Paradigm International Journal of Mathematics and Computer in Engineering 4(1) (2026) 45–64

International Journal of Mathematics and Computer in Engineering

https://reference-global.com/journal/IJMCE

Original Study

A study on different classes of differential equations by semi-analytical and numerical techniques

Sachin K. Narayana[1], Suguntha Devi Kannadasan[1†], Kumbinarasaiah Srinivasa[1]

[1]Department of Mathematics, Bangalore University, Bengaluru, 560056, India

Communicated by Haci Mehmet Baskonus; Received: 16.09.2024; Accepted: 12.01.2025; Online: 02.02.2026

Abstract

This study applies the Homotopy analysis method (HAM) and Haar wavelet transform (HWT) in order to provide an innovative technique for approximating to the nonlinear ordinary differential equations (ODEs), a system of ODEs, and partial differential equations (PDEs). HAM is a potent semi-analytical method that works well when studying linear and nonlinear problems. HWT is a numerical technique that effectively discretizes differential equations (DEs) simultaneously. A robust analytical method builds a family of equations that smoothly transforms the original nonlinear equation into a straightforward linear issue using the topological concept of homotopy. This allows the derivation of extremely precise series solutions. Real-world application problems are solved to analyze the correctness and effectiveness of the projected system.

Keywords: Ordinary differential equations, HAM, Haar wavelet, PDEs, HWT.
AMS 2020 codes: 35F25; 65H20; 65T60; 34A34; 34A12; 90C30.

1 Introduction

Almost every domain relies on the nonlinear ODEs. Their applications arise in the modeling biological, chemical, mechanical, and electrical applications. With the help of calculus, it became apparent that not all ODEs and PDEs could be analytically solved. To tackle this challenge, numerical methods have been devised to approximate solutions for ODEs. Renowned techniques like Heun, Runge-Kutta, Adams-Bashforth, linear multistep, and Euler forward and backward methods emerged to address this need. In addition to analytical and numerical techniques, we have a semi-analytical method to solve such equations [1].

In this paper, we consider one among the semi-analytical methods called HAM to solve the differential equations. In this frame, the initial-value problems (IVPs) of the first order ODEs is taken in the following form

$$v'(y) = g(z, v(z)), \quad v(z_0) = v_0, \tag{1}$$

and also the second-order IVPs of ODEs is considered as

$$v''(z) = g(z, v(z), v'(z)), \quad v(z_0) = v_0, \quad v'(z_0) = v'_0. \tag{2}$$

†Corresponding author.
Email address: suganthadevik77@gmail.com

Paradigm ISSN 2956-7068 doi:10.2478/ijmce-2026-0004 Paradigm

The n system of ODEs is studied as

$$\left.\begin{aligned} v_1'(z) &= g_1(z, v_1(z), v_2(z), \cdots, v_n(z)),\\ v_2'(z) &= g_2(z, v_1(z), v_2(z), \cdots, v_n(z)),\\ v_3'(z) &= g_3(z, v_1(z), v_2(z), \cdots, v_n(z)),\\ &\vdots\\ v_n'(z) &= g_n(z, v_1(z), v_2(z), \cdots, v_n(z)), \end{aligned}\right\} \tag{3}$$

subject to $v_i(0) = v_{i_0}$, for $i = 0, 1, 2, \cdots, n$.

The time-dependent PDEs being,

$$v_z(z,t) + v_t(z,t) = f(z, v, t), \quad v(z, 0) = s(z), \tag{4}$$

is examined via this work. These problems are frequently utilised to represent real-world problems in engineering, social sciences, physics, and economics; they have garnered a lot of interest from scholars in the field of numerical analysis. It's also crucial to remember that a first-order system may be used to represent chemical kinetic problems, a population model for logistic growth, extremely stiff oscillatory problems, a SIR model, and other problems of a similar nature. Since many of these kinds of problems are thought to lack exact solutions, numerical approximations are obtained. Many experts have used various techniques to provide the numerical solutions, particularly, multi-step methods such as Runge-Kutta methods [2–6], and Hybrid block methods [7, 8]. Since some numerical techniques lead to numerical instability while addressing stiff systems, these issues are regarded as complicated problems. Many experts have worked very hard to develop more effective techniques for addressing stiff systems like A-EBDF: an adaptive method [9], Carrol [10] in a matricial exponentially fitted scheme, modified extended backward differentiation [11], Haar wavelet and single term Haar wavelet series [12, 13], Matrix free MEBDF method [14], multistep multi derivative hybrid methods [15], stability and accuracy of one-step methods [16], a semi-implicit midpoint rule [17], wavelet methods [18–21], variational iteration method [22], fractional order PDEs [23–25] and so on.

Here, we consider the semi-analytical method for solving DEs called HAM which has the independent parameter, in contrast to perturbation approaches that depend on small or large parameters. The choice of linear operator, auxiliary function, and control convergence parameter is entirely up to the user. This approach often solves high nonlinearity differential equations [26]. Moreover, HAM is an analytical method that generates a sequence of convergent linear equations from a nonlinear one by applying homotopy, or the distortion of one continuous function into another, to solve nonlinear ordinary or partial differential equations. Liao introduced the HAM first in 1992, and it underwent additional changes in 1997 with the addition of the auxiliary parameter h. This non-zero parameter gives the series control over its convergence. We are free to select the auxiliary linear operator, h, the convergence control value, and the initial approximation of the solution because HAM is predicated on the idea of homotopy. This feature distinguishes HAM from other approaches as it allows to select the base functions of the high-order deformation equation's solution as well as its equation type [27]. An analytical approximation technique for highly nonlinear systems is the HAM. In the past, perturbation techniques were often employed. Nevertheless, perturbation techniques heavily rely on the presence of small physical characteristics, and in addition, perturbation approximations frequently diverge as perturbation quantity increases. However, since the HAM is based on homotopy, a fundamental idea in topology, unlike perturbation techniques, it has nothing to do with the presence of small or large physical factors. In particular, the HAM offers a practical means of ensuring that the solution series will converge. The methodology may be used with other approaches for solving nonlinear differential equations, such as Pade approximation and spectral methods [28]. The HAM is an analytic approximation method as opposed to the discrete computing approach of Homotopy continuation. On the other hand, the HAM demonstrates that a nonlinear system may be divided into endless linear systems that can be solved analytically using the Homotopy parameter merely in theory. In the computer era, the HAM was developed as an analytical approximation method to "compute with functions instead of numbers". Combining the HAM with a computer algebra program such as Maple or Mathematica allows to get analytic approximations of a higher-order strongly nonlinear problem easily [29].

The rest of this paper is organized as follows. In Section 2, the preliminaries of the wavelets and the convergence theorems are introduced. In Section 3, HAM for ODE, the systems of ODE and PDE are explained. In Section 4, the application of problems are investigated and results are discussed in terms of tables and graphs. Finally, Section 5 concludes the paper.

2 Preliminaries

In this part of the paper, we present some important definitions and theorems.
Wavelet: A real-valued function $\Psi(\xi)$ satisfies the following conditions [30, 31]:

$$\int_{-\infty}^{\infty}\Psi(\xi)d\xi = 0, \qquad \text{and} \qquad \int_{-\infty}^{\infty}|\Psi(\xi)|^2 d\xi = 1.$$

This means that $\Psi(\xi)$ is an oscillatory function having unit energy and zero mean.

