+3kt\sigma)\right)}. \tag{24}$$

Solution 3: The dark soliton solution corresponding to Family 3 is obtained

$$q(\mathrm{x},\mathrm{t})=\left(-\frac{\sqrt{A^2+B^2}\sqrt{-1-6k\sigma}}{\sqrt{2+4k\gamma_1\sigma}}\right)\tanh\left(\frac{1}{4}\sqrt{A^2+B^2}\left(4kt+2x+\left(A^2+B^2\right)t\sigma+6k^2t\sigma\right)\right)$$
$$\times\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\left(A^2(t+3kt\sigma)+B^2(t+3kt\sigma)+2\left(k^2t+kx+k^3t\sigma-W(t)\sigma\right)\right)}. \tag{25}$$

Solution 4: The singular periodic solution corresponding to Family 4 is obtained

$$q(\mathrm{x},\mathrm{t})=\left(\frac{\sqrt{(-1-6k\sigma)\left(-A^2+C^2\right)}}{\sqrt{2+4k\gamma_1\sigma}}\right)\tan\left(\frac{1}{4}\sqrt{-A^2+C^2}\left(4kt+2x+\left(A^2-C^2\right)t\sigma+6k^2t\sigma\right)\right)$$
$$\times\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\left(A^2(t+3kt\sigma)-C^2(t+3kt\sigma)+2\left(k^2t+kx+k^3t\sigma-W(t)\sigma\right)\right)}. \tag{26}$$

Solution 5: When $B^2 - C^2 > 0$, the dark soliton solution corresponding to Family 5 is obtained

$$q(\mathrm{x},\mathrm{t}) = \left(\frac{\sqrt{(-1-6k\sigma)\left(B^2-C^2\right)}}{\sqrt{2+4k\gamma_1\sigma}}\right)\tanh\left(\frac{1}{4}\sqrt{B^2-C^2}\left(4kt+2x+\left(B^2-C^2\right)t\sigma+6k^2t\sigma\right)\right) \times \mathrm{e}^{-\frac{1}{2}\mathrm{i}\left(-C^2t+2k^2t+2kx-3C^2kt\sigma+2k^3t\sigma+B^2(t+3kt\sigma)-2\sigma W(t)\right)}. \tag{27}$$

Solution 6: The singular soliton solution corresponding to Family 8 is obtained

$$q(\mathrm{x},\mathrm{t}) = \left(-\frac{\sqrt{2}\sqrt{-1-6k\sigma}}{(2kt+x+3k^2t\sigma)\sqrt{1+2k\gamma_1\sigma}}\right) \times \mathrm{e}^{-\mathrm{i}k\left(kt+x+k^2t\sigma\right)+\mathrm{i}\sigma W(t)}. \tag{28}$$

Solution 7: The dark soliton solution corresponding to Family 11 is obtained

$$q(\mathrm{x},\mathrm{t}) = \left(\frac{A\sqrt{-1-6k\sigma}}{\sqrt{2+4k\gamma_1\sigma}} - \frac{(A-B)\left(-1+(A+B)\mathrm{e}^{B\left(x-t\left(-2k-\frac{1}{2}\left(B^2+6k^2\right)\sigma\right)\right)}\right)\sqrt{-1-6k\sigma}}{\left(-1+(A-B)\mathrm{e}^{B\left(x-t\left(-2k-\frac{1}{2}\left(B^2+6k^2\right)\sigma\right)\right)}\right)\sqrt{2+4k\gamma_1\sigma}}\right) \times \mathrm{e}^{\mathrm{i}\left(-kx+W(t)\sigma-t\sigma^2+t\left(-k^2-k^3\sigma+\sigma^2-\frac{1}{2}B^2(1+3k\sigma)\right)\right)}. \tag{29}$$

Solution 8: The dark soliton solution corresponding to Family 12 is obtained

$$q(\mathrm{x},\mathrm{t}) = \left(\frac{C\sqrt{-1-6k\sigma}}{\sqrt{2+4k\gamma_1\sigma}} + \frac{(-B+C)\left(1+(B+C)\mathrm{e}^{B\left(x-t\left(-2k-\frac{1}{2}\left(B^2+6k^2\right)\sigma\right)\right)}\right)\sqrt{-1-6k\sigma}}{\left(-1+(B-C)\mathrm{e}^{B\left(x-t\left(-2k-\frac{1}{2}\left(B^2+6k^2\right)\sigma\right)\right)}\right)\sqrt{2+4k\gamma_1\sigma}}\right) \times \mathrm{e}^{\mathrm{i}\left(-kx+W(t)\sigma-t\sigma^2+t\left(-k^2-k^3\sigma+\sigma^2-\frac{1}{2}B^2(1+3k\sigma)\right)\right)}. \tag{30}$$

Solution 9: The singular soliton solution corresponding to Family 13 is obtained

$$q(\mathrm{x},\mathrm{t}) = \left(\frac{A\sqrt{-1-6k\sigma}}{\sqrt{2+4k\gamma_1\sigma}} - \frac{(A+B)\left(-A+B+\mathrm{e}^{B\left(x-t\left(-2k-\frac{1}{2}\left(B^2+6k^2\right)\sigma\right)\right)}\right)\sqrt{-1-6k\sigma}}{\left(-A-B+\mathrm{e}^{B\left(x-t\left(-2k-\frac{1}{2}\left(B^2+6k^2\right)\sigma\right)\right)}\right)\sqrt{2+4k\gamma_1\sigma}}\right) \times \mathrm{e}^{\mathrm{i}\left(-kx+W(t)\sigma-t\sigma^2+t\left(-k^2-k^3\sigma+\sigma^2-\frac{1}{2}B^2(1+3k\sigma)\right)\right)}. \tag{31}$$

Solution 10: The dark soliton solution corresponding to Family 14 is obtained

$$q(\mathrm{x},\mathrm{t}) = \left(\frac{A\sqrt{-1-6k\sigma}}{\sqrt{2+4k\gamma_1\sigma}} + \frac{2AC\mathrm{e}^{A\left(x-t\left(-2k-\frac{1}{2}\left(A^2+6k^2\right)\sigma\right)\right)}\sqrt{-1-6k\sigma}}{\left(1-C\mathrm{e}^{A1\left(x-t\left(-2k-\frac{1}{2}\left(A^2+6k^2\right)\sigma\right)\right)}\right)\sqrt{2+4k\gamma_1\sigma}}\right) \times \mathrm{e}^{\mathrm{i}\left(-kx+W(t)\sigma-t\sigma^2+t\left(-k^2-k^3\sigma+\sigma^2-\frac{1}{2}A^2(1+3k\sigma)\right)\right)}. \tag{32}$$

