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### Original Study

# On the traveling wave solutions to the complex nonlinear second-order modified unstable (1+1)-dimensional Schrödinger equation

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### Abstract

The present work applies the improved Bernoulli sub-equation function approach to efficiently generate traveling wave solutions for the (1+1)-dimensional second-order modified unstable complex nonlinear Schrödinger equation. The objective is to provide suitable particulars and semi-analytic solutions for the specified model. The proposed model is converted into a second-order nonlinear ordinary differential equation employing a traveling wave transformation. Several original wave solutions to the studied model are obtained, such as periodic, traveling waves, breather waves, and combined multiple forms. These solutions are expressed using a variety of complex exponential rational functions. Various solutions are shown using two- and three-dimensional graphics, using relevant arbitrary parameters to demonstrate their physical and dynamic consequences. Two-dimensional graphs are used to explain the impact of time evolution and its effectiveness in the structure of the solutions. The peculiar fact is that this is the first time a particular mentioned model is examined using this semi-analytic method performs as proof of the work's originality.

**Keywords:** The (1+1)-dimensional nonlinear Schrödinger equation, improved Bernoulli sub-equation function approach, traveling wave solution, time evolution.

**AMS 2020 codes:** 65Mxx; 35Dxx; 45Kxx; 35C07.

## 1 Introduction

It is important to keep in mind that semi-analytic solutions to non-linear partial differential equations fulfill a crucial role in enlightening an extensive selection of unusual and complicated features that are expressed in a range of applied scientific domains. Investigators from all around the globe are attempting to employ these procedures, as more partial differential equations with applications in physics, engineering, and biology have recently been solved using these methods. An additional contention posits that by augmenting the models with computations, these methods assure readers that they generate apt and accurate affirmations. The numerous evolution methods are applied for instance the modified tanh – function and the extended rational sinh – cosh function methods [1], the modified exponential function method [2], in innumerable scientific and mechanical fields, including physical engineering [3], health care biology [4], imagery and digital signal processing [5], and so on. The nonlinear partial differential equations (NLPDEs) are essential and indispensable, excluding the above quotations, Bernoulli sub-equation function methods, the modified exponential rational function method, and two variable ( $G'/G, 1/G$ ) – expansion techniques [6–9] have been operated.

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The complex dynamical model studied in this research is the modified unstable nonlinear (1+1)-dimensional Schrödinger model. The investigated equation has the subsequent form:

$$-\gamma \frac{\partial^2 u(x,t)}{\partial x \partial t} + \frac{\partial^2 u(x,t)}{\partial x \partial x} + i \frac{\partial u(x,t)}{\partial t} + 2\lambda u(x,t)|u(x,t)|^2 = 0, \quad i = \sqrt{-1}, \quad (1)$$

where  $\gamma, \lambda$  are non zero parameters to be determined later. As the studied model regulates some instabilities of modified wave-trains, it is a general equation of the family of nonlinear integrable systems in nonlinear Schrödinger system of equations. This equation also describes the temporal evolution of perturbations in mediums that are nominally stable or unstable. The term  $\gamma \frac{\partial^2 u}{\partial x \partial t}$  dominates the ill-posed-ness of this dynamic model. In order to provide a comprehensive response on the usefulness of the research, we may assert the following: The Schrödinger equation is a fundamental principle in the field of quantum mechanics. Quantum dynamics is the study of how the quantum state of a physical system evolves over time, offering a conceptual structure for comprehending the actions of particles at the atomic and subatomic scales. The solutions to the Schrödinger equation enable physicists to forecast the characteristics and actions of particles, including their energy levels, probability of locating them in certain places, and their interactions with other particles and fields.

Many experts have conducted investigations into the various interpretations of the dynamical model (1) employing more methodological approaches, which include modified extended mapping technique [10], the  $(G'/G)$ -expansion technique [11], the rational-expansion approach on Jacobi elliptic function [12], the modified version of the Kudryashov method and the expansion mod of the sine-Gordon approach [13], and so on.