Theorem 1. *As long as the series $v_i(y) = \sum_{m=0}^{m} v_{i_m}(y)$ converges, it must be the exact solution of equation (14). Where $v_{i_m}(y)$ is governed by the m^{th} order deformation equation [32].*

Theorem 2. *Let $\psi_0, \psi_1, \psi_2, \cdots$ be the components of solution of equation (14). The series solution $\sum_{k=0}^{\infty}\psi_k(t)$ converges if $\exists\ 0 < \gamma < 1$ such that $||\psi_{k+1}|| \leq \gamma||\psi_k||$, $\forall\ k \geq k_0$ for some $k_0 \in \mathbb{N}$ [33].*

Theorem 3. *Assume that the series solution $\sum_{k=0}^{\infty}\psi_k(t)$ is convergent to the solution $v(y)$, if the truncation series $\sum_{k=0}^{m}\psi_k(t)$ is the approximation to the solution $v_i(y)$, then the maximum absolute truncation error is calculated as, $||v_i(y) - \sum_{k=0}^{m}\psi_k(t)|| \leq \frac{1}{1-\gamma}\,\gamma^{m+1}\,||\psi_0(t)||$ [33].*

Theorem 4. *Suppose that the functions $D_*^{\alpha}v_i(y)$ are the approximation of $D_*^{\alpha}v_i(y)$ obtained using Haar wavelets, then we have an exact upper bound as follows:*
$||D_*^{\alpha}v_i(y) - D_*^{\alpha}v_{i,k}(y)||_E \leq \frac{M}{\Gamma(m-\alpha).(m-\alpha)}\ \frac{1}{[1-2^{2(\alpha-m)}]^{\frac{1}{2}}}\ \frac{1}{k^{m-\alpha}}$, *where* $||v_i(y)||_E = (\int_0^1 (v_i)^2(y)dy)^{\frac{1}{2}}$ *[34].*

3 Description of method

3.1 HAM for ODE

Let us consider the ODE with different physical conditions,

$$\mathscr{M}[v(y)] = 0, y \geq 0. \tag{5}$$

Where $\mathscr{M}$ is the differential operator, and v(y) is the function to be determined.

3.1.1 Zeroth order deformation equation

Let $v_0(y)$ be the initial approximation to the actual solution of equation (5). The zeroth deformation equation is constructed by using the auxiliary function $\mathscr{H}$(y) ($\neq 0$) and auxiliary parameter h ($\neq 0$) as [31, 35],

$$(1-q)\mathscr{S}[\psi(y;q) - v_0(y)] = qh\mathscr{H}(y)\mathscr{M}[\psi(y;q)]. \tag{6}$$

Where, $\psi(y;q)$ is an unknown function, $\mathscr{S}$ is a linear operator. When q=0, equation (6) becomes, $\psi(y;0)$ = v_0 (y) and at q=1, equation (6) becomes, $\psi(y;1)$ = $v(y)$. So as the q varies from 0 to 1, the function $\psi(y;q)$ varies from initial approximation v_0 (y) to the actual solution $v(y)$. Defining the mth order deformation derivatives,

$$v_m(y) = \frac{1}{m!}\frac{\partial^m \psi(y;q)}{\partial q^m}. \tag{7}$$

Expanding $\psi(y;q)$ using the Taylor series with respect to (w.r.t.) q. We get,

$$\psi(y;q) = v_0(y) + \sum_{m=1}^{\infty} v_m(y)q^m. \tag{8}$$

As we know, $\psi(y;q)$ becomes the desired solution at $q = 1$. At q=1, equation (8) becomes

$$\psi(y;1) = v(y) = v_0(y) + \sum_{m=1}^{\infty} v_m(y). \tag{9}$$

Similarly, deformation equation of m^{th} order is obtained as,

$$\mathscr{S}[v_m(y) - \chi_m v_{m-1}(y)] = h\mathscr{H}(y)\mathscr{R}_m(v_{m-1}(y)). \tag{10}$$

Where,

$$\chi_m = \begin{cases} 0 & \text{if} \quad m \leq 1 \\ 1 & Otherwise \end{cases} \tag{11}$$

$$\mathscr{R}_m(v_{m-1}(y)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}[\mathscr{M}[\psi(y;q)]]}{\partial q^{m-1}}. \tag{12}$$

Thus $v_1(y)$, $v_2(y)$, $v_3(y), \cdots$ can be attained on solving equation (10). The m^{th} order approximation of v (y) [36] is given by

$$v(y) = \sum_{m=0}^{m} v_m(y). \tag{13}$$

Equation (13) is the semi-analytical solution of equation (5).

3.2 HAM for the system of ODE

Let us consider the system of stiff ODEs with different physical conditions [37],

$$[v_i'(y)] = g_i(y, v_1, v_2, \cdots, v_n), \qquad i = 1,2,3,\cdots,n, \quad y \geq 0. \tag{14}$$

subject to the condition:

$$v_i(0) = a_i, \qquad i = 1,2,3,\cdots,n. \tag{15}$$

3.2.1 Zeroth order deformation equation

Let $v_{i_0}(y)$, $i = 1,2,3,\cdots,n$ be the initial approximation to the actual solution of equation (14). The zeroth deformation equations are taking the auxiliary functions $\mathscr{H}$(y) ($\neq 0$) and auxiliary parameter h ($\neq 0$) as [35,38],

$$(1-q)\mathscr{S}_i[\psi_i(y;q) - v_{i_0}(y)] = qh\mathscr{H}(y)\mathscr{M}_i[\psi_i(y;q)], \quad i = 1,2,3,\cdots,n, \tag{16}$$

subject to the conditions:

$$\psi_i(0;q) = a_i, \quad i = 1,2,3,\cdots,n. \tag{17}$$

Where, $\psi_i(y;q)$ are unknown functions, $\mathscr{S}_i$ are the Linear operators. When q=0, equation (16) becomes, $\psi_i(y;0)$ = $v_{i_0}(y)$ and at q=1, equation (16) becomes $\psi_i(y;1)$ = $v_i(y)$. So as the q varies from 0 to 1, the function $\psi_i(y;q)$

varies from initial approximation $v_{i_0}(y)$ to the actual solution $v_i(y)$, $i = 1,2,3,\cdots,n$. Defining the mth order deformation derivatives,

$$v_{i_m}(y) = \frac{1}{m!}\frac{\partial^m \psi_i(y;q)}{\partial q^m}, \quad i = 1,2,3,\cdots,n. \tag{18}$$

Expanding $\psi_i(y;q)$ using Taylor series w.r.t. q, $\quad i = 1,2,3,\cdots,n$. We get,

$$\psi_i(y;q) = v_{i_0}(y) + \sum_{m=1}^{\infty} v_{i_m}(y)q^m \quad i = 1,2,3,\cdots,n. \tag{19}$$

As we know at $q = 1$ $\psi_i(y;q)$ becomes the required solution, equation (19) at q=1 becomes

$$\psi_i(y;1) = v_i(y) = v_{i_0}(y) + \sum_{m=1}^{\infty} v_{i_m}(y), \quad i = 1,2,3,\cdots,n. \tag{20}$$

Similarly, the m^{th} order deformation is given by

$$\mathscr{S}[v_{i_m}(y) - \chi_m v_{i_{m-1}}(y)] = h\mathscr{H}(y)R_{i,m}(v_{i_{m-1}}(y)), \quad i = 1,2,3,\cdots,n. \tag{21}$$

Where,

$$\chi_m = \begin{cases} 0 & \text{if} \quad m \leq 1, \\ 1 & Otherwise. \end{cases} \tag{22}$$

$$R_{i,m}(v_{i_{m-1}}(y)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}[\mathscr{M}[\psi_i(y;q)]]}{\partial q^{m-1}}, \quad i = 1,2,3,\cdots,n. \tag{23}$$

Thus $v_{i_1}(y)$, $v_{i_2}(y)$, $v_{i_3}(y),\cdots$ can be obtained from solving equation (21). The m^{th} order approximation of v_i (y) [39] is given by

$$v_i(y) = \sum_{m=0}^{m} v_{i_m}(y). \tag{24}$$

Equation (24) is the semi-analytical solution of equation (14).