Solution 11: The singular soliton solution corresponding to Family 15 is obtained

$$q(\mathrm{x},\mathrm{t}) = \left(\frac{C\sqrt{-1-6k\sigma}}{\sqrt{2+4k\gamma_1\sigma}} - \frac{\sqrt{-1-6k\sigma}(2+C(x+kt(2+3k\sigma)))}{\sqrt{2+4k\gamma_1\sigma}(x+kt(2+3k\sigma))}\right) \times \mathrm{e}^{\mathrm{i}\left(-kx+W(t)\sigma-t\sigma^2+t\left(-k^2-k^3\sigma+\sigma^2\right)\right)}. \tag{33}$$

Solution 12: The singular soliton solution corresponding to Family 17 is obtained

$$q(\mathrm{x},\mathrm{t}) = \left(-\frac{2\sqrt{-1-6k\sigma}}{\sqrt{2+4k\gamma_1\sigma}(x+kt(2+3k\sigma))}\right) \times \mathrm{e}^{\mathrm{i}\left(-kx+W\sigma-t\sigma^2+t\left(-k^2-k^3\sigma+\sigma^2\right)\right)}. \tag{34}$$

Solution 13: The singular soliton solution corresponding to Family 18 is obtained

$$q(\mathrm{x,t})=\left(\frac{C\sqrt{-1-6k\sigma}}{\sqrt{2+4k\gamma_1\sigma}}\right)\tan\left(\frac{1}{2}C\left(x-t\left(\frac{C^2\sigma}{2}-k(2+3k\sigma)\right)\right)\right)$$
$$\times\, e^{\mathrm{i}\left(-kx+W(t)\sigma-t\sigma^2+t\left(-k^2-k^3\sigma+\sigma^2+\frac{1}{2}C^2(1+3k\sigma)\right)\right)}. \tag{35}$$

Set 2: In the case

$$a_0=\frac{\sqrt{\frac{3}{2}}A}{\sqrt{-\gamma_1-\gamma_2}}, a_1=\frac{\sqrt{\frac{3}{2}}(-B+C)}{\sqrt{-\gamma_1-\gamma_2}}, k=-\frac{(\gamma_1+\gamma_2-3)}{6\gamma_2\sigma},$$
$$\omega=-\frac{\left(A^2+B^2-C^2\right)(3-\gamma_1+\gamma_2)}{4\gamma_2}+\frac{(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{216\gamma_2{}^3\sigma^2}+\sigma^2, \tag{36}$$
$$v=-\frac{9+\gamma_1{}^2-2\gamma_1(3+\gamma_2)+3\gamma_2\left(2+\gamma_2\left(-1+2\left(A^2+B^2-C^2\right)\sigma^2\right)\right)}{12\gamma_2{}^2\sigma},$$

where $-\gamma_1-\gamma_2>0, \gamma_2\neq 0$, and $\sigma\neq 0$, the outcomes solutions are:
Solution 1: When $-A_1{}^2-B_1{}^2+C_1{}^2>0$, the following singular periodic solution corresponding to Family 1 is obtained

$$q(x,t)=\left(\sqrt{\frac{3}{2}}\frac{\sqrt{-A^2-B^2+C^2}}{\sqrt{-\gamma_1-\gamma_2}}\right)\times e^{\frac{1}{6}\mathrm{i}\left(-\frac{3\left(A^2+B^2-C^2\right)t(3-\gamma_1+\gamma_2)}{2\gamma_2}+\frac{t(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{36\gamma_2^3\sigma^2}+\frac{x(-3+\gamma_1+\gamma_2)}{\gamma_2\sigma}+6W(t)\sigma\right)}$$
$$\times\tan\left(\frac{1}{2}\sqrt{-A^2-B^2+C^2}\left(x+\frac{t\left(9+\gamma_1^2-2\gamma_1(3+\gamma_2)+3\gamma_2\left(2+\gamma_2\left(-1+2\left(A^2+B^2-C^2\right)\sigma^2\right)\right)\right)}{12\gamma_2^2\sigma}\right)\right). \tag{37}$$

Solution2: When $A^2+B^2-C^2>0$, the dark soliton solution corresponding to Family 2 is obtained

$$q(x,t)=\left(-\sqrt{\frac{3}{2}}\frac{\sqrt{A^2+B^2-C^2}}{\sqrt{-\gamma_1-\gamma_2}}\right)\times e^{\frac{1}{6}\mathrm{i}\left(-\frac{3\left(A^2+B^2-C^2\right)t(3-\gamma_1+\gamma_2)}{2\gamma_2}+\frac{t(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{36\gamma_2{}^3\sigma^2}+\frac{x(-3+\gamma_1+\gamma_2)}{\gamma_2\sigma}+6W(t)\sigma\right)}$$
$$\times\tanh\left(\frac{1}{2}\sqrt{A^2+B^2-C^2}\left(x+\frac{t\left(9+\gamma_1{}^2-2\gamma_1(3+\gamma_2)+3\gamma_2\left(2+\gamma_2\left(-1+2\left(A^2+B^2-C^2\right)\sigma^2\right)\right)\right)}{12\gamma_2{}^2\sigma}\right)\right). \tag{38}$$

Solution 3: The dark soliton solution corresponding to Family 3 is obtained

$$q(\mathrm{x,t})=\left(-\sqrt{\frac{3}{2}}\frac{\sqrt{A^2+B^2}}{\sqrt{-\gamma_1-\gamma_2}}\right)\times e^{\frac{1}{6}\left(-\frac{3\left(A^2+B^2\right)t(3-\gamma_1+\gamma_2)}{2\gamma_2}+\frac{t(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{36\gamma_2{}^3\sigma^2}+\frac{x(-3+\gamma_1+\gamma_2)}{\gamma_2\sigma}+\sigma W(t)\sigma\right)}$$
$$\times\tanh\left(\frac{1}{2}\sqrt{A^2+B^2}\left(x+\frac{t\left(9+\gamma_1{}^2-2\gamma_1(3+\gamma_2)+3\gamma_2\left(2+\gamma_2\left(-1+2\left(A^2+B^2\right)\sigma^2\right)\right)\right)}{12\gamma_2{}^2\sigma}\right)\right). \tag{39}$$

Solution 4: The singular periodic solution corresponding to Family 4 is obtained

$$q(\mathrm{x,t})=\left(\sqrt{\frac{3}{2}}\frac{\sqrt{-A^2+C^2}}{\sqrt{-\gamma_1-\gamma_2}}\right)\times e^{\frac{1}{6}\mathrm{i}\left(-\frac{3\left(A^2+C^2\right)t(3-\gamma_1+\gamma_2)}{2\gamma_2}+\frac{t(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{36\gamma_2{}^3\sigma^2}+\frac{x(-3+\gamma_1+\gamma_2)}{\gamma_2\sigma}+6W(t)\sigma\right)}$$
$$\times\tan\left(\frac{1}{2}\sqrt{-A^2+C^2}\left(x+\frac{t\left(9+\gamma_1^2-2\gamma_1(3+\gamma_2)+3\gamma_2\left(2+\gamma_2\left(-1+2\left(A^2-C^2\right)\sigma^2\right)\right)\right)}{12\gamma_2^2\sigma}\right)\right). \tag{40}$$

