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Original Study

Signless Laplacian spectrum of power graph of certain finite non-commutative groups

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Abstract

In this study, we investigate the signless Laplacian spectrum of power graphs of different finite non-commutative groups. Initially, we obtain the spectrum of the signless Laplacian matrix of power graph of elementary abelian groups whose orders are powers of a prime number. The signless Laplacian spectrum of the smallest sporadic group, the Mathieu group M_{11}, is then computed. We also find the signless Laplacian eigenvalues of $\mathscr{P}(Q_{2^{k+2}})$, where $Q_{2^{k+2}}$ represents the generalized quaternion group. For $\mathscr{P}(\mathrm{Dic}_{4n})$, where Dic_{4n} is the dicyclic group, we finally give bounds on the signless Laplacian spectral radius.

Keywords: Finite groups, non-commutative groups, power graph, signless Laplacian eigenvalues.
AMS 2020 codes: 05C25; 05C50; 20K01; 15A18; 15A39.

1 Introduction

In [1], Kelarev and Quinn began studying the power graph of a finite semigroup. The *directed power graph* of a semigroup S, $\mathscr{P}(S)$, has vertex set as S. There is a directed edge from u to v if v is some power of u. The *undirected power graph* $\mathscr{P}(G)$ of a finite group G was introduced in [2]. $\mathscr{P}(G)$ has vertex set as G, and two vertices u and v are adjacent if and only if $u = v^m$ or $v = u^n$ for some m, n. From [2], it is known that $\mathscr{P}(G)$ is always connected and it is complete if and only if G is a cyclic group of order 1 or p^m where p is a prime number and m is a positive integer. In [3], it was proved that non-isomorphic finite groups may have isomorphic power graphs, but finite abelian groups with isomorphic power graphs must be isomorphic. Moreover, it was conjectured that two finite groups with isomorphic power graphs must have the same number of elements of each order. In [4], the conjecture was proved. In [5], several structural properties of power graph of finite groups have been studied. Various interesting results on the power graph of finite groups and semigroups have been studied in [6].

Given a simple graph $\mathscr{G}$, the *signless Laplacian* matrix $Q(\mathscr{G})$ of $\mathscr{G}$ is defined as $Q(\mathscr{G}) = \mathrm{Deg}\,(\mathscr{G}) + A(\mathscr{G})$. Here, $A(\mathscr{G})$ is the adjacency matrix of $\mathscr{G}$, and $\mathrm{Deg}\,(\mathscr{G})$ is the diagonal matrix of vertex degrees. The signless Laplacian matrix (see [7–9]) $Q(\mathscr{G})$ has all its eigenvalues as non-negative. The set of eigenvalues of the adjacency matrix and signless Laplacian matrix of $\mathscr{G}$ respectively form the adjacency, and signless Laplacian *spectrum* of $\mathscr{G}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ denote the eigenvalues of $Q(\mathscr{G})$ having multiplicities $m_1, m_2, \cdots, m_k$.

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Then the spectrum of $Q(\mathscr{G})$, denoted by $\sigma(\mathscr{G})$, is described as follows: $\sigma(\mathscr{G}) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}$. Thus, $\sigma(K_n) = \begin{pmatrix} 2(n-1) & n-2 \\ 1 & n-1 \end{pmatrix}$ for $n \geq 2$, where, K_n is the complete graph. The largest eigenvalue of Q is known as the *Q-spectral radius* of G. We shall denote the finite cyclic group by $\mathbb{Z}_n$, the dihedral group by D_n, the dicyclic group by Dic_{4n}, and the generalized quaternion group by $Q_{2^{k+2}}$. The Laplacian spectrum of $\mathscr{P}(\mathbb{Z}_n)$ and $\mathscr{P}(\mathrm{D}_n)$ was studied in [10]. Various other structural properties of power graph of finite groups have been studied in [11–13]. For the past few years, scholars have been interested in studying the spectrum of graphs connected to algebraic structures in [14–19]. The adjacency spectra of power graph of various finite groups have been studied in [20]. In [21], Banerjee and Adhikari initiated the study of eigenvalues of the signless Laplacian matrix of $\mathscr{P}(\mathbb{Z}_n)$. A recent survey on power graph of finite groups has been studied in [22].

The above works motivate us to calculate the signless Laplacian spectrum of the power graph of certain finite non-commutative groups. We compute the signless Laplacian spectrum of the power graph of elementary abelian groups of prime power order, Mathieu group M_{11}, and $Q_{2^{k+2}}$. Finally, we provide bounds on the signless Laplacian spectral radius of $\mathscr{P}(\mathrm{Dic}_{4n})$. In [23] and [24], some definitions of standard terms related to graph theory and group theory used in this paper have been given. Throughout the paper, the complement of a graph $\mathscr{G}$ is denoted by $\overline{\mathscr{G}}$. The characteristic polynomial of a matrix M is denoted by $\Theta(M;x)$.