3.3 HAM for PDE

Let us consider the PDE with different physical conditions,

$$\mathscr{M}[v(y,t)] = 0, \quad y,t \geq 0. \tag{25}$$

Where $\mathscr{M}$ is the differential operator, and $v(y,t)$ is the function to be determined.

3.3.1 Zeroth order deformation equation

Let $v_0(y,t)$ be the initial approximation to the actual solution of equation (25). The zeroth deformation equation is constructed using the auxiliary function $\mathscr{H}$(y,t) ($\neq 0$) and auxiliary parameter h ($\neq 0$) as [31, 35],

$$(1-q)\mathscr{S}[\psi(y,t;q) - v_0(y,t)] = qh\mathscr{H}(y,t)\mathscr{M}[\psi(y,t;q)]. \tag{26}$$

Where, $\psi(y,t;q)$ is unknown function, $\mathscr{S}$ is Linear operator.
When q=0, equation (26) becomes $\psi(y,t;0) = v_0\ (y,t)$. At q=1, equation (26) becomes $\psi(y,t;1) = v(y,t)$. So as the q varies from 0 to 1, the function $\psi(y,t;q)$ varies from initial approximation $v_0\ (y,t)$ to the actual solution $v(y,t)$. Defining the mth order deformation derivatives,

$$v_m(y,t) = \frac{1}{m!}\frac{\partial^m \psi(y,t;q)}{\partial q^m}. \tag{27}$$

Expanding $\psi(y,t,q)$ using the Taylor series w.r.t. q. We get,

$$\psi(y,t;q) = v_0(y,t) + \sum_{m=1}^{\infty} v_m(y,t)q^m. \tag{28}$$

As we know, $\psi(y,t;q)$ becomes the desired solution at $q = 1$. At q=1, equation (28) becomes

$$\psi(y,t;1) = v(y,t) = v_0(y,t) + \sum_{m=1}^{\infty} v_m(y,t). \tag{29}$$

Similarly, deformation equation of m^{th} order is obtained as,

$$\mathscr{S}[v_m(y,t) - \chi_m v_{m-1}(y,t)] = h\mathscr{H}(y)\mathscr{R}_m(v_{m-1}(y,t)), \tag{30}$$

where,

$$\chi_m = \begin{cases} 0 & \text{if} \quad m \leq 1, \\ 1 & Otherwise. \end{cases} \tag{31}$$

$$\mathscr{R}_m(v_{m-1}(y,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}[\mathscr{M}[\psi(y,t;q)]]}{\partial q^{m-1}}. \tag{32}$$

Thus $v_1(y,t)$, $v_2(y,t)$, $v_3(y,t), \cdots$ can be attained on solving equation (30). The m^{th} order approximation of v (y) [32, 36, 39] is given by

$$v(y) = \sum_{m=0}^{m} v_m(y,t). \tag{33}$$

Equation (33) is the semi-analytical solution of equation (25).

4 Applications

We use some package programs to solve the following problems by HAM.

4.1 Application 4.1

Let us consider the following ODE with different physical conditions

$$x^2u''(x) + xu'(x) + (x^2 - 0.25)u(x) = 0; \quad u(1) = \sqrt{\frac{2}{\pi}}\sin(1), \quad u'(1) = \frac{2\cos(1) - \sin(1)}{\sqrt{2\pi}}. \tag{34}$$

On applying HAM to equation (34), the m^{th} order deformation is given by,

$$D^2[u_m(x) - \chi_m u_{m-1}(x)] = h\mathscr{R}_m(u_{m-1}(x)). \tag{35}$$

Where, $D^2 = \frac{d^2}{dx^2}$,

$$\mathscr{R}_m(u_{m-1}(x)) = x^2\frac{d^2u_{m-1}}{dx^2} + x\frac{du_{m-1}}{dx} + (x^2 - 0.25)u_{m-1}, \tag{36}$$

subject to,

$$u_m(1) = 0, \qquad u'_m(1) = 0. \tag{37}$$

Integrating twice on either sides of equation (35), we get

$$u_m(x) = \chi_m u_{m-1}(x) + h\int_0^x\int_0^x [\mathscr{R}_m(u_{m-1}(x))]dxdx + C_1 + C_2x, \quad m \geq 1.$$

The integration constants C_1 and C_2 are calculated using equation (37). The HAM series solution when $u_0(x) = 0.671397 + 0.0954005(x^2 - x)$ and $h = -1$ is given by

$$u(x) = 0.410465 + x(0.171623 + x(0.301729 + x(-0.00759606 + x(-0.299954 + x(-0.0113261 + x(0.12512 + x(-0.00184181 + x(-0.0160226 + x(0.00206268 + x(-0.00288598 + x(0.0000767687 + x(-0.0000536825 + (4.24682 \times 10{-}7 - 1.9414 \times 10{-}7x)x + \cdots)))))))))))).$$

Table 1 Comparison of solutions obtained from HAM, HWT, and their absolute errors (AE) with Exact solution for Application 4.1.

x	Exact	HAM	HWT	HAM Error	HWT Error
0.5	0.541049	0.540974	0.54147	7.554×10^{-5}	5.0×10^{-4}
0.6	0.581622	0.581618	0.58234	4.062×10^{-6}	7.2×10^{-4}
0.7	0.614361	0.614361	0.61534	3.496×10^{-7}	9.8×10^{-4}
0.8	0.639926	0.639926	0.64121	2.917×10^{-7}	1.28×10^{-3}
0.9	0.658813	0.658813	0.66043	2.933×10^{-7}	1.62×10^{-3}
1.0	0.671397	0.671397	0.67340	2.929×10^{-7}	2×10^{-3}
1.1	0.677989	0.677989	0.68041	2.904×10^{-7}	2.42×10^{-3}
1.2	0.678866	0.678865	0.68175	2.838×10^{-7}	2.88×10^{-3}
1.3	0.674286	0.674289	0.67767	3.718×10^{-6}	3.38×10^{-3}
1.4	0.663534	0.664524	0.66844	9.9×10^{-4}	3.92×10^{-3}

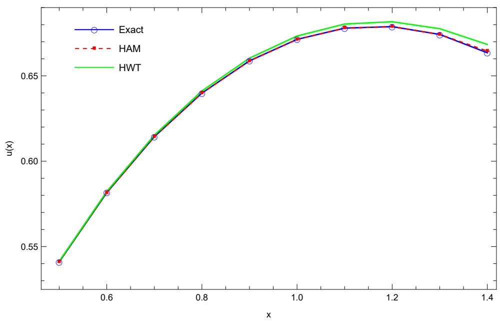

Fig. 1 Comparison of the exact solution with HAM and HWT solutions for Application 4.1.

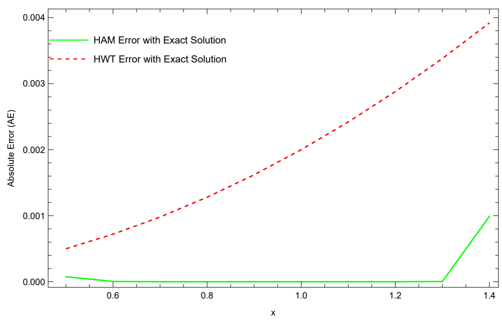

Fig. 2 Error analysis of HAM and HWT solutions for Application 4.1.