Here, we try to give some specific semi-analytic methods that play a major role in solving mathematical, physical, and engineering models, such as: Laplace transformation method has been applied to the features of the Caputo-fractional derivatives [14], the improved version of Bernoulli combined with the methods related to the hyperbolic trigonometric functions [15–17], the new extended direct algebraic method employed [18], the sine-Gordon expansion method and its rational version have been applied [19], the development version of the ultra-spherical wavelet method utilized [20], employed a Lie symmetry analysis [21], the Hirota's bilinear method is considered [22], describing auxiliary equation approach [23], the bilinear method [24], the Hirota bilinear method and Bell polynomials [25,26], if you are seeking a unique form of solution that logically is localized in every possible direction, semi-analytic solutions and their interactions are an amazing place to start [27–30]. The utilized method is trustworthy, dependable, and more practical for solving different mathematical models. To provide evidence of the prior declaration, one can inspect how many researchers applied this method and other different performances, herein are some sources [31–38], diversified arrangements for multi-dimensional space Schrödinger equation became an attractive model to the researcher for investigation [39–42].

The subsequent structure of the keeping article is thus outlined as follows: The literature about the dynamical model is discussed in Section 1. The construction of the suggested approach is briefly presented in Section 2. The recommended technique is implemented on the provided model in Section 3, and the resulting solutions are visually shown. Discussing the outcomes and physical meaning of the illustrated figures are demonstrated in Section 4. Ultimately, the concluding views are expressed in the last part of Section 5.

## 2 Structure and explanation of the employed method

The improved Bernoulli sub-equation function method (IBSEFM) is presented briefly in the subsequent steps:

**Step 1.** We consider the general form of NLPDEs given as

$$\mathcal{P}(\mathcal{E}, \mathcal{E}_x, \mathcal{E}_t, \mathcal{E}_{xt}, \mathcal{E}_{xx}, \dots) = 0, \quad (2)$$

wherein  $\mathcal{E} = \mathcal{E}(x, t)$ . Setting

$$\mathcal{E}(x, t) = \mathcal{U}(\mathcal{L}), \quad \mathcal{L} = \delta_1 x - \delta_2 t, \quad (3)$$

here the constants  $\delta_i$  for  $i = 1, 2$  are arbitrary parameters. Upon substituting equation (3) into equation (2), the resulting expression is as follows:

$$\mathcal{N}(\mathcal{U}, \mathcal{U}', \mathcal{U}'', \dots) = 0, \quad (4)$$

where

$$\mathcal{U} = \mathcal{U}(\mathcal{L}), \mathcal{U}' = \frac{d\mathcal{U}}{d\mathcal{L}}, \mathcal{U}'' = \frac{d^2\mathcal{U}}{d\mathcal{L}^2}, \dots$$

**Step 2.** Let us consider that the solution of equation (4) might be expressed in the following form:

$$\mathcal{U}(\mathcal{L}) = \frac{\omega_0 + \omega_1 \mathcal{T} + \omega_2 \mathcal{T}^2 + \dots + \omega_n \mathcal{T}^n}{v_0 + v_1 \mathcal{T} + v_2 \mathcal{T}^2 + \dots + v_m \mathcal{T}^m}. \quad (5)$$

Here, must be both  $\omega_m, v_n$  are different from zero, and they should be determined afterwards. Also,  $\mathcal{T}(\mathcal{L})$  satisfies Bernoulli ODE:

$$\mathcal{T}' = \alpha \mathcal{T} + \beta \mathcal{T}^{\mathcal{M}}, \alpha \neq 0, \beta \neq 0, \mathcal{M} \in \mathbb{R} - \{0, 1\}. \quad (6)$$

By combining equation (5) with equation (6) and inserting in equation (4), the following result is obtained:

$$\Phi(\mathcal{T}(\mathcal{L})) = \varepsilon_s \mathcal{T}(\mathcal{L})^s + \dots + \varepsilon_1 \mathcal{T}(\mathcal{L}) + \varepsilon_0 = 0. \quad (7)$$

Utilizing the concepts of balance, one can come up with a formula for  $n, m$ , and  $M$  by comparing the highest-order derivatives with the terms of the highest degree in equation (4).

**Step 3.** When the coefficients of  $\Phi(\mathcal{T}(\mathcal{L}))$  were set to zero, the following algebraic system of equations was produced:

$$\varepsilon_i = 0, i = 0, \dots, s. \quad (8)$$

The values of  $\alpha, \beta, v_0, \dots, v_n, \omega_0, \dots, \omega_m$ , and so solutions of (4) are discovered if one knows how to solve equation (6). These values are obtained by solving the system (8).