The rest of this paper is organized as follows: In Section 2, we present some definitions and theorems. In Section 3, we provide some applications of studying the signless Laplacian spectrum of power graph of various finite groups in brief. We determine the eigenvalues of the signless Laplacian matrix of the power graph of elementary abelian group of order p^n, and the smallest sporadic group M_{11}, and the generalized quaternion group $Q_{2^{k+2}}$. Moreover, we provide lower and upper bounds on the largest eigenvalue of the signless Laplacian matrix of dicyclic group Dic_n. In Section 4, we introduce the main contribution of the paper and also discuss some related work which can be considered for future research.

2 Preliminaries

In this section of the paper, we present some theorems and definitions used in this paper.

Definition 1. If G is a commutative group and all elements of G other than the identity element have the same order, then G is said to be an elementary abelian group.

If G is an elementary abelian group with identity element e, then for any element $g(\neq e)$ of G, the order of g will be p for some prime number.

Theorem 1. *Suppose G is an elementary abelian group having order p^n. If $\ell = \frac{p^n-1}{p-1}$, then the signless Laplacian spectrum of $\mathscr{P}(G)$ is given as follows:*

$$\sigma(\mathscr{P}(G)) = \begin{pmatrix} 2p-3 & \dfrac{\ell a^2 + t + \sqrt{a^4\ell^2 - 2a^2\ell t + 4a^2\ell + t^2}}{2} & \dfrac{\ell a^2 + t - \sqrt{a^4\ell^2 - 2a^2\ell t + 4a^2\ell + t^2}}{2} \\ \ell - 1 & 1 & 1 \end{pmatrix}.$$

Here, $a = \sqrt{p-1}, t = 2p-3$.

Proof. Given $\ell = \frac{p^n-1}{p-1}$, using [25, Theorem 8], we find that $\mathscr{P}(G)$ is isomorphic to $\mathscr{G}[K_1, \underbrace{K_{p-1}, K_{p-1}, \cdots, K_{p-1}}_{\ell \text{ times}}]$, where $\mathscr{G} = K_1 + \overline{K}_\ell$. We note that $\mathscr{G}$ has $\ell + 1$ vertices. If we denote the vertices of $\mathscr{G}$ by $\{1, 2, 3, 4, \cdots, \ell, \ell+1\}$, then $\mathscr{G}$ looks like in Figure 1.

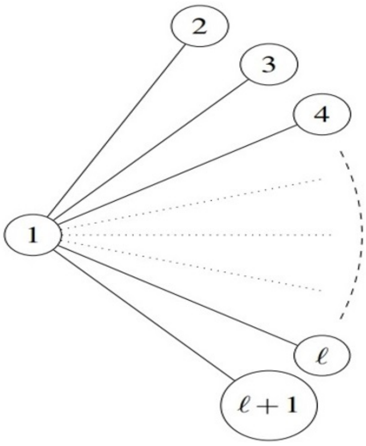

Fig. 1 $\mathscr{G} = K_1 + \overline{K}_\ell$.

Definition 2. [26, Page 59] The dicyclic group of order $4n$, denoted by Dic_{4n}, is represented as follows:

$$\mathrm{Dic}_{4n} = \langle a, b : a^{2n} = e, a^n = b^2, ab = ba^{-1} \rangle.$$

If $n = 2$, then Dic_{4n} is known as the quaternion group. Moreover, if $n = 2^k$ for some natural number k, then Dic_{4n} is known as the generalized quaternion group, and it is denoted by $Q_{2^{k+2}}$.

3 Applications

The signless Laplacian eigenvalues of a graph are associated to the graphs structural properties. In the case of graphs derived from algebraic structures such as power graph which has been studied in this paper, spectral analysis with regard to signless Laplacian matrix can help address a variety of algebraic structure-related concerns. Certain invariants related to graph connectivity and structure can also be derived from the signless Laplacian spectrum. The signless Laplacian spectrum can be used to study partitioning and cut problems in power graph of finite groups, where we are interested in separating a graph into subgraphs with minimal edge cuts.

3.1 Signless Laplacian spectrum of power graph of finite groups

Here, we shall determine the signless Laplacian spectrum of power graph of elementary abelian group, the smallest sporadic group M_{11}, and the generalized quaternion group $Q_{2^{k+2}}$. From Definition 1 and Theorem 1, we note that $N_{\mathscr{G}}(i) = \{1\}$ for all $2 \leq i \leq \ell+1$. Moreover, $N_{\mathscr{G}}(1) = \{2, 3, 4, \cdots, \ell, \ell+1\}$. Thus, we observe that $N_1 = \sum_{i=2}^{\ell+1}(p-1) = \ell(p-1)$. Moreover, $N_i = 1$ for $2 \leq i \leq \ell+1$. We further note that K_{p-1} is a $p-2$ regular graph whose signless Laplacian eigenvalues are $2(p-2)$ having multiplicity 1, and $p-3$ having multiplicity $p-2$.