The HWT [40–42] is also used to solve the problem studied. The exact solution equation (34) is $u(x) = \sqrt{\frac{2}{\pi x}}\sin(x)$ [43]. Table 1 contains the numerical values of the solutions obtained from HAM and HWT and also their AE with exact solutions. The tables and graphs explain the results. Geometric comparisons of the Exact, HAM, and HWT solutions are shown in Figure 1. Figure 2 presents a graphic representation of the error analysis of HAM and HWT solutions with the exact solution.

4.2 Application 4.2

Let us consider the following ODE with the initial condition [43]

$$u'(x) = 0.5(1-u(x)), \quad u(0) = 0.5. \tag{38}$$

On applying HAM to equation (38). The m^{th} order deformation is given by,

$$D[u_m(x) - \chi_m u_{m-1}(t)] = h\mathscr{R}_m(u_{m-1}(t)). \tag{39}$$

Where, $D = \frac{d}{dx}$,

$$\mathscr{R}_m(u_{m-1}(t)) = \frac{du_{m-1}}{dx} + 0.5u_{m-1} - 0.5(1-\chi_m), \tag{40}$$

subject to,

$$u_m(0) = 0. \tag{41}$$

Integrating on the either sides of equation (39), we get

$$u_m(x) = \chi_m u_{m-1}(x) + h\int_0^x [\mathscr{R}_m(u_{m-1}(x))]dx + C_1, \quad m \geq 1.$$

C_1 is a integration constant calculated using equation (41). Taking $u_0(x) = 0.5$ and setting $h = -1$ successively we obtain,
$u_1(x) = 0.25x,$
$u_2(x) = -0.0625x^2,$
$u_3(x) = 0.0104167x^3,$
$u_4(x) = 0.00130208x^4,$
$\vdots$

Collectively, HAM series solution $u(x) = \sum_{m=0}^{\infty} u_m(x)$ is given by
$u(x) = 0.5 + 0.25x - 0.0625x^2 + 0.0104167x^3 - 0.00130208x^4 + 0.000130208x^5 \cdots.$

Table 2 Comparison of solutions obtained from HAM, HWT, and their AE with ND Solver solution for Application 4.2.

x	ND Solver	HAM	HWT	HAM Error	HWT Error
0	0.5	0.5	0.5	0.	0
0.1	0.52438	0.52438	0.52439	2.048×10^{-8}	1.120×10^{-6}
0.2	0.54758	0.54758	0.54759	1.718×10^{-8}	4.480×10^{-6}
0.3	0.56964	0.56964	0.56966	1.413×10^{-8}	1.008×10^{-5}
0.4	0.59063	0.59063	0.59065	9.894×10^{-9}	1.792×10^{-5}
0.5	0.61059	0.61059	0.61063	2.164×10^{-8}	2.800×10^{-5}
0.6	0.62959	0.62959	0.62963	1.587×10^{-8}	4.032×10^{-5}
0.7	0.64765	0.64765	0.64771	4.028×10^{-8}	5.488×10^{-5}
0.8	0.66483	0.66483	0.66491	1.566×10^{-8}	7.168×10^{-5}
0.9	0.68118	0.68118	0.68128	1.790×10^{-8}	9.072×10^{-5}
1.0	0.69673	0.69673	0.69685	7.461×10^{-10}	1.120×10^{-4}

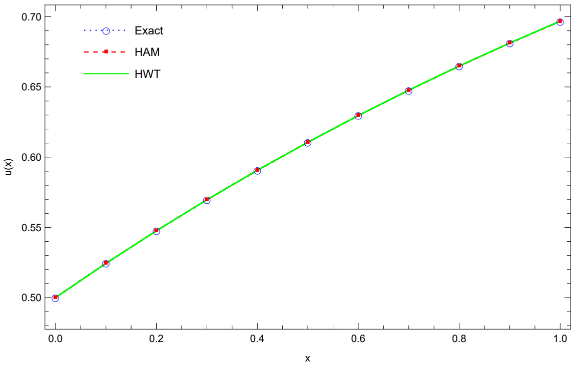

Fig. 3 Comparison of the ND solver solution with HAM and HWT solutions for Application 4.2.

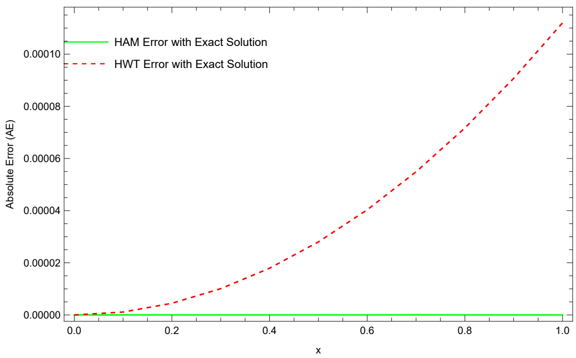

Fig. 4 Error analysis of HAM and HWT solutions for Application 4.2.

This previously stated problem is similarly resolved with the HWT. The numerical values of the solutions derived from HAM and HWT, along with their AE with exact solutions, are shown in Table 2. The tables and graphs provide an explanation of the findings. Figure 3 displays geometric comparisons of the Exact, HAM, and HWT solutions. A graphic illustration of the error analysis of the HWT and HAM answers with the exact solution is shown in Figure 4.

4.3 Application 4.3

Let us consider the following ODE system as follows

$$\left.\begin{aligned} u_1'(x) &= -1000u_1(x) + 10000u_2(x)^4, \quad u_1(0) = 1, \\ u_2'(x) &= u_1'(x) - u_2(x)^4 - u_2(x), \quad u_2(0) = 1, \end{aligned}\right\} \tag{42}$$

having the exact solution as $u_1(x) = e^{-4x}$ and $u_2(x) = e^{-x}$ [43]. Applying HAM to equation (42), the m^{th} order deformation are given by,

$$\left.\begin{aligned} D[u_{1,m}(x) - \chi_m u_{1,m-1}(x)] &= h\mathscr{R}_{1,m}(u_{1,m-1}(x)), \\ D[u_{2,m}(x) - \chi_m u_{2,m-1}(x)] &= h\mathscr{R}_{2,m}(u_{2,m-1}(x)). \end{aligned}\right\} \tag{43}$$

Where, $D = \frac{d}{dx}$, and

$$\left.\begin{aligned} \mathscr{R}_{1,m}(u_{1,m-1}(x)) &= D[u_{1,m-1}(x)] + 1000u_{1,m-1}(x) - 10000\sum_{j=0}^{m-1} u_{2,m-1-j}\sum_{i=0}^{j} u_{2,j-i}\sum_{k=0}^{i} u_{2,k}u_{2,i-k}, \\ \mathscr{R}_{2,m}(u_{2,m-1}(x)) &= D[u_{1,m-1}(x)] + \sum_{j=0}^{m-1} u_{2,m-1-j}\sum_{i=0}^{j} u_{2,j-i}\sum_{k=0}^{i} u_{2,k}u_{2,i-k} + u_{2,m-1}(x), \end{aligned}\right\} \tag{44}$$

subject to,

$$u_{1,m}(0) = 0, \quad u_{2,m}(0) = 0. \tag{45}$$

Integrating on the either sides of equation (43), we get

$$u_{1,m}(x) = \chi_m u_{1,m-1}(x) + h\int_0^x [\mathscr{R}_{1,m}(u_{1,m-1}(x))]dx + C_1, \quad m \geq 1,$$
$$u_{2,m}(x) = \chi_m u_{2,m-1}(x) + h\int_0^x [\mathscr{R}_{2,m}(u_{2,m-1}(x))]dx + C_2, \quad m \geq 1.$$