**Step 4.** The commanded solutions for equation (6) have the following manners:

$$\begin{aligned} \mathcal{T}(\mathcal{K}) &= \left[ -\frac{\beta}{\alpha} + \frac{\mathcal{C}}{e^{\alpha(\mathcal{M}-1)\mathcal{K}}} \right]^{\frac{1}{1-\mathcal{M}}}, \alpha \neq \beta, \\ \mathcal{T}(\mathcal{K}) &= \left[ \frac{(\mathcal{C}-1) + (\mathcal{C}+1) \tanh\left(\frac{\alpha(1-\mathcal{M})\mathcal{K}}{2}\right)}{1 - \tanh\left(\frac{\alpha(1-\mathcal{M})\mathcal{K}}{2}\right)} \right]^{\frac{1}{1-\mathcal{M}}}, \alpha = \beta, \mathcal{C} \in \mathbb{R}. \end{aligned} \quad (9)$$

### 3 Outcomes and implications of the specified method

Consider equation (1) has a solution in the form of

$$\mathcal{E}(\eta) = e^{i\theta} V(\eta), \theta = \beta_1 x - \beta_2 t, \eta = \delta_1 x - \delta_2 t. \quad (10)$$

The following produces are straight away obtained by plugging wave transformation formula (10) into equation (1):

$$\begin{aligned} &e^{i(\beta_1 x - \beta_2 t)} (2\lambda V^3 + \delta_1 (\gamma \delta_2 + \delta_1) V'' + \beta_2 (V + i\gamma \delta_1 V') - \beta_1^2 V) \\ &+ e^{i(\beta_1 x - \beta_2 t)} (\beta_1 (-\beta_2 \gamma V + i(\gamma \delta_2 + 2\delta_1) V') - i\delta_2 V') = 0. \end{aligned} \quad (11)$$

After reestablishing equation (11), it may be interpreted by spreading the real and complex components into two formulations:

$$2\lambda V^3 + \gamma\delta_1\delta_2 V'' + \delta_1^2 V'' - \beta_1\beta_2\gamma V - \beta_1^2 V + \beta_2 V = 0, \quad (12)$$

and

$$(\delta_1(\beta_2\gamma + 2\beta_1) + \delta_2(\beta_1\gamma - 1))V' = 0, \quad (13)$$

from equation (13) one achieves:

$$\delta_1(\beta_2\gamma + 2\beta_1) + \delta_2(\beta_1\gamma - 1) = 0, \quad (14)$$

the equation (14) is a selected equation to be added to the obtained systems of algebraic equations in all cases.

Considering the balancing principles connecting the highest order and a greater degree of the nonlinear component in equation (12) we can obtain an essential connection between  $n, m$  and  $M$  which is as follows:

$$n = M + m - 1. \quad (15)$$

For the recommendation of the values of positive integers in equation (15), we have the following cases:

**Case 1.** In equation (15), if  $m = 1$  and  $M = 3$ , then  $n = 3$ , so equation (5) and equation (6) are taking the following forms:

$$V(\eta) = \frac{\omega_0 + \omega_1 \mathcal{T} + \omega_2 \mathcal{T}^2 + \omega_3 \mathcal{T}^3}{v_0 + v_1 \mathcal{T}}, \quad (16)$$

and

$$\mathcal{T}' = \alpha \mathcal{T} + \beta \mathcal{T}^3, \quad \alpha \neq 0, \beta \neq 0, \quad (17)$$

hence,

$$\begin{aligned} V' &= (\alpha \mathcal{T} + \beta \mathcal{T}^3) \left( \frac{\omega_1 + 2\omega_2 \mathcal{T} + 3\omega_3 \mathcal{T}^2}{v_0 + v_1 \mathcal{T}} - \frac{v_1 (\omega_0 + \omega_1 \mathcal{T} + \omega_2 \mathcal{T}^2 + \omega_3 \mathcal{T}^3)}{(v_0 + v_1 \mathcal{T})^2} \right) \\ &= \frac{\Psi(\mathcal{T})}{\Omega(\mathcal{T})}, \end{aligned} \quad (18)$$

notice that there should be at least  $\omega_3 \neq 0, v_1 \neq 0$ , one can reach this:

$$V'' = \frac{\Omega(\mathcal{T})\Psi'(\mathcal{T}) - \Psi(\mathcal{T})\Omega'(\mathcal{T})}{(\Omega(\mathcal{T}))^2}. \quad (19)$$

By lodging equations (16)-(19) into equation (12), we acquire the following algebraic system of equations by correlating to zero all the coefficients of the same powers of  $\mathcal{T}^i$  for  $i = 0, 1, 2, \dots$ .

$$\mathcal{C}_i = 0, \text{ where } \mathcal{C}_i \text{ is the coefficient of } \mathcal{T}^i, \text{ for any } i = 0, \dots, 10. \quad (20)$$

The following sub-cases are constructed from the solving of the system (20), where  $\alpha \neq \beta$ .