Hence, $\sigma(K_{p-1}) = \begin{pmatrix} 2(p-2) & p-3 \\ 1 & p-2 \end{pmatrix}$. Also, $\sigma(K_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Using [27, Theorem 2.1] and the above information, the signless Laplacian spectrum of $\mathscr{P}(G)$ consists of $p-2$ having multiplicity $\ell(p-2)$, and remaining eigenvalues are contained in the spectrum of $C_{\mathscr{P}(G)}$ where,

$$C_{\mathscr{P}(G)} = \begin{pmatrix} \ell(p-1) & \sqrt{p-1} & \sqrt{p-1} & \sqrt{p-1} & \cdots\cdots & \sqrt{p-1} & \sqrt{p-1} \\ \sqrt{p-1} & 2p-3 & 0 & 0 & \cdots\cdots & 0 & 0 \\ \sqrt{p-1} & 0 & 2p-3 & 0 & \cdots\cdots & 0 & 0 \\ \sqrt{p-1} & 0 & 0 & 2p-3 & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \\ cdots & & & & & & \\ \sqrt{p-1} & 0 & 0 & 0 & \cdots\cdots & 2p-3 & 0 \\ \sqrt{p-1} & 0 & 0 & 0 & \cdots\cdots & 0 & 2p-3 \end{pmatrix}_{(\ell+1)\times(\ell+1)}.$$

For the sake of convenience, we shall write matrix $C_{\mathscr{P}(G)}$ as follows:

$$C_{\mathscr{P}(G)} = \begin{pmatrix} \ell a^2 & a & a & a & \cdots\cdots & a & a \\ a & t & 0 & 0 & \cdots\cdots & 0 & 0 \\ a & 0 & t & 0 & \cdots\cdots & 0 & 0 \\ a & 0 & 0 & t & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ a & 0 & 0 & 0 & \cdots\cdots & t & 0 \\ a & 0 & 0 & 0 & \cdots\cdots & 0 & t \end{pmatrix}_{(\ell+1)\times(\ell+1)},$$

where $a = \sqrt{p-1}$ and $t = 2p-3$. We shall now find the characteristic polynomial of $C_{\mathscr{P}(G)}$.

$$\begin{aligned} \Theta(C_{\mathscr{P}(G)};x) &= \det(xI - C_{\mathscr{P}(G)}) \\ &= \det \begin{pmatrix} x-\ell a^2 & -a & -a & -a & \cdots\cdots\cdots & -a & -a \\ -a & x-t & 0 & 0 & \cdots\cdots\cdots & 0 & 0 \\ -a & 0 & x-t & 0 & \cdots\cdots\cdots & 0 & 0 \\ -a & 0 & 0 & x-t & \cdots\cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots \\ -a & 0 & 0 & 0 & \cdots\cdots\cdots & x-t & 0 \\ -a & 0 & 0 & 0 & \cdots\cdots\cdots & 0 & x-t \end{pmatrix}_{(\ell+1)\times(\ell+1)}. \end{aligned}$$

We shall evaluate the above determinant using an inductive process. We define $\mathscr{M}_{(\ell+1)\times(\ell+1)} = \Big(xI - C_{\mathscr{P}(G)}\Big)_{(\ell+1)\times(\ell+1)}$. Moreover, we shall obtain $\mathscr{M}_{\ell\times\ell}$ from $\mathscr{M}_{(\ell+1)\times(\ell+1)}$ by deleting the $(\ell+1)^{\text{th}}$ row and column. Thus,

$$\mathscr{M}_{\ell\times\ell} = \begin{pmatrix} x-\ell a^2 & -a & -a & -a & \cdots\cdots\cdots & -a \\ -a & x-t & 0 & 0 & \cdots\cdots\cdots & 0 \\ -a & 0 & x-t & 0 & \cdots\cdots\cdots & 0 \\ -a & 0 & 0 & x-t & \cdots\cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ -a & 0 & 0 & 0 & \cdots\cdots\cdots & x-t \end{pmatrix}_{\ell\times\ell},$$

$$\text{and } \mathscr{M}_{2\times 2} = \begin{pmatrix} x-\ell a^2 & -a \\ -a & x-t \end{pmatrix}_{2\times 2}.$$

The matrices $\mathscr{M}_{(\ell-i)\times(\ell-i)}$ for $1\le i\le \ell-2$ can be obtained along similar lines as shown above. Now, expanding $\mathscr{M}_{(\ell+1)\times(\ell+1)}$ along the $(\ell+1)^{\text{th}}$ row, we obtain,

$$\det(\mathscr{M}_{(\ell+1)\times(\ell+1)}) = (x-t)\det(\mathscr{M}_{\ell\times\ell}) - a^2(x-t)^{\ell-1}. \tag{1}$$