The integration constants C_1 and C_2 are calculated using equation (45). Taking $u_{1,0}(x) = 1$, $u_{2,0}(x) = 1$ and setting $h = -1$ successively we obtain,

$u_{1,1}(x) = -4x,$
$u_{1,2}(x) = \frac{(-4x)^2}{2},$
$u_{1,3}(x) = \frac{(-4x)^3}{6},$
$u_{1,4}(x) = \frac{(-4x)^4}{6}, \cdots$
$u_{2,1}(x) = -x,$
$u_{2,2}(x) = \frac{(-x)^2}{2},$
$u_{2,3}(x) = \frac{(-x)^3}{6},$
$u_{2,4}(x) = \frac{(-x)^4}{24},$
$\vdots$

Collectively, HAM series solution $u_1(x) = \sum_{m=0}^{\infty} u_{1,m}(x)$ and $u_2(x) = \sum_{m=0}^{\infty} u_{2,m}(x)$ is given by,

$u_1(x) = 1 - 4x + 8x^2 - 32\frac{x^3}{3} + 32\frac{x^4}{3} - \cdots = e^{-4x}.$
$u_2(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots = e^{-x}.$

Table 3 Comparison of solutions obtained from HAM, HWT, and their AE with Exact solution for u_1 in Application 4.3.

x	Exact	HAM	HWT	HAM Error	HWT Error
0	1	1	1	0	0
0.1	0.67032	0.67032	0.67032	0	1.120×10^{-7}
0.2	0.44932	0.44932	0.44933	0	8.960×10^{-7}
0.3	0.30119	0.30119	0.30120	0	3.024×10^{-6}
0.4	0.20189	0.20189	0.20190	0	7.168×10^{-6}
0.5	0.13533	0.13533	0.13535	0	1.140×10^{-5}
0.6	0.09071	0.09071	0.09074	0	2.419×10^{-5}
0.7	0.06081	0.06081	0.06084	0	3.841×10^{-5}
0.8	0.04076	0.04076	0.04082	0	5.734×10^{-5}
0.9	0.02732	0.02732	0.02740	0	8.164×10^{-5}
1.0	0.01831	0.01831	0.01842	0	1.120×10^{-4}

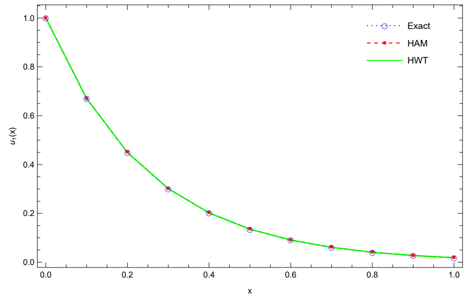

Fig. 5 Comparison of the exact solution with HAM and HWT solutions for $u_1(x)$ in Application 4.3.

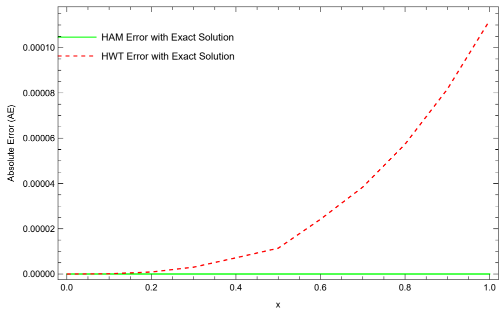

Fig. 6 Error analysis of HAM and HWT solutions for $u_1(x)$ in Application 4.3.

Table 4 Comparison of solutions obtained from HAM, HWT, and their AE with Exact solution for u_2 in Application 4.3.

x	Exact	HAM	HWT	HAM Error	HWT Error
0	1	1	1	0	0
0.1	0.90483	0.90483	0.90484	0	5.672×10^{-8}
0.2	0.81873	0.81873	0.81873	0	9.075×10^{-7}
0.3	0.74081	0.74081	0.74082	0	4.594×10^{-6}
0.4	0.67032	0.67032	0.67057	0	1.452×10^{-5}
0.5	0.60653	0.60653	0.60657	0	3.545×10^{-5}
0.6	0.54881	0.54881	0.54889	0	7.350×10^{-5}
0.7	0.49658	0.49658	0.49672	0	1.361×10^{-4}
0.8	0.44932	0.44932	0.44956	0	2.323×10^{-4}
0.9	0.40656	0.40656	0.40694	0	3.721×10^{-4}
1.0	0.36787	0.36787	0.36845	0	5.672×10^{-4}

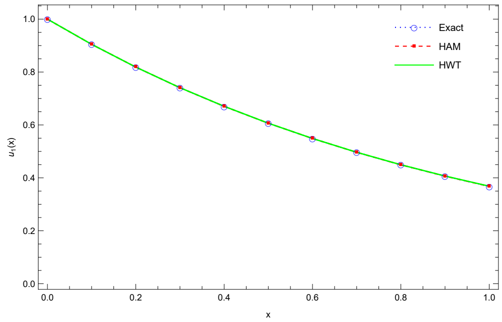

Fig. 7 Comparison of the exact solution with HAM and HWT solutions for $u_2(x)$ in Application 4.3.

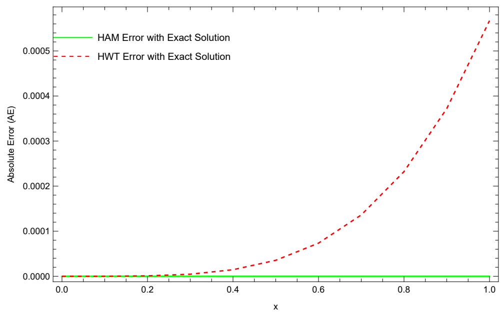

Fig. 8 Error analysis of HAM and HWT solutions for $u_2(x)$ in Application 4.3.

The above-mentioned problem is additionally solved using the HWT. Table 3 and 4 present the numerical values of the solutions of u_1 and u_2, respectively, together with their exact solutions and AE. Exact, HAM, and HWT solution geometric comparisons are displayed in Figure 5 and 7 for u_1 and u_2, respectively. An error analysis of the HAM and HWT solutions is shown graphically in Figure 6 and 8 for u_1 and u_2 respectively.

4.4 Application 4.4

Let us consider the following PDE with the condition

$$u_t + \frac{1}{2}(u^2)_x - u(1-u) = 0; \quad u(x,0) = e^{-x}, \tag{46}$$

having the exact solution $u(x,t) = e^{t-x}$. By applying HAM to equation (46), the m^{th} order deformation is given by,

$$D[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h\mathscr{R}_m(u_{m-1}(x,t)). \tag{47}$$

Where, $D = \frac{\partial}{\partial t}$, and

$$\mathscr{R}_m(u_{m-1}(x,t)) = D[u_{m-1}(x,t)] + \frac{1}{2}\frac{\partial}{\partial x}\left[\sum_{j=0}^{m-1} u_{m-1-j}(x,t)u_j(x,t)\right] - u_{m-1}(x,t) + \sum_{j=0}^{m-1} u_{m-1-j}(x,t)u_j(x,t), \tag{48}$$

subject to,

$$u_m(x,0) = 0. \tag{49}$$

Integrating on the either sides of equation (47), we get

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h\int_0^t [\mathscr{R}_m(u_{m-1}(x,t))]dt + C_1, \quad m \geq 1.$$

The integration constants C_1 and C_2 are calculated using equation (49). Taking $u_0(x,t) = e^{-x}$ and setting $h = -1$ successively we obtain,

$u_1(x,t) = e^{-x}t,$

$u_2(x,t) = e^{-x}\frac{t^2}{2},$

$u_3(x,t) = e^{-x}\frac{t^3}{6},$

$u_4(x,t) = e^{-x}\frac{t^4}{24},$

$\vdots$

Collectively, HAM series solution $u(x,t) = \sum_{m=0}^{\infty} u_m(x,t)$ is given by,

$$u(x,t) = e^{-x} + te^{-x} + \frac{t^2}{2}e^{-x} + \frac{t^3}{6}e^{-x} + \frac{t^4}{24}e^{-x} + \frac{t^5}{120}e^{-x} + \cdots = e^{(t-x)}.$$

This problem is also solved again by HWT, and the results obtained are compared using tables and graphs. Tables 5,6,7,8,9 and Table 10 contain the numerical values and their errors with exact solutions of the above problem for different values of x and t. Figures 9, 11, 13, and 15 compare the HWT and HAM solutions with exact solutions, and Figures 10, 12, 14, and 16 represent the graphical interpretation for different values of x and t. Figure 17 depicts the 3-dimensional analysis of the HAM and HWT solutions with Exact solutions.