Solution 5: The dark soliton solution corresponding to Family 5 is obtained

$$
\begin{aligned}
q(\mathrm{x,t}) = {} & \left(-\sqrt{\frac{3}{2}}\frac{\sqrt{B^2-C^2}}{\sqrt{-\gamma_1-\gamma_2}}\right) \times \mathrm{e}^{\frac{1}{6}\mathrm{i}\left(\frac{3\left(A^2-C^2\right)t(3-\gamma_1+\gamma_2)}{2\gamma_2}+\frac{t(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{36\gamma_2{}^3\sigma^2}+\frac{x(-3+\gamma_1+\gamma_2)}{\gamma_2\sigma}+6w(t)\sigma\right)} \\
& \times \tanh\left(\frac{1}{2}\sqrt{B^2-C^2}\left(x+\frac{t\left(9+\gamma_1{}^2-2\gamma_1\left(3+\gamma_2\right)+3\gamma_2\left(2+\gamma_2\left(-1+2\left(A^2-C^2\right)\sigma^2\right)\right)\right)}{12\gamma_2{}^2\sigma}\right)\right).
\end{aligned} \tag{41}
$$

Solution 6: The singular soliton solution corresponding to Family 8 is obtained

$$
\begin{aligned}
q(\mathrm{x,t}) = {} & \left(-\frac{12\sqrt{6}\gamma_2{}^2\sigma}{\sqrt{-\gamma_1-\gamma_2}\left(t\left(9+\gamma_1{}^2+6\gamma_2-3\gamma_2{}^2-2\gamma_1\left(3+\gamma_2\right)\right)+12x\gamma_2{}^2\sigma\right)}\right) \\
& \times \mathrm{e}^{\frac{\mathrm{i}(-3+\gamma_1+\gamma_2)\left(t\left(9+\gamma_1{}^2+12\gamma_2-5\gamma_2{}^2-2\gamma_1\left(3+2\gamma_2\right)\right)+36x\gamma_2{}^2\sigma\right)}{216\gamma_2{}^3\sigma^2}+\mathrm{i}\sigma W(t)}.
\end{aligned} \tag{42}
$$

Solution 7: The dark soliton solution corresponding to Family 11 is obtained

$$
\begin{aligned}
q(\mathrm{x,t}) = {} & \left(\frac{A\sqrt{-1-6k\sigma}}{\sqrt{2+4k\gamma_1\sigma}}-\frac{\sqrt{\frac{3}{2}}(A-B)\left(-1+(A+B)\mathrm{e}^{B\left(x+\frac{t\left(9+\gamma_1^2+6\gamma_2-2\gamma_1\left(3+\gamma_2\right)+\gamma_2^2\left(-3+6B^2\sigma^2\right)\right)}{12\gamma_2^2\sigma}\right)}\right)}{\left(-1+(A-B)\mathrm{e}^{B\left(x+\frac{t\left(9+\gamma_1^2+6\gamma_2-2\gamma_1\left(3+\gamma_2\right)+\gamma_2^2\left(-3+6B^2\sigma^2\right)\right)}{12\gamma_2{}^2\sigma}\right)}\right)\sqrt{-\gamma_1-\gamma_2}}\right) \\
& \mathrm{e}^{\mathrm{i}\left(\frac{x(-3+\gamma_1+\gamma_2)}{6\gamma_2\sigma}+W(t)\sigma-t\sigma^2+t\left(-\frac{B^2(3-\gamma_1+\gamma_2)}{4\gamma_2}+\frac{(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{216\gamma_2{}^3\sigma^2}+\sigma^2\right)\right)}.
\end{aligned} \tag{43}
$$

Solution 8: The dark soliton solution corresponding to Family 12 is obtained

$$
\begin{aligned}
q(\mathrm{x,t}) = {} & \left(\frac{\sqrt{\frac{3}{2}}C}{\sqrt{-\gamma_1-\gamma_2}}+\frac{\sqrt{\frac{3}{2}}(-B+C)\left(1+(B+C)\mathrm{e}^{B\left(x+\frac{t+9+\gamma_1^2+6\gamma_2-2\gamma_1\left(3+\gamma_2\right)+\gamma_2^2\left(-3+6B^2\sigma^2\right)\right)}{12\gamma_2^2\sigma}\right)}\right)}{\left(-1+(B-C\mathrm{e}^{B\left(x+\frac{t\left(9+\gamma_1^2+6\gamma_2-2\gamma_1\left(3+\gamma_2\right)+\gamma_2^2\left(-3+6B^2\sigma^2\right)\right)}{12\gamma_2^2\sigma}\right)}\right)\sqrt{-\gamma_1-\gamma_2}}\right) \\
& \mathrm{e}^{\left(\frac{x(-3+\gamma_1+\gamma_2)}{6\gamma_2\sigma}+w(t)\sigma-t\sigma^2+t\left(-\frac{B^2(3-\gamma_1+\gamma_2)}{4\gamma_2}+\frac{(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{216\gamma_2^2\sigma^2}+\sigma^2\right)\right)}.
\end{aligned} \tag{44}
$$

Solution 9: The singular soliton solution corresponding to Family 13 is obtained

$$
\begin{aligned}
q(\mathrm{x,t}) = {} & \left(\frac{\sqrt{\frac{3}{2}}A}{\sqrt{-\gamma_1-\gamma_2}}-\frac{\sqrt{\frac{3}{2}}(A+B)\left(-A+B+\mathrm{e}^{B\left(x+\frac{t+9+\gamma_1^2+6\gamma_2-2\gamma_1\left(3+\gamma_2\right)+\gamma_2^2\left(-3+6B^2\sigma^2\right)\right)}{12\gamma_2^2\sigma}\right)}\right)}{\left(-A-B+\mathrm{e}^{B\left(x+\frac{t\left(9+\gamma_1^2+6\gamma_2-2\gamma_1\left(3+\gamma_2\right)+\gamma_2^2\left(-3+6B^2\sigma^2\right)\right)}{12\gamma_2^2\sigma}\right)}\right)\sqrt{-\gamma_1-\gamma_2}}\right) \\
& \mathrm{e}^{\left(\frac{x(-3+\gamma_1+\gamma_2)}{6\gamma_2\sigma}+w(t)\sigma-t\sigma^2+t\left(-\frac{B^2(3-\gamma_1+\gamma_2)}{4\gamma_2}+\frac{(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{216\gamma_2^2\sigma^2}+\sigma^2\right)\right)}.
\end{aligned} \tag{45}
$$