In order to evaluate $\det(\mathscr{M}_{\ell\times\ell})$, we shall expand along the ℓ^{th} row of $\mathscr{M}_{\ell\times\ell}$. We observe,

$$\det(\mathscr{M}_{\ell\times\ell}) = (x-t)\det(\mathscr{M}_{(\ell-1)\times(\ell-1)}) - a^2(x-t)^{\ell-2}. \tag{2}$$

On using equation (2) into equation (1), we obtain,

$$\det(\mathscr{M}_{(\ell+1)\times(\ell+1)}) = (x-t)^2\det(\mathscr{M}_{(\ell-1)\times(\ell-1)}) - 2a^2(x-t)^{\ell-1}. \tag{3}$$

If we continue the above process, we shall obtain

$$\begin{aligned}\det(\mathscr{M}_{(\ell+1)\times(\ell+1)}) &= (x-t)^{\ell-1}\det(\mathscr{M}_{2\times 2}) - a^2(\ell-1)(x-t)^{\ell-1} \\ &= (x-t)^{\ell-1}\Big(\det(\mathscr{M}_{2\times 2}) - a^2(\ell-1)\Big) \\ &= (x-t)^{\ell-1}\Big(x^2 - x(\ell a^2+t) + t\ell a^2 - a^2 - a^2(\ell-1)\Big) \\ &= (x-t)^{\ell-1}\Big(x^2 - x(\ell a^2+t) + a^2(t\ell-1-\ell+1)\Big) \\ &= (x-t)^{\ell-1}\Big(x^2 - x(\ell a^2+t) + a^2\ell(t-1)\Big).\end{aligned}$$

The roots of $x^2 - x(\ell a^2+t) + a^2\ell(t-1) = 0$ are $\dfrac{\ell a^2+t\pm\sqrt{a^4\ell^2-2a^2\ell t+4a^2\ell+t^2}}{2}$.

Since the roots of $\Theta(C_{\mathscr{P}(G)};x)=0$ are $2p-3$ having multiplicity $\ell-1$ and $\dfrac{\ell a^2+t\pm\sqrt{a^4\ell^2-2a^2\ell t+4a^2\ell+t^2}}{2}$ having multiplicity 1, the result follows. We know that any finite simple group will be either cyclic, or alternating, or it will belong Lie groups, or else it is one of 26 exceptions known as the *sporadic groups*. Among the 26 sporadic groups, 5 were discovered by the famous mathematician Émile Léonard Mathieu in 1860. The smallest sporadic group M_{11}, known as the Mathieu group of degree 11, has order $7920 = 2^4\times 3^2\times 5\times 11$. For more information about M_{11}, it has been presented in [28]. The graph $\mathscr{P}(M_{11})$ has been studied in detail in [29]. The structure of $\mathscr{P}(M_{11})$ has been depicted in [29, Figure 3].

Theorem 2. *The signless Laplacian spectrum of the proper power graph of the sporadic group M_{11} is given as follows:*

$$\sigma(\mathscr{P}(M_{11}) = \begin{pmatrix} 0 & 2 & 3 & 5 & 5 & 7 & 8 & 11 & \alpha_1 & \alpha_2 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 55 & 1243 & 660 & 2970 & 396 & 432 & 785 & 330 & 144 & 144 & 165 & 165 & 165 & 165 \end{pmatrix}.$$

Here, α_1, α_2 are roots of the polynomial equation $x^2 - 21x + 70 = 0$, and $\beta_1, \beta_2, \beta_3, \beta_4$ are roots of the polynomial equation $x^4 - 52x^3 + 849x^2 - 5110x + 8232 = 0$.

Proof.

From [29], it is known that the $\mathscr{P}(M_{11})$ consists of 165 copies of a graph L, 55 copies of $\mathscr{P}(\mathbb{Z}_3)$, 396 copies of $\mathscr{P}(\mathbb{Z}_5)$, and 144 copies of $\mathscr{P}(\mathbb{Z}_{11})$. All of them are connected to each other through the identity element of M_{11}. Consequently, the proper power graph of M_{11} consists of 165 copies of L^*, 55 copies of $\mathscr{P}(\mathbb{Z}_2)$, 396 copies of $\mathscr{P}(\mathbb{Z}_4)$, and 144 copies of $\mathscr{P}(\mathbb{Z}_{10})$. The graph of L^* is shown in Figure 2.

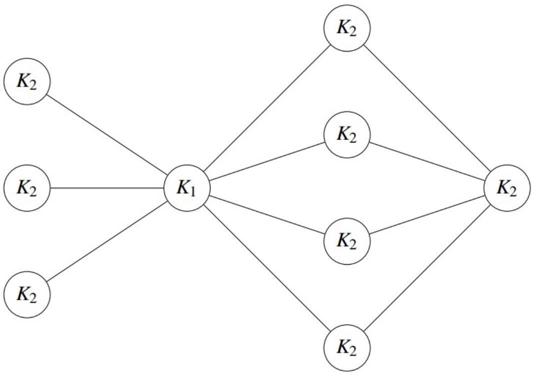

Fig. 2 The graph L^*.