Table 5 Comparison of solutions obtained from HAM, HWT, and their AE with Exact solution for $x = 0$ in Application 4.4.

t	Exact	HAM	HWT	HAM Error	HWT Error
0	1	1	1	0	0
0.1	1.10517	1.10517	1.10517	0	0
0.2	1.22140	1.22140	1.22140	0	0
0.3	1.34986	1.34986	1.34986	0	0
0.4	1.49182	1.49182	1.49182	0	0
0.5	1.64872	1.64872	1.64872	0	0
0.6	1.82212	1.82212	1.82212	0	0
0.7	2.01375	2.01375	2.01375	0	0
0.8	2.22554	2.22554	2.22554	0	0
0.9	2.45960	2.45960	2.45960	0	0
1.0	2.71828	2.71828	2.71828	0	0

Table 6 Comparison of solutions obtained from HAM, HWT, and their AE with Exact solution for $x = 1$ in Application 4.4.

t	Exact	HAM	HWT Error	HAM Error	HWT
0	0.36787	0.36787	0.36787	0	0
0.1	0.40657	0.40657	0.41224	0	5.672×10^{-3}
0.2	0.44932	0.44932	0.47202	0	2.268×10^{-2}
0.3	0.49658	0.49658	0.54763	0	5.104×10^{-2}
0.4	0.54881	0.54881	0.63956	0	9.075×10^{-2}
0.5	0.60653	0.60653	0.74833	0	1.418×10^{-1}
0.6	0.67032	0.67032	0.87451	0	2.042×10^{-1}
0.7	0.74081	0.74081	1.01870	0	2.779×10^{-1}
0.8	0.81873	0.81873	1.18175	0	3.630×10^{-1}
0.9	0.90483	0.90483	1.36432	0	4.594×10^{-1}
1.0	1	1	1.56721	0	5.672×10^{-1}

Table 7 Comparison of solutions obtained from HAM, HWT, and their AE with Exact solution for $t = 0$ in Application 4.4.

x	Exact	HAM	HWT Error	HAM Error	HWT
0	1	1	1	0	0
0.1	0.90483	0.90483	0.90483	0	0
0.2	0.81873	0.81873	0.81873	0	0
0.3	0.74081	0.74081	0.74081	0	0
0.4	0.67032	0.67032	0.67032	0	0
0.5	0.60653	0.60653	0.60653	0	0
0.6	0.54881	0.54881	0.54881	0	0
0.7	0.49658	0.49658	0.49658	0	0
0.8	0.44932	0.44932	0.44932	0	0
0.9	0.40656	0.40656	0.40656	0	0
1.0	0.36787	0.36787	0.36787	0	0

Table 8 Comparison of solutions obtained from HAM, HWT, and their AE with Exact solution for $t = 1$ in Application 4.4.

x	Exact	HAM	HWT Error	HAM Error	HWT
0	2.71828	2.71828	2.71828	0	0
0.1	2.45960	2.45960	2.46021	0	5.672×10^{-4}
0.2	2.22554	2.22554	2.23012	0	4.357×10^{-3}
0.3	2.01375	2.01375	2.0291	0	1.531×10^{-2}
0.4	1.82212	1.82212	1.8584	0	3.630×10^{-2}
0.5	1.64872	1.64872	1.7196	0	7.090×10^{-2}
0.6	1.49182	1.49182	1.6143	0	1.225×10^{-1}
0.7	1.34986	1.34986	1.5444	0	1.945×10^{-1}
0.8	1.22140	1.22140	1.5118	0	2.904×10^{-1}
0.9	1.10517	1.10517	1.5187	0	4.134×10^{-1}
1.0	1	1	1.5672	0	5.672×10^{-1}

Table 9 Comparison of absolute error AE of the HAM and HWT solutions with Exact solution for $t = 0.1$, $t = 0.01$, and $t = 0.001$ in Application 4.4.

x	Error at $t = 0.1$		Error at $t = 0.01$		Error at $t = 0.001$	
	HAM	HWT	HAM	HWT	HAM	HWT
0	0	0	0	0	0	0
0.1	0	5.672×10^{-6}	0	5.672×10^{-8}	0	5.672×10^{-10}
0.2	0	4.537×10^{-5}	0	4.537×10^{-7}	0	4.537×10^{-9}
0.3	0	1.531×10^{-4}	0	1.531×10^{-6}	0	1.531×10^{-8}
0.4	0	3.630×10^{-4}	0	3.630×10^{-6}	0	3.630×10^{-8}
0.5	0	7.090×10^{-4}	0	7.090×10^{-6}	0	7.090×10^{-8}
0.6	0	1.225×10^{-3}	0	1.225×10^{-5}	0	1.225×10^{-7}
0.7	0	1.945×10^{-3}	0	1.945×10^{-5}	0	1.945×10^{-7}
0.8	0	2.904×10^{-3}	0	2.904×10^{-5}	0	2.904×10^{-7}
0.9	0	4.134×10^{-3}	0	4.134×10^{-5}	0	4.134×10^{-7}
1.0	0	5.672×10^{-3}	0	5.672×10^{-5}	0	5.672×10^{-7}

Table 10 Comparison of absolute error AE of the HAM and HWT solutions with Exact solution for $x = 0.1$, $x = 0.01$, and $x = 0.001$ in Application 4.4.

t	Error at $x = 0.1$		Error at $x = 0.01$		Error at $x = 0.001$	
	HAM	HWT	HAM	HWT	HAM	HWT
0	0	0	0	0	0	0
0.1	0	5.671×10^{-6}	0	5.672×10^{-9}	0	5.671×10^{-12}
0.2	0	2.268×10^{-5}	0	2.268×10^{-8}	0	2.268×10^{-11}
0.3	0	5.104×10^{-5}	0	5.104×10^{-8}	0	5.104×10^{-11}
0.4	0	9.075×10^{-5}	0	9.075×10^{-8}	0	9.075×10^{-11}
0.5	0	1.418×10^{-4}	0	1.418×10^{-7}	0	1.418×10^{-10}
0.6	0	2.041×10^{-4}	0	2.041×10^{-7}	0	2.041×10^{-10}
0.7	0	2.779×10^{-4}	0	2.779×10^{-7}	0	2.779×10^{-10}
0.8	0	3.630×10^{-4}	0	3.630×10^{-7}	0	3.630×10^{-10}
0.9	0	4.594×10^{-4}	0	4.594×10^{-7}	0	4.594×10^{-10}
1.0	0	5.672×10^{-4}	0	5.672×10^{-7}	0	5.672×10^{-10}

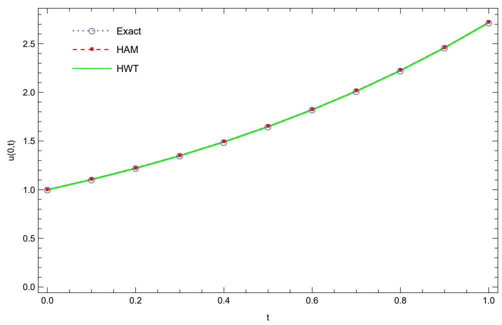

Fig. 9 Comparison of the exact solution with HAM and HWT solutions at $x = 0$ of Application 4.4.