Solution 10: The dark soliton solution corresponding to Family 14 is obtained

$$q(\mathrm{x},\mathrm{t})=\left(\frac{\sqrt{\frac{3}{2}}A}{\sqrt{-\gamma_1-\gamma_2}}+\frac{\sqrt{6}AC\mathrm{e}^{A\left(x+\frac{t\left(9+\gamma_1^2+6\gamma_2-2\gamma_1(3+\gamma_2)+\gamma_2^2\left(-3+6B^2\sigma^2\right)\right)}{12\gamma_2^2\sigma}\right)}}{\left(1-C\mathrm{e}^{A\times+\frac{t+\gamma_1^2+6\gamma_2-2\gamma_1(3+\gamma_2)+\gamma_2^2\left(-3+6B^2\sigma^2\right)\right)}{12\gamma_2^2\sigma}}\right)\right)\sqrt{-\gamma_1-\gamma_2}}\right) e^{i\left(\frac{x(-3+\gamma_1+\gamma_2)}{6\gamma_2\sigma}+w(t)\sigma-t\sigma^2+t\left(-\frac{A^2(3-\gamma_1+\gamma_2)}{4\gamma_2}+\frac{(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{216\gamma_2^2\sigma^2}+\sigma^2\right)\right)}. \tag{46}$$

Solution 11: The singular soliton solution corresponding to Family 15 is obtained

$$q(\mathrm{x},\mathrm{t})=\left(\frac{\sqrt{\frac{3}{2}}C}{\sqrt{-\gamma_1-\gamma_2}}-\frac{\sqrt{\frac{3}{2}}\left(2+C\left(x-\frac{t(-3+\gamma_1+\gamma_2)(3-\gamma_1+3\gamma_2)}{12{\gamma_2}^2\sigma}\right)\right)}{\sqrt{-\gamma_1-\gamma_2}\left(x-\frac{t(-3+\gamma_1+\gamma_2)(3-\gamma_1+3\gamma_2)}{12{\gamma_2}^2\sigma}\right)}\right) \times \mathrm{e}^{\left(\frac{x(-3+\gamma_1+\gamma_2)}{6\gamma_2\sigma}+W\sigma-t\sigma^2+t\left(\frac{(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{216\gamma_2^3\sigma^2}+\sigma^2\right)\right)}. \tag{47}$$

Solution 12: The singular soliton solution corresponding to Family 17 is obtained

$$q(\mathrm{x},\mathrm{t})=\left(-\frac{\sqrt{6}}{\sqrt{-\gamma_1-\gamma_2}\left(x-\frac{t(-3+\gamma_1+\gamma_2)(3-\gamma_1+3\gamma_2)}{12\gamma_2^2\sigma}\right)}\right) \times \mathrm{e}^{\left(\frac{x(-3+\gamma_1+\gamma_2)}{6\gamma_2\sigma}+w\sigma-t\sigma^2+t\left(\frac{(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{216\gamma_2^3\sigma^2}+\sigma^2\right)\right)}. \tag{48}$$

Solution 13: The singular periodic solution corresponding to Family 18 is obtained

$$q(x,t)=\left(\sqrt{\frac{3}{2}}\frac{C}{\sqrt{-\gamma_1-\gamma_2}}\right)\tan\left(\frac{1}{2}C\left(x+\frac{t\left(9+\gamma_1^2-2\gamma_1(3+\gamma_2)-3y2\left(-2+\gamma_2+2C^2\gamma_2\sigma^2\right)\right)}{12\gamma_2^2\sigma}\right)\right) \times \mathrm{e}^{\left(\frac{x-3+\gamma_1+\gamma_2)}{6\gamma_2\sigma}+W(t)\sigma-t\sigma^2+t\left(\frac{C^2(3-\gamma_1+\gamma_2)}{4\gamma_2}+\frac{(-3+\gamma_1-5\gamma_2)(-3+\gamma_1+\gamma_2)^2}{216\gamma_2^3\sigma^2}+\sigma^2\right)\right)}. \tag{49}$$

3.2 Application of $(G'/G,1/G)$ -expansion method

In this part, we will apply the $(G'/G,1/G)$ -expansion method to find the exact traveling wave solutions of Eq.(1). Balancing the terms of Q^3 and Q'' in Eq.(18) and Eq.(20) gives $n=1$. Putting n into Eq.(16), we derive

$$Q(\xi)=a_0+a_1\Phi(\xi)+\mathrm{B}_1\psi(\xi), \tag{50}$$

where a_0,a_1,B_1 are constants to be determined, such that $a_1^2+\mathrm{B}_1^2\neq 0$. To analyze NLEE based on the sign of δ, three distinct cases must be considered. Let us now elaborate on these three scenarios:
Case 1: When $\delta<0$.
By putting Eq.(50) into Eq.(18) and Eq.(20) and applying Eq.(8) and Eq.(10), the lefthand side of equation Eq.(18) and Eq.(20) transforms into a polynomial in ψ and ϕ. Collecting each coefficient of this polynomial to zero yields in a system of algebraic equations involving $a_0,a_1,B_1,\rho,\delta,A_1$, and A_2. Utilizing package program, we obtain the desired solutions as follows

Set 1:

$$
\begin{aligned}
&a_0=0, a_1=a_1, B_1=\frac{a_1\delta\sqrt{\rho^2+\left(-A_1^2+A_2^2\right)\delta^2}}{(-\delta)^{3/2}}, \gamma_1=-\frac{6\left(3-2a1^2\right)k^2+\delta+2a_1^2\delta+12kv}{8a_1^2k(2k+v)},\\
&\omega=\frac{-24k^6-20k^4\delta+\delta^3-2k^2\left(-16+\delta^2\right)+24k^5v-40k^3\delta v+2k\left(16+3\delta^2\right)v+8v^2}{2\left(-6k^2+\delta\right)^2},\\
&\gamma_2=-\frac{\left(1+2a_1^2\right)\left(6k^2-\delta\right)}{8a_1^2k(2k+v)}, \sigma=-\frac{2(2k+v)}{6k^2-\delta},
\end{aligned}
\tag{51}
$$

where $\rho^2+\left(-A_1^2+A_2^2\right)\delta^2>0, a_1\neq 0$, and $k\neq 0$. Substituting the values of Eq.(51) into Eq.(50) and applying Eq.(17), the solution for Eq.(1) is obtained as follows

$$
\begin{aligned}
q(\mathrm{x},\mathrm{t})=&\left(\frac{a_1\delta\sqrt{\rho^2+(-A_1{}^2+A_2{}^2)\delta^2}}{(-\delta)^{3/2}\left(\frac{\rho}{\delta}+A_1\cosh(\sqrt{-\delta}(x-tv))+A_2\sinh(\sqrt{-\delta}(x-tv))\right)}\right.\\
&\left.+\frac{a_1\left(A_2\sqrt{-\delta}\cosh(\sqrt{-\delta}(x-tv))+A_1\sqrt{-\delta}\sinh(\sqrt{-\delta}(x-tv))\right)}{\frac{\rho}{\delta}+A_1\cosh(\sqrt{-\delta}(x-tv))+A_2\sinh(\sqrt{-\delta}(x-tv))}\right)\\
&\times \mathrm{e}^{\left(-kx-\frac{4t(2k+v)^2}{\left(6k^2-\delta\right)^2}+\frac{t\left(-24k^6-20k^4\delta+\delta^3-2k^2\left(-16+\delta^2\right)+24k^5\bar{v}-40k^3\delta v+2k\left(16+3\delta^2\right)v+8v^2\right)}{2\left(-6k^2+\delta\right)^2}-\frac{2(2k+v)\omega(t)}{6k^2-\delta}\right)}.
\end{aligned}
\tag{52}
$$

Eq.(52) describes a singular soliton solution.