**Case 1.1.** The following is a set of the obtained parameters:

$$\delta_2 = -\frac{\omega_3^2 \lambda}{4v_1^2 \gamma \beta^2 \delta_1} - \frac{\delta_1}{\gamma}, \beta_2 = \frac{\frac{\sqrt{2}(\omega_3^2 \lambda - 4v_1^2 \beta^2 \delta_1^2) \sqrt{\omega_3^2 \gamma^2 \delta_1^2 \lambda (\omega_3^2 \alpha^2 \gamma^2 \lambda - 2v_1^2 \beta^2)}}{\omega_3^2 v_1^2 \beta^2 \delta_1^2 \lambda} - 8\gamma}{4\gamma^3},$$

$$\omega_0 = 0, \omega_1 = \frac{\omega_3 \alpha}{2\beta}, \omega_2 = 0, v_0 = 0, \beta_1 = \frac{\frac{\sqrt{2} \sqrt{\omega_3^2 \gamma^2 \delta_1^2 \lambda (\omega_3^2 \alpha^2 \gamma^2 \lambda - 2v_1^2 \beta^2)}}{\omega_3^2 \lambda} + \gamma}{\gamma^2}.$$
(21)

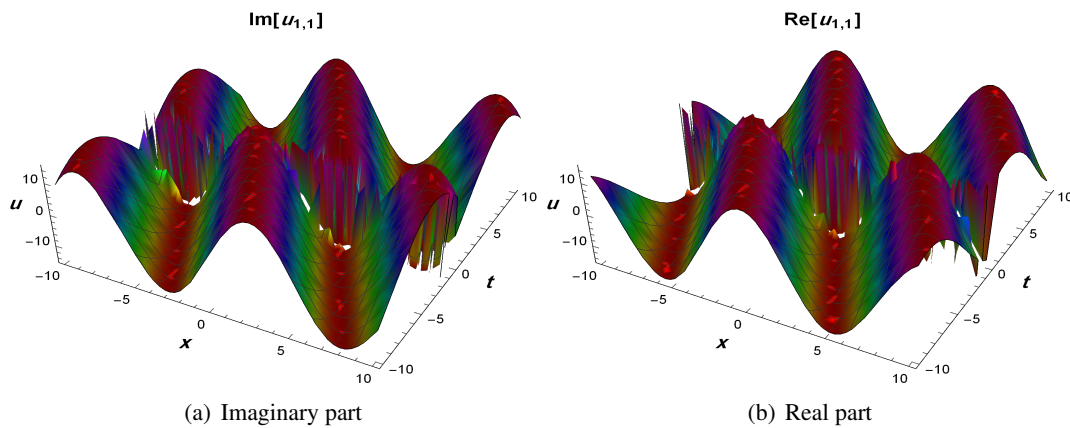
With the cited values of the parameters in equation (21), the following solution is attained where  $\mathcal{C} = 1$ :

$$u_{1,1}(x,t) = \frac{\omega_3}{2v_1\beta} \left( \frac{2\alpha\beta}{\alpha \left( \frac{\alpha \left( -\frac{\omega_3^2 \lambda t}{v_1^2 \beta^2} - 4\delta_1^2(t+\gamma x) \right)}{2\gamma\delta_1} \right) - \beta} + \alpha \right) e^{i \left( \frac{\frac{\sqrt{2}\mathcal{B} (4v_1^2 \beta^2 \delta_1^2(t+\gamma x) - \omega_3^2 \lambda t)}{\omega_3^2 v_1^2 \beta^2 \delta_1^2 \lambda} + 4\gamma(2t+\gamma x)}{4\gamma^3} \right)},$$
(22)