Using [21, Theorem 3.1], we have $\sigma(\mathscr{P}(\mathbb{Z}_2)) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, and $\sigma(\mathscr{P}(\mathbb{Z}_4)) = \begin{pmatrix} 6 & 2 \\ 1 & 3 \end{pmatrix}$. Moreover, using [21, Theorem 5.4], the signless Laplacian spectrum of $\mathscr{P}(\mathbb{Z}_{10})$ is given as follows:

$$\sigma(\mathscr{P}(\mathbb{Z}_{10})) = \begin{pmatrix} 8 & 7 & p_1 & p_2 \\ 5 & 3 & 1 & 1 \end{pmatrix},$$

where p_1, p_2 are roots of the quadratic equation $x^2 - 21x + 70 = 0$. Now, we find that $L^* = \mathscr{G}_{L^*}[K_6, K_6, K_6, K_1, K_2, K_2, K_2, K_2, K_2]$, where $\mathscr{G}_{L^*}$ is of the following Figure 3.

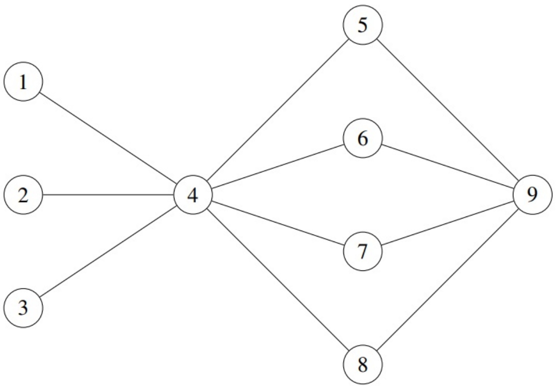

Fig. 3 The graph $\mathscr{G}_{L^*}$ for L^*.

We have $N_{\mathscr{G}_{L^*}}(i) = \{4\}$ for $1 \leq i \leq 3$, $N_{\mathscr{G}_{L^*}}(i) = \{4,9\}$ for $5 \leq i \leq 8$, $N_{\mathscr{G}_{L^*}}(4) = \{1,2,3,5,6,7,8\}$, and $N_{\mathscr{G}_{L^*}}(9) = \{5,6,7,8\}$

Here, $n_1 = n_2 = n_3 = 6$, $n_4 = 1, n_5 = n_6 = n_7 = n_8 = n_9 = 2$. Consequently, $N_i = 1$ for all $1 \leq i \leq 3$, and $N_i = 1+2 = 3$ for all $5 \leq i \leq 8$. Moreover, $N_4 = 6+6+6+2+2+2+2 = 26$, and $N_9 = 2+2+2+2 = 8$. Now, $\sigma(K_6) = \begin{pmatrix} 10 & 4 \\ 1 & 5 \end{pmatrix}$, and $\sigma(K_2) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. Thus, using [27, Theorem 2.1], the signless Laplacian eigenvalues of L^* are 5 having multiplicity 15, 3 having multiplicity 4, 8 having multiplicity 1, and rest are contained in the following:

$$C_{\mathscr{G}_{L^*}} = \begin{pmatrix} 11 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 11 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{6} & \sqrt{6} & \sqrt{6} & 26 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} & 5 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 5 & 0 & 0 & 2 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 5 & 0 & 2 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 10 \end{pmatrix}_{9\times 9}.$$

We have:

$$\begin{aligned} \Theta(C_{\mathscr{G}_{L^*}}, x) &= x^9 - 89x^8 + 3299x^7 - 67465x^6 + 842381x^5 - 6670379x^4 \\ &\quad + 33500857x^3 - 102819755x^2 + 174632150x - 124509000 \\ &= (x-11)^2(x-5)^3(x^4 - 52x^3 + 849x^2 - 5110x + 8232). \end{aligned} \tag{4}$$

Hence, using 4, we obtain

$$\begin{aligned} \Theta(L^*, x) &= (x-5)^{15} \times (x-3)^4 \times (x-8) \times \Theta(C_{\mathscr{G}_{L^*}}, x) \\ &= (x-5)^{15}(x-3)^4(x-8) \times \Big((x-11)^2(x-5)^3(x^4 - 52x^3 + 849x^2 - 5110x + 8232) \Big) \\ &= (x-3)^4(x-5)^{18}(x-8)(x-11)^2(x^4 - 52x^3 + 849x^2 - 5110x + 8232). \end{aligned} \tag{5}$$