Set 2:

$$
\begin{aligned}
&a_0=0, a_1=\frac{1}{\sqrt{-2-\frac{8\gamma_2}{3}}}, B_1=0, \gamma_1=3\left(1+\gamma_1\right), \sigma=\frac{2v}{\delta},\\
&\omega=-\frac{\delta}{2}-\frac{\delta^2}{27v^2}+\frac{4v^2}{\delta^2}, k=-\frac{\delta}{3v}, A_1=\frac{\sqrt{\rho^2+A_2{}^2\delta^2}}{\delta},
\end{aligned}
\tag{53}
$$

where $-2-\frac{8\gamma_2}{3}>0$, and $v\neq 0$. Substituting the values of Eq.(53) into Eq.(50) and applying Eq.(17), the solution for Eq.(1) is obtained as follows:

$$
\begin{aligned}
\mathrm{q(x,t)}=&\left(\frac{A_2\sqrt{-\delta}\cosh(\sqrt{-\delta}\,(x-t\nu))+\frac{\sqrt{-\delta}\sqrt{\rho^2+A_2{}^2\delta^2}\sinh(\sqrt{-\delta}(x-t\nu))}{\delta}}{\sqrt{-2-\frac{8\gamma_2}{3}}\left(\frac{\rho}{\delta}+\frac{\sqrt{\rho^2+A_2{}^2\delta^2}\cosh(\sqrt{-\delta}(x-t\nu))}{\delta}+A_2\sinh(\sqrt{-\delta}\,(x-t\nu))\right)}\right)\\
&\times \mathrm{e}^{\mathrm{i}\left(\frac{x\delta}{3v}-\frac{4tv^2}{\delta^2}+t\left(-\frac{\delta}{2}-\frac{\delta^2}{27v^2}+\frac{4v^2}{\delta^2}\right)+\frac{2vW(t)}{\delta}\right)}.
\end{aligned}
\tag{54}
$$

Eq.(54) describes a singular soliton solution.

Set 3:

$$
\begin{aligned}
&a_0=0, a_1=\frac{\sqrt{-6k^2+\delta}}{\sqrt{4k^2\left(3+4\gamma_2\right)-2\delta+8k\gamma_2 v}}, B_1=0, \gamma_1=\frac{18k^2\left(1+\gamma_2\right)+\left(-3+\gamma_2\right)\delta+12k\gamma_2 v}{6k^2-\delta},\\
&,\omega=\frac{-24k^6-20k^4\delta+\delta^3-2k^2\left(-16+\delta^2\right)+24k^5v-40k^3\delta v+2k\left(16+3\delta^2\right)v+8v^2}{2\left(-6k^2+\delta\right)^2},\\
&\sigma=-\frac{2(2k+v)}{6k^2-\delta}, A_1=\frac{\sqrt{\rho^2+A_2^2\delta^2}}{\delta},
\end{aligned}
\tag{55}
$$

where $-6k^2+\delta>0$, and $4k^2(3+4\gamma_2)-2\delta+8k\gamma_2 v>0$.
Substituting the values of Eq.(55) into Eq.(50) and applying Eq.(17), the solution for Eq.(1) is obtained as follows

$$q(\mathrm{x},\mathrm{t})=\left(\frac{\sqrt{-\delta\left(-6k^2+\delta\right)}\left(A_2\delta\cosh(\sqrt{-\delta}(x-tv))+\sqrt{\rho^2+A_2{}^2\delta^2}\sinh(\sqrt{-\delta}(x-tv))\right)}{\sqrt{4k^2(3+4\gamma_2)-2\delta+8k\gamma_2 v}\left(\rho+\sqrt{\rho^2+A_2{}^2\delta^2}\cosh(\sqrt{-\delta}(x-tv))+A_2\delta\sinh(\sqrt{-\delta}(x-tv))\right)}\right)$$
$$\times\,\mathrm{e}^{-\frac{\mathrm{i}\left(4k^4t+4k^2t\delta+t\delta^2-2k\delta(x-3tv)+4k^3(3x-tv)+4(2k+v)W(t)\right)}{12k^2-2\delta}}. \tag{56}$$

Eq.(56) describes a dark soliton solution.
Case 2: When $\delta>0$.
By putting Eq.(50) into Eq.(18) and Eq.(20) and applying Eq.(8) and Eq.(12), the lefthand side of Eq.(18) and Eq.(20) transforms into a polynomial in ψ and ϕ. Collecting each coefficient of this polynomial to zero yields in a system of algebraic equations involving $a_0, a_1, B_1, \rho, \delta, A_1$, and A_2. Via package program, we obtain solutions as follows

Set 1:

$$\begin{aligned}&a_0=0, a_1=-\frac{1}{\sqrt{-2+4k\gamma_2}}, B_1=-\frac{\sqrt{-\rho^2+A_1^2\delta^2+A_2^2\delta^2}}{\sqrt{-2\delta+4k\gamma_2\delta}}, \gamma_1=3-(1+6k)\gamma_2,\\&\sigma=1, v=-k(2+3k)+\frac{\delta}{2}, \omega=\frac{1}{2}(2+\delta+k(-2k(1+k)+3\delta)),\end{aligned} \tag{57}$$

where $-2+4k\gamma_2>0$, and $-\rho^2+A_1^2\delta^2+A_2^2\delta^2>0$.
Substituting the values of Eq.(57) into Eq.(50) and applying Eq.(17), the solution for Eq.(1) is obtained as follows:

$$q(\mathrm{x},\mathrm{t})=\left(\begin{array}{c}-\frac{\sqrt{-\rho^2+A_1{}^2\delta^2+A_2{}^2\delta^2}}{\sqrt{-2\delta+4k\gamma_2\delta}\left(\frac{\rho}{\delta}+A_1\cos\left(\left(x-t\left(-k(2+3k)+\frac{\delta}{2}\right)\right)\sqrt{\delta}\right)+A_2\sin\left(\left(x-t\left(-k(2+3k)+\frac{\delta}{2}\right)\right)\sqrt{\delta}\right)\right)}\\-\frac{A_2\sqrt{\delta}\cos\left[\left(x-t\left(-k(2+3k)+\frac{\delta}{2}\right)\right)\sqrt{\delta}\right]-A_1\sqrt{\delta}\sin\left[\left(x-t\left(-k(2+3k)+\frac{\delta}{2}\right)\right)\sqrt{\delta}\right]}{\sqrt{-2+4k\gamma_2}\left(\frac{\rho}{\delta}+A_1\cos\left(\left(x-t\left(-k(2+3k)+\frac{\delta}{2}\right)\right)\sqrt{\delta}\right)+A_2\sin\left(\left(x-t\left(-k(2+3k)+\frac{\delta}{2}\right)\right)\sqrt{\delta}\right)\right)}\end{array}\right)$$
$$\times\,\mathrm{e}^{\mathrm{i}\left(-t-kx+\frac{1}{2}t(2+\delta+k(-2k(1+k)+3\delta))+W(t)\right)}. \tag{58}$$

Eq.(58) describes a singular periodic soliton solution.

Set 2:

$$\begin{aligned}&a_0=0, a_1=\frac{\sqrt{\frac{3}{2}}}{\sqrt{-\gamma_1-\gamma_2}}, B_1=\frac{\sqrt{\frac{3}{2}}\sqrt{\rho^2-\left(A_1^2+A_2^2\right)\delta^2}}{\sqrt{(\gamma_1+\gamma_2)\,\delta}}, v=-\frac{9+\gamma_1^2-2\gamma_1(3+\gamma_2)-3\gamma_2(-2+\gamma_2+2\gamma_2\delta)}{12\gamma_2^2},\\&\omega=\frac{-27+\gamma_1{}^3-3\gamma_1{}^2(3+\gamma_2)-9\gamma_1(-3+\gamma_2(-2+\gamma_2+6\gamma_2\delta))+\gamma_2(-27+\gamma_2(27+211\gamma_2+54(3+\gamma_2)\delta))}{216\gamma_2{}^3},\\&\sigma=1, k=-\frac{-3+\gamma_1+\gamma_2}{6\gamma_2},\end{aligned} \tag{59}$$

where $-\gamma_1-\gamma_2>0, \rho^2-\left(A_1{}^2+A_2{}^2\right)\delta^2>0$, and $\gamma_2\neq 0$. Substituting the values of Eq.(59) into Eq.(50) and applying Eq.(17), the solution for Eq.(1) is obtained as follows:

$$q(\mathrm{x,t}) = \left(\frac{\sqrt{\frac{3}{2}}\sqrt{-\rho^2 + \left(A_1{}^2 + A_2{}^2\right)\delta^2}}{\sqrt{-\left(\left(\gamma_1+\gamma_2\right)\delta\right)}\left(\frac{\rho}{\delta} + A_1\cos\left(\sqrt{\delta}\left(x+D_1t\right)\right) + A_2\sin\left(\sqrt{\delta}\left(x+D_1t\right)\right)\right)} + \frac{\sqrt{\frac{3}{2}}\left(A_2\sqrt{\delta}\cos\left(\sqrt{\delta}\left(x+D_1t\right)\right)\right) - A_1\sqrt{\delta}\sin\left(\sqrt{\delta}\left(x+D_1t\right)\right)}{\sqrt{-\gamma_1-\gamma_2}\left(\frac{\rho}{\delta} + A_1\cos\left(\sqrt{\delta}\left(x+D_1t\right)\right) + A_2\sin\left(\sqrt{\delta}\left(x+D_1t\right)\right)\right)} \right) \times \mathrm{e}^{\left(-t + \frac{x(-3+\gamma_1+\gamma_2)}{6\gamma_2} + \frac{t\left(-27+\gamma_1^3-3\gamma_1^2(3+\gamma_2)-9\gamma_1(-3+\gamma_2(-2+\gamma_2+6\gamma_2\delta))+\gamma_2(-27+\gamma_2(27+211\gamma_2+54(3+\gamma_2)\delta))\right)}{216\gamma_2^3} + W(t)\right)}, \tag{60}$$

where $D_1 = \frac{(9+\gamma_1{}^2-2\gamma_1(3+\gamma_2)-3\gamma_2(\gamma_2-2+2\gamma_2\delta))}{12\gamma_2{}^2}$. Eq.(60) describes a singular periodic soliton solution.

Case 3: When $\delta = 0$.

By putting Eq.(50) into Eq.(18) and Eq.(20) and applying Eq.(8) and Eq.(14), the lefthand side of equation Eq.(18) and Eq.(20) transforms into a polynomial in ψ and ϕ. Collecting each coefficient of this polynomial to zero yields in a system of algebraic equations involving a_0, a_1, B_1, ρ, A_1, and A_2. By using package program, we obtain solutions as follows:

Set 1:

$$\begin{aligned} &a_0 = 0, a_1 = a_1, B_1 = a_1\sqrt{A_1^2 - 2A_2\rho}, \gamma_1 = \frac{3}{2} - \frac{3}{4a_1^2}, \gamma_2 = -\frac{3}{2} - \frac{3}{4a_1^2}, \\ &\sigma = \frac{\sqrt{9\omega+\sqrt{24+81\omega^2}}}{3\sqrt{2}}, v = \frac{\sqrt{2}}{\sqrt{9\omega+\sqrt{24+81\omega^2}}}, k = -\frac{\sqrt{2}}{\sqrt{9\omega+\sqrt{24+81\omega^2}}}, \end{aligned} \tag{61}$$

where $A_1^2 - 2A_2\rho > 0$, and $9\omega + \sqrt{24+81\omega^2} > 0$. Substituting the values of Eq.(61) into Eq.(50) and applying Eq.(17), the solution for Eq.(1) is obtained as follows:

$$q(\mathrm{x,t}) = \left(-\frac{2a_1\sqrt{9\omega+\sqrt{24+81\omega^2}}\left(-\sqrt{2}t\rho + \left(A_1 + \sqrt{A_1{}^2 - 2A_2\rho} + \rho x\right)\sqrt{9\omega+\sqrt{24+81\omega^2}}\right)}{\left(-2t^2\rho + 2\sqrt{2}t\left(A_1+\rho x\right)\sqrt{9\omega+\sqrt{24+81\omega^2}} - \left(2A_2 + x\left(2A_1+\rho x\right)\right)\left(9\omega+\sqrt{24+81\omega^2}\right)\right)} \right) \times \mathrm{e}^{\frac{1}{18}\mathrm{i}\left(9t\omega - t\sqrt{24+81\omega^2} + \frac{18\sqrt{2}x}{\sqrt{9\omega+\sqrt{24+81\omega^2}}} + 3\sqrt{2}\sqrt{9\omega+\sqrt{24+81\omega^2}}W(t)\right)}. \tag{62}$$

Eq.(62) describes singular soliton solution.