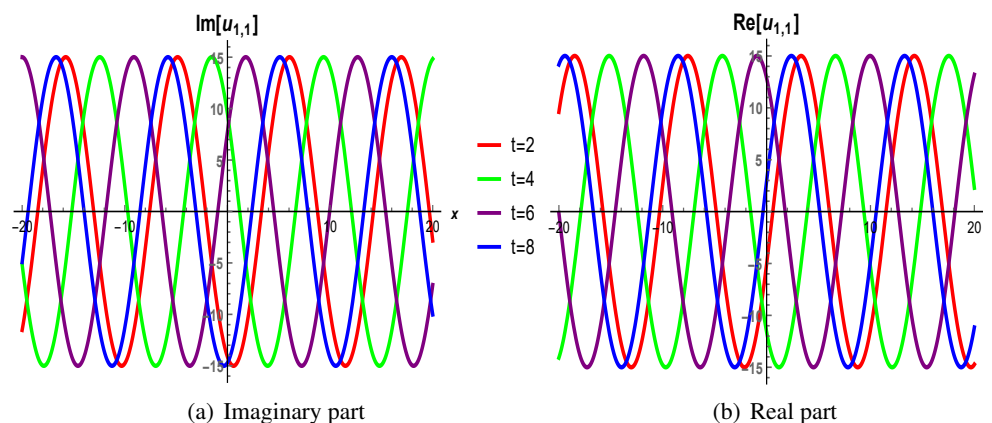
where

$$\mathcal{B} = \sqrt{\omega_3^2 \gamma^2 \delta_1^2 \lambda (\omega_3^2 \alpha^2 \gamma^2 \lambda - 2v_1^2 \beta^2)}.$$

Profile of the obtained solution in equation (22) where  $\gamma = \frac{1}{2}, \beta = \frac{1}{4}, \alpha = -\frac{3}{2}, \lambda = -\frac{2}{3}, v_1 = \frac{1}{3}, \omega_3 = \frac{5}{3}, \delta_1 = \frac{2}{3}$ , and  $-10 \leq x \leq 10, -10 \leq t \leq 10$  are graphed below



**Fig. 1** Three-dimensional figures of (22).



**Fig. 2** Two-dimensional time evolution graphs of equation (22) where values of  $t$  are presented in the legend.

**Case 1.2.** Then following coefficients are obtained:

$$\begin{aligned}\beta_2 &= \frac{-2\delta_1 \left( \sqrt{\delta_1 (2\alpha^2 \gamma^2 \delta_1 (\gamma\delta_2 + \delta_1) + 1)} + \sqrt{\gamma\delta_2 + \delta_1} \right) - \gamma\delta_2 \sqrt{\delta_1 (2\alpha^2 \gamma^2 \delta_1 (\gamma\delta_2 + \delta_1) + 1)}}{\gamma^2 \delta_1 \sqrt{\gamma\delta_2 + \delta_1}}, \\ \beta &= \frac{\omega_3 \sqrt{\lambda}}{2\sqrt{-v_1^2 \delta_1 (\gamma\delta_2 + \delta_1)}}, \omega_0 = \frac{\alpha v_0 \sqrt{-v_1^2 \delta_1 (\gamma\delta_2 + \delta_1)}}{v_1 \sqrt{\lambda}}, \omega_1 = \frac{\alpha \sqrt{-v_1^2 \delta_1 (\gamma\delta_2 + \delta_1)}}{\sqrt{\lambda}}, \\ \beta_1 &= \frac{\frac{\sqrt{\delta_1 (2\alpha^2 \gamma^2 \delta_1 (\gamma\delta_2 + \delta_1) + 1)}}{\sqrt{\gamma\delta_2 + \delta_1}} + 1}{\gamma}, \omega_2 = \frac{\omega_3 v_0}{v_1},\end{aligned}\quad (23)$$