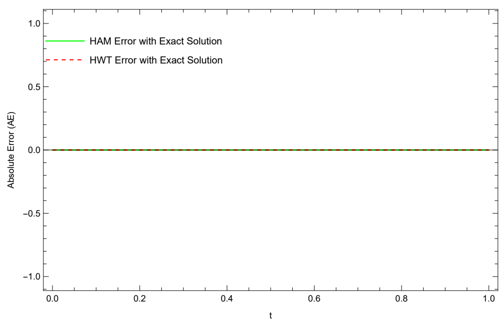

Fig. 10 Error analysis of HAM and HWT solutions at $x = 0$ of Application 4.4.

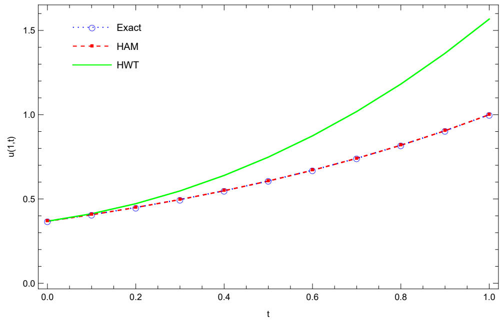

Fig. 11 Comparison of the exact solution with HAM and HWT solutions at $x = 1$ of Application 4.4.

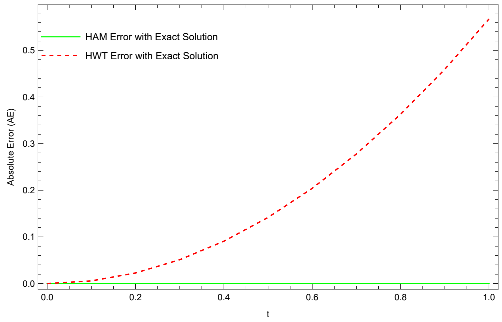

Fig. 12 Error analysis of HAM and HWT solutions at $x = 1$ of Application 4.4.

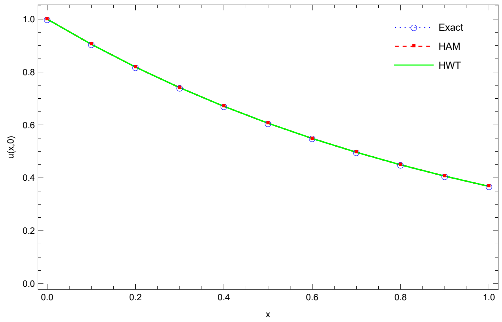

Fig. 13 Comparison of the exact solution with HAM and HWT solutions at $t = 0$ of Application 4.4.

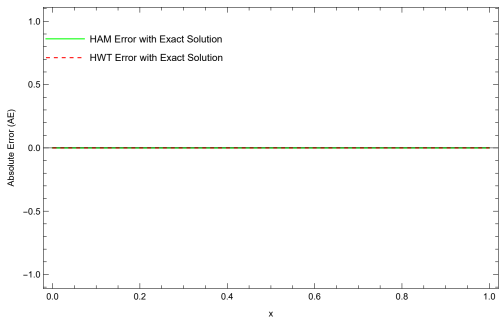

Fig. 14 Error analysis of HAM and HWT solutions at $t = 0$ of Application 4.4.

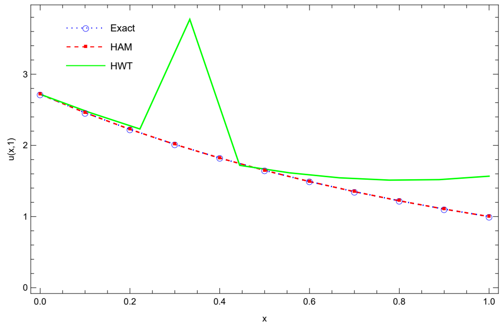

Fig. 15 Comparison of the exact solution with HAM and HWT solutions at $t = 1$ of Application 4.4.

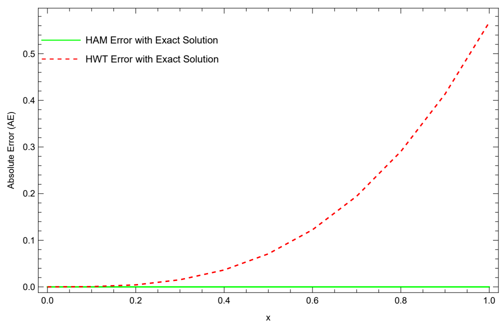

Fig. 16 Error analysis of HAM and HWT solutions at $t = 1$ of Application 4.4.

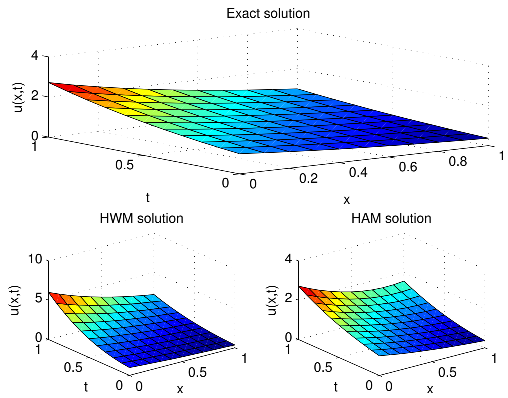

Fig. 17 3D plot of HAM and HWT solutions for various values of x and t of Application 4.4.

5 Conclusions

In this research, we suggested a scheme for solving the ODEs, system of ODEs, and PDEs using the homotopy analysis and a Haar wavelet methodology. Theorems were used to discuss convergence analysis. To exhibit the efficiency and efficacy of the suggested scheme, tables and figures describing the results were provided, along with numerical examples illustrating the method's effectiveness. The error analysis reveals that the HAM solutions are more accurate than HWT and all other techniques found in literature, and the obtained results agree with the exact or ND solver solutions. Results show that the suggested technique can potentially solve complex ODE and PDE problems efficiently.

6 Declarations

6.1 Conflict of interest:

Not applicable.

6.2 Funding:

Not applicable.

6.3 Author's contribution:

S.K.N.-Conceptualization, Methodology, Writing-Original Draft, Writing-Review. S.D.K.- Editing, Formal Analysis, Supervision. K.S.-Resources, Methodology, Editing, Validation. All authors read and approved and submitted the final version of this manuscript.

6.4 Acknowledgement:

The authors deeply appreciate the reviewers for their helpful and constructive suggestions, which can help further improve this paper.

6.5 Data availability statement:

All data that support the findings of this study are included within the article.