Set 2:

$$\begin{aligned} &a_0 = 0, a_1 = \frac{1}{\sqrt{-2-\frac{4\gamma_2}{3}}}, B_1 = \frac{\sqrt{-\frac{3A_1^2}{2} + 3A_2\rho}}{\sqrt{3+2\gamma_2}}, \gamma_1 = 3+\gamma_2, \\ &\sigma = -\frac{1}{3k}, v = -k, \omega = \frac{1-6k^4}{9k^2}, \end{aligned} \tag{63}$$

where $-2-\frac{4\gamma_2}{3} > 0$, and $-\frac{3A_1^2}{2} + 3A_2\rho > 0$. Substituting the values of Eq.(63) into Eq.(50) and applying Eq.(17), the solution for Eq.(1) is obtained as follows:

$$q(x,t) = \left(\frac{A_1 + \rho(kt+x)}{\left(A_2 + A_1(kt+x) + \frac{1}{2}\rho(kt+x)^2\right)\sqrt{-2-\frac{4\gamma_2}{3}}} + \frac{\sqrt{-\frac{3A_1{}^2}{2}+3A_2\rho}}{\left(A_2 + A_1(kt+x) + \frac{1}{2}\rho(kt+x)^2\right)\sqrt{3+2\gamma_2}} \right) \times e^{\left(-\frac{t}{9k^2}+\frac{\left(1-6k^4\right)t}{9k^2}-kx-\frac{W(t)}{3k}\right)}. \tag{64}$$

Eq.(64) describes singular soliton solution.

4 Stability analysis of the solutions

The stability of the obtained traveling wave solutions was analyzed using the Hamiltonian technique [36,37].

$$U = \frac{1}{2}\int_{-\infty}^{\infty} |q(x,t)|^2 dx, \tag{65}$$

where $q(x,t)$ denotes a traveling wave solution, and U represents the momentum associated with the Hamiltonian system (HS). The corresponding stability criterion is given by

$$\frac{\partial U}{\partial v} > 0. \tag{66}$$

The wave velocity is represented by v. By considering the parameter values $\gamma_1 = -1, \gamma_2 = -1, \sigma = -1, A = 4, B = 1, W(t) = \sin(t)$, $-10 < x < 10$, and evaluating at $t = 1$, we obtain:

$$\left.\frac{\partial U}{\partial v}\right|_{v=9} = 0.399894 > 0,$$

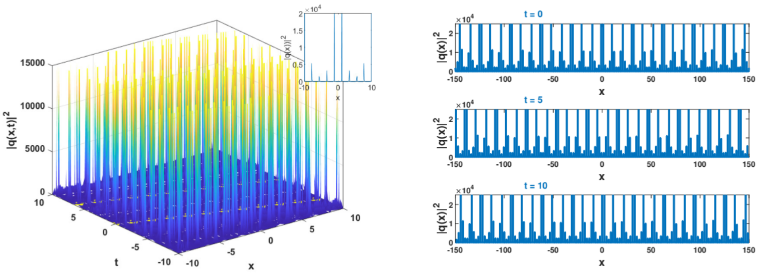
which denotes that Eq.(39) is stable. And by considering the parameter values $A_1 = 1, A_2 = 0, \delta = -1, \rho = 3, k = 1, a_1 = -2, W(t) = \sin(t)$, $-10 < x < 10$, and evaluating at $t = 1$, we obtain:

$$\left.\frac{\partial U}{\partial v}\right|_{v=4} = -0.00336642 < 0,$$

which is denotes that Eq.(52) is unstable. However, other solutions' stability can be tested in a similar way.

5 Graphical representation of the results

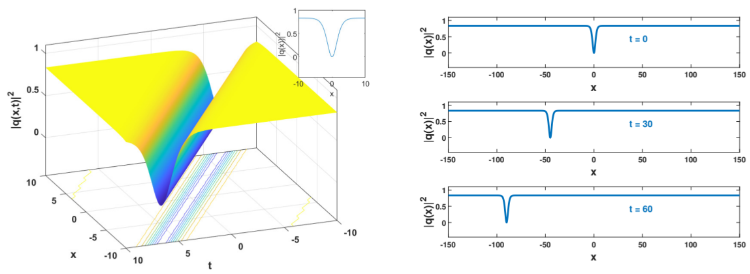
This section presents 3D, 2D, and contour graphs for various obtained solutions to clarify the characteristics of the retrieved solutions. Eq.(23) possesses a singular periodic solution and illustrated in Figure 1 for $\sigma = -1, k = 1, \gamma_1 = -1, A = 4, B = 1, C = 5, W(t) = \sin(t)$. Eq.(24) possesses a dark soliton solution and illustrated in Figure 2, when $\sigma = 1, k = -1, \gamma_1 = -1, A = 4, B = 1, C = 4, W(t) = \sin(t)$. Eq.(28) possesses a singular soliton solution and illustrated in Figure 3, when $\sigma = -1, k = 1, \gamma_1 = -1, W(t) = \sin(t)$. A dark soliton solution for equation (56) with parameters $k = 4, \gamma_2 = -4, A_2 = 1, v = 1, \delta = -1, \rho = 3, W(t) = \sin(t)$ is illustrated in Figure 4. Eq.(60) possesses a singular periodic solution and illustrated in Figure 5, when $\gamma_1 = -1, \gamma_2 = -1, A_1 = 1, A_2 = 3, \delta = 1, \rho = 1, W(t) = \sin(t)$. Eq.(62) possesses a singular soliton solution, illustrated in Figure 6, when $A_1 = -1, A_2 = 0, \rho = -10, a_1 = 1, \omega = 1, W(t) = \sin(t)$. These investigations demonstrate that the evolution of soliton solutions over time in optical fibers is influenced by the effect of dispersion. By selecting specific integer values for the system parameters, one can control the emergence and behavior of these solutions, which retain their shape and velocity during propagation.



(a) 3D, and 2D graphics.

(b) The effect of time.

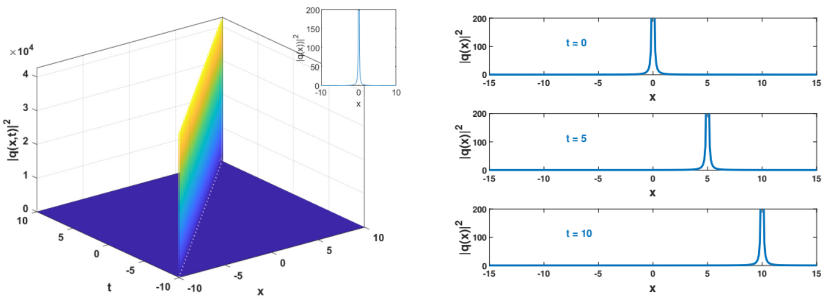
Fig. 1 Graphical representations of $|q(x,t)|^2$ in Eq.(23).