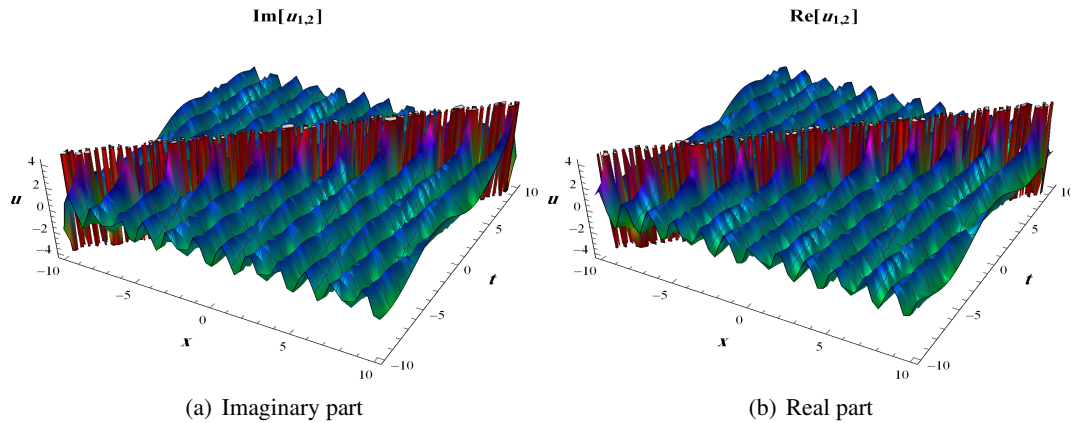
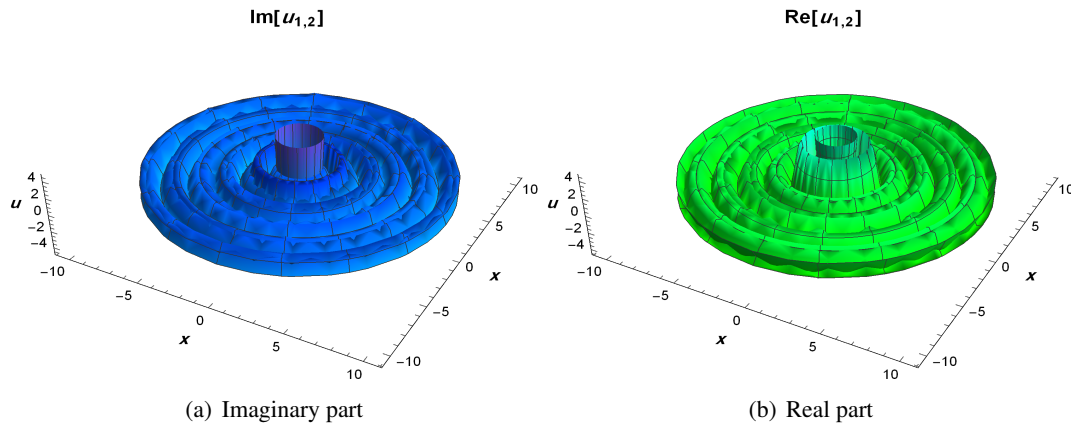
with the referenced values in equation (23), the following solution has been acquired:

$$u_{1,2}(x,t) = -\frac{\alpha \left( 2\mathcal{C} \alpha v_1^2 \delta_1 (\gamma\delta_2 + \delta_1) - \omega_3 \sqrt{\lambda} \sqrt{-v_1^2 \delta_1 (\gamma\delta_2 + \delta_1)} e^{2\alpha(\delta_1 x - \delta_2 t)} \right) e^{i\mathcal{B}}}{2\mathcal{C} \alpha v_1 \sqrt{\lambda} \sqrt{-v_1^2 \delta_1 (\gamma\delta_2 + \delta_1)} - \omega_3 v_1 \lambda e^{2\alpha(\delta_1 x - \delta_2 t)}} \quad (24)$$

where

$$\mathcal{B} = \frac{\left( \gamma\delta_2 t \sqrt{\delta_1 (2\alpha^2 \gamma^2 \delta_1 (\gamma\delta_2 + \delta_1) + 1)} + \delta_1 \left( \sqrt{\delta_1 (2\alpha^2 \gamma^2 \delta_1 (\gamma\delta_2 + \delta_1) + 1)} + \sqrt{\gamma\delta_2 + \delta_1} \right) (2t + \gamma x) \right)}{\gamma^2 \delta_1 \sqrt{\gamma\delta_2 + \delta_1}}.$$

Profile of the obtained solution in equation (24) where  $\alpha = \frac{4}{5}$ ,  $\delta_1 = \frac{2}{3}$ ,  $\delta_2 = \frac{3}{4}$ ,  $\mathcal{C} = \frac{4}{3}$ ,  $\lambda = -\frac{1}{3}$ ,  $v_1 = \frac{3}{5}$ ,  $\omega_3 = \frac{3}{2}$ ,  $\gamma = \frac{1}{2}$  and  $-10 \leq x \leq 10$ ,  $-10 \leq t \leq 10$  are graphed in the following:

**Fig. 3** Three-dimensional figures of equation (24).**Fig. 4** Three-dimensional revolving plot of equation (24).

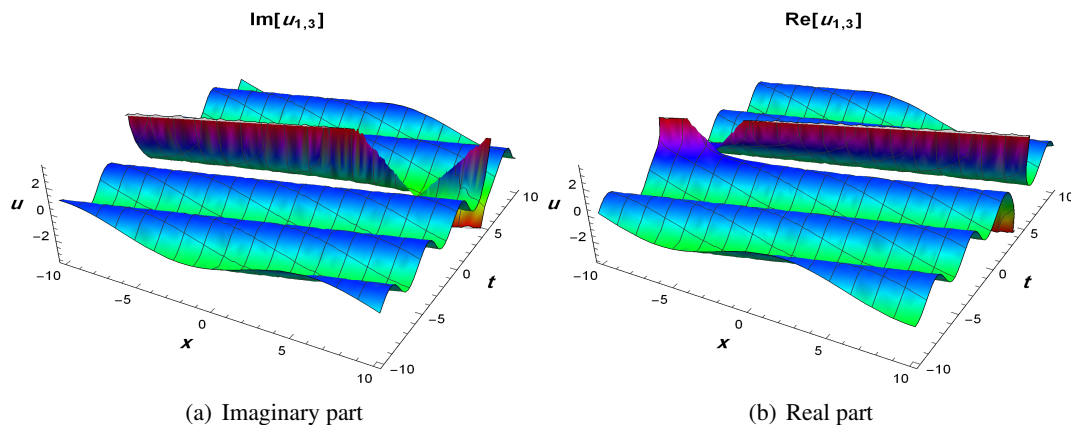
**Case 1.3.** We will get the following coefficients:

$$\begin{aligned}\omega_0 &= \frac{iv_0\sqrt{-\beta^2(\beta_2(\beta_1\gamma-1)+\beta_1^2)}}{\sqrt{2}\beta\sqrt{\lambda}}, \omega_2 = \frac{i\sqrt{2}\omega_0\sqrt{-\beta^2(\beta_2(\beta_1\gamma-1)+\beta_1^2)}}{\alpha\sqrt{\lambda}}, \\ v_1 &= -\frac{i\omega_3\alpha\sqrt{\lambda}}{\sqrt{2}\sqrt{-\beta^2(\beta_2(\beta_1\gamma-1)+\beta_1^2)}}, \delta_1 = -\frac{i\sqrt{\beta_1\gamma-1}\sqrt{\beta_2(\beta_1\gamma-1)+\beta_1^2}}{\sqrt{2}\sqrt{-\alpha^2(\gamma(\beta_2\gamma+\beta_1)+1)}}, \\ \omega_1 &= \frac{\omega_3\alpha}{2\beta}, \delta_2 = \frac{i(\beta_2\gamma+2\beta_1)\sqrt{\beta_2(\beta_1\gamma-1)+\beta_1^2}}{\sqrt{2}\beta_1\gamma-2\sqrt{-\alpha^2(\gamma(\beta_2\gamma+\beta_1)+1)}},\end{aligned}\quad (25)$$

the considered parameters in equation (25) generate the following solution

$$u_{1,3}(x,t) = \frac{i\sqrt{-\beta^2(\beta_2(\beta_1\gamma-1)+\beta_1^2)}\left(\frac{1}{\beta} - \frac{2}{\beta - \mathcal{C}\alpha e^{\frac{i\sqrt{2}\alpha\sqrt{\beta_2(\beta_1\gamma-1)+\beta_1^2}(\beta_2\gamma+\beta_1(2t+\gamma x)-x)}}{\sqrt{\beta_1\gamma-1}\sqrt{-\alpha^2(\gamma(\beta_2\gamma+\beta_1)+1)}}}\right)e^{i(\beta_1x-\beta_2t)}}{\sqrt{2}\sqrt{\lambda}}. \quad (26)$$

Profile of the obtained solution in equation (26) where  $\mathcal{C} = \frac{1}{4}, \beta = \frac{2}{3}, \beta_1 = \frac{2}{5}, \beta_2 = \frac{4}{3}, \gamma = \frac{3}{4}, \lambda = \frac{2}{5}, \alpha = \frac{5}{6}$ , and  $-10 \leq x \leq 10, -10 \leq t \leq 10$  are plotted in the following:



**Fig. 5** Three-dimensional figures of equation (26).

**Case 2.** In equation (15), if  $m = 1$  and  $M = 4$ , then  $n = 4$ , so equations (5) and (6) appear in the following forms:

$$V(\eta) = \frac{\omega_0 + \omega_1 \mathcal{T} + \omega_2 \mathcal{T}^2 + \omega_3 \mathcal{T}^3 + \omega_4 \mathcal{T}^4}{v_0 + v_1 \mathcal{T}}, \quad (27)$$

and

$$\mathcal{T}' = \alpha \mathcal{T} + \beta \mathcal{T}^4, \quad \alpha \neq 0, \beta \neq 0, \quad (28)$$