Using the above information, the characteristic polynomial of the signless Laplacian matrix of the proper power graph of M_{11} is given as follows:

$$\begin{aligned} \Theta(Q(\mathscr{P}(M_{11})), x) &= \Theta(Q(\mathscr{P}(\mathbb{Z}_2)), x) \times \Theta(Q(\mathscr{P}(\mathbb{Z}_4)), x) \times \Theta(Q(\mathscr{P}(\mathbb{Z}_{10})), x) \times \Theta(Q(L^*), x) \\ &= \Big(x(x-2) \Big)^{55} \times \Big((x-6)(x-2)^3 \Big)^{396} \\ &\quad \times \Big((x-8)^5(x-7)^3(x^2 - 21x + 70) \Big)^{144} \\ &\quad \times \Big((x-3)^4(x-5)^{18}(x-8)(x-11)^2 \\ &\quad \times (x^4 - 52x^3 + 849x^2 - 5110x + 8232) \Big)^{165} \\ &= x^{55}(x-2)^{1243}(x-3)^{660}(x-5)^{2970}(x-6)^{396}(x-7)^{432} \\ &\quad \times (x-8)^{785}(x-11)^{330} \Big((x^2 - 21x + 70) \Big)^{144} \\ &\quad \times (x^4 - 52x^3 + 849x^2 - 5110x + 8232)^{165}. \end{aligned}$$

Thus, the signless Laplacian spectrum of the proper power graph of M_{11} consists of 0 having multiplicity 55, 2 having multiplicity 1243, 3 having multiplicity 660, 5 having multiplicity 2970, 6 having multiplicity 396, 7 having multiplicity 432, 8 having multiplicity 785, 11 having multiplicity 330, α_1, α_2 each having multiplicity 144, and $\beta_1, \beta_2, \beta_3, \beta_4$ each having multiplicity 165, where α_1, α_2 are roots of the polynomial equation $x^2 - 21x + 70 = 0$, and $\beta_1, \beta_2, \beta_3, \beta_4$ are roots of the polynomial equation $x^4 - 52x^3 + 849x^2 - 5110x + 8232 = 0$.

Theorem 3. *The signless Laplacian spectrum of $\mathscr{P}(Q_{2^{k+2}})$ is given as follows:*

$$\sigma(\mathscr{P}(Q_{2^{k+2}})) = \begin{pmatrix} 2n-2 & 4n-2 & 4 & 2 & \alpha_1 & \alpha_2 \\ 2n-3 & 1 & n-1 & n & 1 & 1 \end{pmatrix},$$

where α_1, α_2 are roots of the cubic polynomial $x^3 - 8nx^2 + (16n^2 + 8n - 12)x - 48n^2 + 64n - 16$.

Proof. Using [30, Section 4.1], we find that $\mathscr{P}(Q_{2^{k+2}})$ is constructed from a copy of $\mathscr{P}(\mathbb{Z}_{2n})$, and n copies of the complete graph K_4, which share the identity element e of $Q_{2^{k+2}}$, and the element a^n known as the *involution*. Since $n = 2^k$, hence using [2, Theorem 2.12], we find that $\mathscr{P}(\mathbb{Z}_{2n})$ is the complete graph on $2n$ vertices. Thus, we have

$$\mathscr{P}(Q_{2^{k+2}}) = P_3[K_{2n-2}, K_2, nK_2].$$

Since the characteristic polynomial of C_{P_3} is $x^3 - 8nx^2 + (16n^2 + 8n - 12)x - 48n^2 + 64n - 16$, the result follows.

We now find the signless Laplacian spectrum of power graph of quaternion group.

Corollary 4. *The signless Laplacian spectrum of $\mathscr{P}(Q_8)$ is given as follows:*

$$\sigma(\mathscr{P}(Q_8)) = \begin{pmatrix} 4 & 6 & 2 & 10 \\ 3 & 1 & 3 & 1 \end{pmatrix}.$$

Proof. On putting $n = 2$ in 3, we find that the signless Laplacian spectrum of $\mathscr{P}(Q_8)$ consists of 4 having multiplicity 2, 6 having multiplicity 1, 2 having multiplicity 2, and remaining eigenvalues are roots of the cubic polynomial $x^3 - 16x^2 + 68x - 80$. Since $x^3 - 16x^2 + 68x - 80 = (x-2)(x-4)(x-10)$, the result follows.

3.2 Bounds on signless Laplacian spectral radius of power graph of dicyclic group

In the following theorem, we shall provide an upper bound and a lower bound on the Q-spectral radius of $\mathscr{P}(\mathrm{Dic}_n)$.

Theorem 5. *If $n \geq 3$ is a positive integer, then the Q-spectral radius of $\mathscr{P}(\mathrm{Dic}_n)$ is bounded above by* $\dfrac{3n - 6 + \sqrt{8\alpha + n^2 - 4n + 4}}{2} + 2n + \sqrt{2n}$*. Moreover, the Q-spectral radius of $\mathscr{P}(\mathrm{Dic}_n)$ is bounded below by* $\dfrac{2\alpha + n - 2 + \sqrt{n^2 - 4\alpha^2 + 4\alpha n - 4n + 4}}{2}$*. Here, $\alpha = \phi(n) + 1$.*

Proof. We note that $\mathscr{P}(\mathrm{Dic}_{4n})$ consists of one copy of $\mathscr{P}(\mathbb{Z}_{2n})$, and n copies of the complete graph K_4, which contain the elements e and a^n of Dic_{4n}.