(a) 3D, and 2D graphics.

(b) The effect of time.

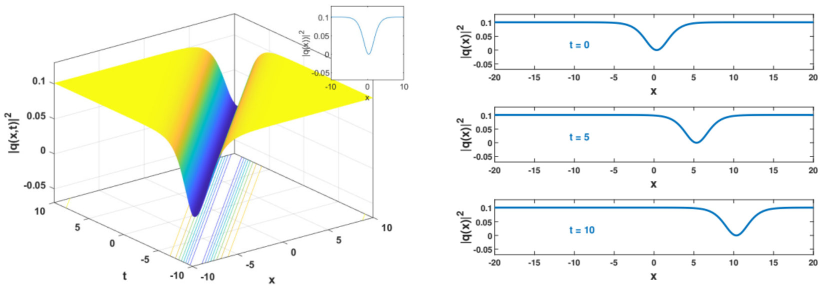
Fig. 2 Graphical representations of $|q(x,t)|^2$ in Eq.(24).



(a) 3D, and 2D graphics.

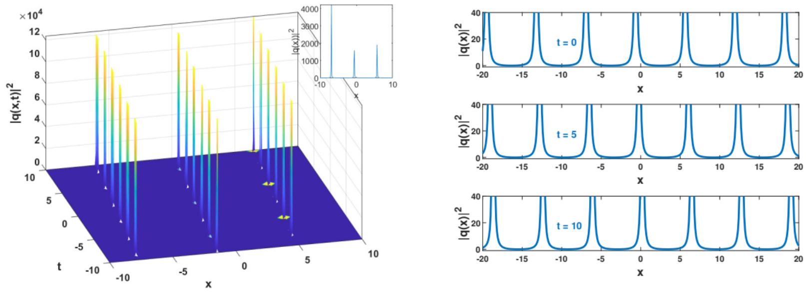
(b) The effect of time.

Fig. 3 Graphical representations of $|q(x,t)|^2$ in Eq. (28).



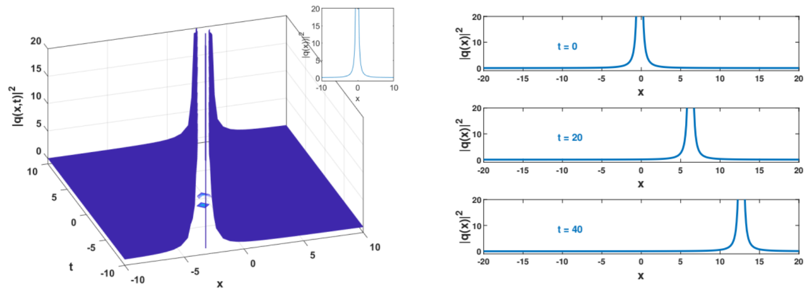
(a) 3D, and 2D graphics. (b) The effect of time.

Fig. 4 Graphical representations of $|q(x,t)|^2$ in Eq. (56).



(a) 3D, and 2D graphics. (b) The effect of time.

Fig. 5 Graphical representations of $|q(x,t)|^2$ in Eq. (60).



(a) 3D, and 2D graphics. (b) The effect of time.

Fig. 6 Graphical representations of $|q(x,t)|^2$ in Eq. (62).

6 Results and discussions

The $\tan\left(\frac{\varpi(\xi)}{2}\right)$-expansion method and the $(G'/G, 1/G)$-expansion method are applied to the SGNLSE with Kerr effect and higher-order nonlinearity. The exact solutions obtained include dark solitons, singular periodic solutions, and singular solitons. Figures 1-6 show 3D and 2D plots of these solutions with appropriate parameter values. Eq.(23) gives a singular periodic solution shown in Figure (1a) as a 3D plot. Figure (1b) shows its time evolution at $t = 0, 5, 10$, where the soliton shifts left over time. This solution is useful in signal processing and nonlinear optics. Eq.(24) describes dark soliton solutions shown in Figure (2a) as a 3D plot. Figure (2b) shows their evolution at $t = 0, 30, 60$, with the soliton moving left. Dark solitons are localized dips in background intensity with a phase jump, maintaining shape and velocity during propagation. Eq.(28) represents singular soliton solutions shown in Figure (3a). Figure (3b) shows evolution at $t = 0, 5, 10$, with soliton shifting right. Singular solitons have infinite peak intensity and help model extreme wave behavior. Eq.(56) also gives dark soliton solutions shown in Figure (4a). Figure (4b) shows evolution at $t = 0, 5, 10$, soliton shifts right. Eq.(60) is a singular periodic solution shown in Figure (5a). Figure (5b) shows time evolution at $t = 0, 5, 10$, soliton shifts left. Eq.(62) shows singular soliton solutions in Figure (6a), with evolution at $t = 0, 20, 40$ in Figure (6b). Soliton shifts right over time.

In this study, the $\tan\left(\frac{\varpi(\xi)}{2}\right)$-expansion and the $(G'/G, 1/G)$-expansion methods are employed to analyze the SGNLSE with Kerr effect and higher-order nonlinearity. A comparison with previously reported results [23] highlights the novelty of the present findings. The derived solutions consist of dark, singular, and singular periodic soliton solutions, which broaden the range of known solution types by introducing generalized trigonometric, hyperbolic, and rational forms. These results represent new contributions that have not been previously reported in literature.

7 Conclusions

In this work, we investigated the SGNLSE incorporating the Kerr effect and higher-order nonlinearity by employing the $\tan\left(\frac{\varpi(\xi)}{2}\right)$-expansion method and the $(G'/G, 1/G)$-expansion method. A diverse set of exact solutions was successfully derived, including singular, singular periodic, and dark soliton solutions. To visualize the dynamics, we included two- and three-dimensional graphical representations that illustrate how nonlinear effects influence the behavior and evolution of these solutions over time. In particular, the figures clearly demonstrate the impact of the time variable on wave structure and stability. Overall, the findings offer new insights into the dynamics governed by NLPDEs and further establish the effectiveness of the proposed methods in mathematical physics.

8 Declarations

8.1 Conflict of interest:

Not applicable.

8.2 Funding:

Not applicable.

8.3 Author's contribution:

A.K.S.-Methodology, Conceptualization, Resources, Writing-Original Draft, Software, Validation, Conceptualization and Formal Analysis, Investigation, Visualization. H.F.I.-Investigation, Resources, Software. All authors read and approved the final submitted version of this manuscript.

8.4 Acknowledgement:

Not applicable.

8.5 Availability of data and materials:

All data that support the findings of this study are included within the article.

8.6 Usage of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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