6.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

References

[1] Henrici P., Discrete Variable Methods in Ordinary Differential Equations, Wiley, USA, 1962.
[2] Lambert J.D., Numerical Methods for Ordinary Differential Systems: The Initial Value Problem, Wiley, USA, 1991.
[3] Murphy G.M., Ordinary Differential Equations and Their Solutions, Dover Publications, USA, 2011.
[4] Hairer E., Wanner G., Nørsett S.P., Solving Ordinary Differential Equations I: Nonstiff problems, Springer Verlag, Germany, 1993.
[5] Butcher J.C., Numerical Methods for Ordinary Differential Equations, Wiley, UK, 2016.
[6] Gragg W.B., Stetter H.J., Generalized multistep predictor-corrector methods, Journal of the ACM, 11(2), 188–209, 1964.
[7] Ramos H., Kalogiratou Z., Monovasilis T., Simos T.E., An optimized two-step hybrid block method for solving general second order initial-value problems, Numerical Algorithms, 72, 1089–1102, 2016.
[8] Rufai M.A., Ramos H., A variable step-size fourth-derivative hybrid block strategy for integrating third-order IVPs, with applications, International Journal of Computer Mathematics, 99(2), 292–308, 2022.
[9] Hojjati G., Rahimi Ardabili M.Y., Hosseini S.M., A-EBDF: an adaptive method for numerical solution of stiff systems of ODEs, Mathematics and Computers in Simulation, 66(1), 33–41, 2004.
[10] Carroll J., A matricial exponentially fitted scheme for the numerical solution of stiff initial-value problems, Computers & Mathematics with Applications, 26(4), 57–64, 1993.
[11] Cash J.R., The integration of stiff initial value problems in ODEs using modified extended backward differentiation formulae, Computers & Mathematics with Applications, 9(5), 645–657, 1983.
[12] Hsiao C.H., Haar wavelet approach to linear stiff systems, Mathematics and Computers in Simulation, 64(5), 561–567, 2004.
[13] Bujurke N.M., Salimath C.S., Shiralashetti S.C., Numerical solution of stiff systems from nonlinear dynamics using single-term Haar wavelet series, Nonlinear Dynamics, 51, 595–605, 2008.

[14] Hosseini S.M., Hojjati G., Matrix free MEBDF method for the solution of stiff systems of ODEs, Mathematical and Computer Modelling, 29(4), 67–77, 1999.

[15] Khalsaraei M.M., Shokri A., Molayi M., The new class of multistep multiderivative hybrid methods for the numerical solution of chemical stiff systems of first order IVPs, Journal of Mathematical Chemistry, 58, 1987–2012, 2020.

[16] Prothero A., Robinson A., On the stability and accuracy of one-step methods for solving stiff systems of ordinary differential equations, Mathematics of Computation, 28(125), 145–162, 1974.

[17] Bader G., Deuflhard P., A semi-implicit mid-point rule for stiff systems of ordinary differential equations, Numerische Mathematik, 41, 373–398, 1983.

[18] Mulimani M., Srinivasa K., Numerical and theoretical analysis of the parabolic partial differential equation through the Bernoulli wavelet collocation scheme, International Journal of Dynamics and Control, 12, 4000–4019, 2024.

[19] Srinivasa K., Mulimani M., Adel W., A numerical approach for solving nonlinear fractional Klein Gordon equation with applications in quantum mechanics, Journal of Nonlinear Complex and Data Science, 25(2), 173–195, 2024.

[20] Heydari M., Avazzadeh Z., Hosseinzadeh N., Haar wavelet method for solving high-order differential equations with multi-point boundary conditions, Journal of Applied and Computational Mechanics, 8(2), 528–544, 2022.

[21] Darweesh A., Al-Khaled K., Al-Yaqeen O.A., Haar wavelets method for solving class of coupled systems of linear fractional Fredholm integro-differential equations, Heliyon, 9(9), e19717, 2023.

[22] Darvishi M.T., Khani F., Soliman A.A., The numerical simulation for stiff systems of ordinary differential equations, Computers & Mathematics with Applications, 54(7-8), 1055–1063, 2007.

[23] Mohammad M., Trounev A., Fractional nonlinear Volterra-Fredholm integral equations involving Atangana-Baleanu fractional derivative: framelet applications, Advances in Difference Equations, 2020(1), 618, 2020.

[24] Alshbool M.H.T., Mohammad M., Isik O., Hashim I., Fractional Bernstein operational matrices for solving integro-differential equations involved by Caputo fractional derivative, Results in Applied Mathematics, 14, 100258, 2022.

[25] Mohammad M., Trounev A., Computational precision in time fractional PDEs: Euler wavelets and novel numerical techniques, Partial Differential Equations in Applied Mathematics, 12, 100918, 2024.

[26] Sachin K.N., Suguntha D.K., Srinivasa K., A semi-analytical solution of stiff equations arising in chemistry problems by homotopy analysis method, International Journal of Dynamics and Control, 13(87), 1–16, 2025.

[27] Morrow L., An investigation of the homotopy analysis method for solving non-linear differential equations, 2014. https://calculate.org.au/wp-content/uploads/sites/6/2014/09/CORRECTED-MORROW-Report.pdf, Accessed: September 16, 2024.

[28] Motsa S.S., Sibanda P., Awad F.G., Shateyi S., A new spectral-homotopy analysis method for the MHD Jeffery-Hamel problem, Computers & Fluids, 39(7), 1219–1225, 2010.

[29] Adomian G., Solving Frontier Problems of Physics: The Decomposition Method, Springer, USA, 2013.

[30] Shiralasetti S., Some results on Haar wavelets matrix through linear algebra, Wavelet and Linear Algebra, 4(2), 49–59, 2017.

[31] Sachin K.N., Suguntha K., Srinivasa K., A semi-analytic and numerical approach to the fractional differential equations, Iranian Journal of Numerical Analysis and Optimization, 14(4), 1069–1105, 2024.

[32] Liao S., Beyond Perturbation: Introduction to the Homotopy Analysis Method, CRC Press, USA, 2004.

[33] Odibat Z.M., A study on the convergence of homotopy analysis method, Applied Mathematics and Computation, 217(2), 782–789, 2010.

[34] Chen Y., Yi M., Yu C., Error analysis for numerical solution of fractional differential equation by Haar wavelets method, Journal of Computational Science, 3(5), 367–373, 2012.

[35] Srinivasa K., Preetham M.P., A study on homotopy analysis method and clique polynomial method, Computational Methods for Differential Equations, 10(3), 774–788, 2022.

[36] Liao S., Advances in the Homotopy Analysis Method, World Scientific, Singapore, 2013.

[37] Zurigat M., Momani S., Odibat Z., Alawneh A., The homotopy analysis method for handling systems of fractional differential equations, Applied Mathematical Modelling, 34(1), 24–35, 2010.

[38] Narayana S.K., Kannadasan S.D., Srinivasa K., A study on the fractional Ebola virus model by the semi-analytic and numerical approach, Computational Methods for Differential Equations, 13(3), 783–801, 2025.

[39] Liao S., Homotopy Analysis Method in Nonlinear Differential Equations, Springer, China, 2012.

[40] Srinivasa K., Yeshwanth R., Haar wavelet approach for the mathematical model on hepatitis B virus, Journal of Computational and Nonlinear Dynamics, 19(9), 091003, 2024.

[41] Srinivasa K., Yeshwanth R., A numerical study of the evolution of smoking habit model through Haar wavelet technique, International Journal of Dynamics and Control, 12, 3179–3197, 2024.

[42] Srinivasa K., Yeshwanth R., Investigation of the wavelet method impact on the mathematical model of global warming effects on marine ecosystems, Journal of Applied Mathematics and Computing, 70, 4601–4627, 2024.

[43] Ramos H., Rufai M.A., An adaptive one-point second-derivative Lobatto-type hybrid method for solving efficiently differential systems, International Journal of Computer Mathematics, 99(8), 1687–1705, 2022.