We shall index the elements of $\mathscr{P}(\mathrm{Dic}_{4n})$ in the following order:

We first list the elements e and a^n. We then list the remaining elements of $\mathbb{Z}_n$ namely $\{a, a^2, a^3, \cdots, a^{n-1}, a^{n+1}, a^{n+2}, \cdots, a^{2n-1}\}$.

We then list the remaining elements of Dic_{4n} in the following order:

$\{b, a^n b, ab, a^{n+1}b, a^2 b, a^{n+2}b, \cdots, a^{n-1}b, a^{2n-1}b\}$.

Using the above indexing, the signless Laplacian matrix of $\mathscr{P}(\mathrm{Dic}_{4n})$ is of the following form:

$$Q(\mathscr{P}(\mathrm{Dic}_{4n})) = \left[\begin{array}{c|c} Q(\mathscr{P}(\mathbb{Z}_n))_{2n\times 2n} + \mathscr{M} & \mathscr{N}_{2n\times 2n} \\ \hline \mathscr{N}^T_{2n\times 2n} & \mathscr{K}_{2n\times 2n} \end{array}\right],$$

where

$$\mathscr{M} = \begin{bmatrix} 2n & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 2n & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathscr{N} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots\cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots\cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathscr{K} = \begin{bmatrix} 3 & 1 & 0 & 0 & \cdots\cdots\cdots\cdots & 0 \\ 1 & 3 & 0 & 0 & \cdots\cdots\cdots\cdots & 0 \\ 0 & 0 & 3 & 1 & \cdots\cdots\cdots\cdots & 0 \\ 0 & 0 & 1 & 3 & \cdots\cdots\cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots\cdots\cdots & 3 & 1 \\ 0 & 0 & 0 & 0 & \cdots\cdots\cdots & 1 & 3 \end{bmatrix}.$$

Using [31, Theorem 2.8.1] and $\sigma(\mathscr{K}) = \begin{pmatrix} 2 & 4 \\ n & n \end{pmatrix}$, we obtain

$$\begin{aligned} \lambda_n(Q(\mathscr{P}(\mathrm{Dic}_n))) &\leq \lambda_n\left(\begin{bmatrix} Q(\mathscr{P}(\mathbb{Z}_n)) + \mathscr{M} & 0 \\ 0 & \mathscr{K} \end{bmatrix}\right) + \lambda_n\left(\begin{bmatrix} 0 & \mathscr{N} \\ \mathscr{N}^T & 0 \end{bmatrix}\right) \\ &= \max\left\{\lambda_n\Big(Q(\mathscr{P}(\mathbb{Z}_n)) + \mathscr{M}\Big), 4\right\} + \lambda_n\left(\begin{bmatrix} 0 & \mathscr{N} \\ \mathscr{N}^T & 0 \end{bmatrix}\right) \\ &\leq \lambda_n(Q(\mathscr{P}(\mathbb{Z}_n)) + \lambda_n(\mathscr{M}) + \lambda_n\left(\begin{bmatrix} 0 & V \\ V^T & (4n-5)I \end{bmatrix}\right) \\ &\leq \lambda_n(Q(\mathscr{P}(\mathbb{Z}_n)) + 2n + \lambda_n\left(\begin{bmatrix} 0 & \mathscr{N} \\ \mathscr{N}^T & 0 \end{bmatrix}\right). \end{aligned} \tag{6}$$

Using *Schur complement* determinant formula [31, Theorem 2.7.1], the matrix $\begin{bmatrix} 0 & \mathscr{N} \\ \mathscr{N}^T & 0 \end{bmatrix}$ has characteristic polynomial as:

$$
\begin{aligned}
\Theta(x) &= \det\left(\begin{bmatrix} xI & -\mathscr{N} \\ -\mathscr{N}^T & xI \end{bmatrix}\right) \\
&= \det(xI) \times \det\left(xI - \frac{\mathscr{N}\mathscr{N}^T}{x}\right) \\
&= \det(xI) \times \det\left(xI - \frac{1}{x}\begin{pmatrix} n\, n\, 0\, 0 \cdots\cdots 0 \\ n\, n\, 0\, 0 \cdots\cdots 0 \\ 0\, 0\, 0\, 0 \cdots\cdots 0 \\ 0\, 0\, 0\, 0 \cdots\cdots 0 \\ 0\, 0\, 0\, 0 \cdots\cdots 0 \\ 0\, 0\, 0\, 0 \cdots\cdots 0 \\ 0\, 0\, 0\, 0 \cdots\cdots 0 \end{pmatrix}\right) \\
&= \det(xI) \times \det\begin{pmatrix} x - \frac{n}{x} & -\frac{n}{x} & 0\, 0 \cdots\cdots 0 \\ -\frac{n}{x} & x - \frac{n}{x} & 0\, 0 \cdots\cdots 0 \\ 0 & 0 & x\, 0 \cdots\cdots 0 \\ 0 & 0 & 0\, x \cdots\cdots 0 \\ 0 & 0 & 0\, 0 \cdots\cdots 0 \\ 0 & 0 & 0\, 0 \cdots\cdots 0 \\ 0 & 0 & 0\, 0 \cdots\cdots x \end{pmatrix} \\
&= x^n \times x^{n-2} \times \left(\left(x - \frac{n}{x}\right)^2 - \left(\frac{n}{x}\right)^2\right) \\
&= x^{2n-2}(x^2 - 2n).
\end{aligned}
\tag{7}
$$

Using 7, we find that the spectrum of $\begin{bmatrix} 0 & \mathscr{N} \\ \mathscr{N}^T & 0 \end{bmatrix}$ consists of 0 having multiplicity $2n-2$ and $\pm\sqrt{2n}$ having multiplicity 1. Hence, the largest eigenvalue of $\begin{bmatrix} 0 & \mathscr{N} \\ \mathscr{N}^T & 0 \end{bmatrix}$ is $\sqrt{2n}$. Therefore, using 6 and [32, Theorem 2.1], we obtain

$$
\begin{aligned}
\lambda_n(Q(\mathscr{P}(\mathrm{Dic}_n))) &\leq \lambda_n(Q(\mathscr{P}(\mathbb{Z}_n)) + 2n + \sqrt{2n} \\
&\leq \frac{3n - 6 + \sqrt{8\alpha + n^2 - 4n + 4}}{2} + 2n + \sqrt{2n}.
\end{aligned}
$$

It is quite trivial to note that the number of edges in $\mathscr{P}(\mathrm{Dic}_n)$ is greater than that in $\mathscr{P}(\mathbb{Z}_n)$. Hence, using *Interlacing Theorem* [33, Theorem 7.8.13], we find that

$$\lambda_n(Q(\mathscr{P}(\mathbb{Z}_n))) \leq \lambda_n(Q(\mathscr{P}(\mathrm{Dic}_n))).$$

Moreover, using [32, Theorem 2.2],

$$\lambda_n(Q(\mathscr{P}(\mathbb{Z}_n))) \geq \frac{2\alpha + n - 2 + \sqrt{n^2 - 4\alpha^2 + 4\alpha n - 4n + 4}}{2}.$$

Combining the two equations, we obtain,

$$\lambda_n(Q(\mathscr{P}(\mathrm{Dic}_n))) \geq \frac{2\alpha + n - 2 + \sqrt{n^2 - 4\alpha^2 + 4\alpha n - 4n + 4}}{2}.$$

Hence, the result follows.

4 Conclusion and future work

The *universal adjacency matrix* of a graph $\mathscr{G}$ denoted by $\mathscr{U}(\mathscr{G})$ is defined as $\mathscr{U}(\mathscr{G}) = \alpha A(\mathscr{G}) + \beta I + \gamma J + \delta \mathrm{Deg}(\mathscr{G})$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha \neq 0$, I is the identity matrix and J is the matrix all of whose entries are 1. It was introduced by Haemers and Omidi in [34] to unify the approach to study the spectral theories of the adjacency, Laplacian, and signless Laplacian spectra of graphs. Interested readers may determine the universal adjacency spectrum of power graph of various finite groups considered in this paper. Moreover, determining energy of graphs associated with algebraic structures has also attracted the attention of researchers over the last few decades, see for example [35] and [36]. The signless Laplacian spectrum of power graph of different finite non-commutative groups was examined in this research article. First, for elementary abelian groups whose orders are powers of a prime integer, we found the spectrum of the signless Laplacian matrix of the power graph. Next, we determined the signless Laplacian spectrum of the Mathieu group M_{11}, which is the smallest sporadic group. Additionally, we got the signless Laplacian eigenvalues of $\mathscr{P}(Q_{2^{k+2}})$, where the generalized quaternion group is denoted by $Q_{2^{k+2}}$. At last, we established constraints on the signless Laplacian spectral radius of $\mathscr{P}(\mathrm{Dic}_{4n})$, where Dic_{4n} is the dicyclic group. We encourage the readers to determine the adjacency, Laplacian, and signless Laplacian energy of power graph of various finite groups considered in this paper.

5 Declaration

5.1 Conflict of interest:

The author states that there is no conflict of interest.

5.2 Funding:

The author states that there is no funding applicable for this study.

5.3 Author's contributions:

S.B.-Writing-Original Draft, Writing-Review Editing, Methodology, Supervision, Validation, Conceptualization, Formal Analysis. All authors read and approved the final submitted version of this manuscript.

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5.5 Availability of data and materials:

All data that support the findings of this study are included within the article.

5.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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