thereafter,

$$V' = (\alpha \mathcal{T} + \beta \mathcal{T}^4) \left( \frac{\omega_1 + 2\omega_2 \mathcal{T} + 3\omega_3 \mathcal{T}^2 + 4\omega_4 \mathcal{T}^3}{v_0 + v_1 \mathcal{T}} - \frac{v_1 (\omega_0 + \omega_1 \mathcal{T} + \omega_2 \mathcal{T}^2 + \omega_3 \mathcal{T}^3 + \omega_4 \mathcal{T}^4)}{(v_0 + v_1 \mathcal{T})^2} \right) \quad (29)$$

$$= \frac{\Psi(\mathcal{T})}{\Omega(\mathcal{T})},$$

there should be  $\omega_4 \neq 0, v_1 \neq 0$ , one can find this:

$$V'' = \frac{\Omega(\mathcal{T})\Psi'(\mathcal{T}) - \Psi(\mathcal{T})\Omega'(\mathcal{T})}{(\Omega(\mathcal{T}))^2}. \quad (30)$$

By appointing equations (27)-(30) into equation (12), we acquire an algebraic system of equations involving coefficients of equation (12). Henceforth, we will try to equate the coefficients of the same powers of  $\mathcal{T}^i$  for  $i = 0, 1, 2, \dots$ , to zero, aiming to obtain the following algebraic system:

$$\mathcal{C}_i = 0, \text{ where } \mathcal{C}_i \text{ is the coefficient of } \mathcal{T}^i, \text{ for any } i = 0, \dots, 13. \quad (31)$$

The following sub-cases have been generated using computer software programs to solve (31), where  $\alpha \neq \beta$ .

**Case 2.1.** We can get the following parameters:

$$\beta_1 = \frac{-\sqrt{2}\sqrt{\delta_1(\gamma\delta_2 + \delta_1)(9\alpha^2\gamma^2\delta_1(\gamma\delta_2 + \delta_1) + 2)} + 2\gamma\delta_2 + 2\delta_1}{2\gamma(\gamma\delta_2 + \delta_1)}, \lambda = -\frac{9v_1^2\beta^2\delta_1(\gamma\delta_2 + \delta_1)}{\omega_4^2}, \quad (32)$$

$$\beta_2 = \frac{\frac{\sqrt{2}(\gamma\delta_2 + 2\delta_1)\sqrt{\delta_1(\gamma\delta_2 + \delta_1)(9\alpha^2\gamma^2\delta_1(\gamma\delta_2 + \delta_1) + 2)}}{\delta_1(\gamma\delta_2 + \delta_1)} - 4}{2\gamma^2}, \omega_0 = \frac{\omega_4\alpha v_0}{2v_1\beta}, \omega_1 = \frac{\omega_4\alpha}{2\beta}, \omega_2 = 0, \omega_3 = \frac{\omega_4 v_0}{v_1},$$

the declared values of parameters in equation (32) generate the following solution:

$$u_{2,1}(x,t) = \frac{\omega_4 \left( \frac{2\beta}{\mathcal{C} e^{-3\alpha(\delta_1 x - \delta_2 t)} - \frac{\beta}{\alpha}} + \alpha \right)}{2v_1\beta} e^{i \left( \frac{x(-\sqrt{2}\mathcal{B} + 2\gamma\delta_2 + 2\delta_1)}{2\gamma(\gamma\delta_2 + \delta_1)} - \frac{t \left( \frac{\sqrt{2}\mathcal{B}(\gamma\delta_2 + 2\delta_1)}{\delta_1(\gamma\delta_2 + \delta_1)} - 4 \right)}{2\gamma^2} \right)}, \quad (33)$$

where

$$\mathcal{B} = \sqrt{\delta_1(\gamma\delta_2 + \delta_1)(9\alpha^2\gamma^2\delta_1(\gamma\delta_2 + \delta_1) + 2)}.$$

Profile of the obtained solution in equation (33) where  $\gamma = \frac{3}{2}$ ,  $\beta = \frac{3}{4}$ ,  $\alpha = \frac{1}{2}$ ,  $v_1 = \frac{4}{5}$ ,  $\omega_4 = \frac{5}{3}$ ,  $\delta_1 = \frac{1}{2}$ ,  $\delta_2 = \frac{2}{5}$ ,  $\mathcal{C} = \frac{3}{4}$ , and  $-10 \leq x \leq 10$ ,  $-10 \leq t \leq 10$ , are figured in the following:

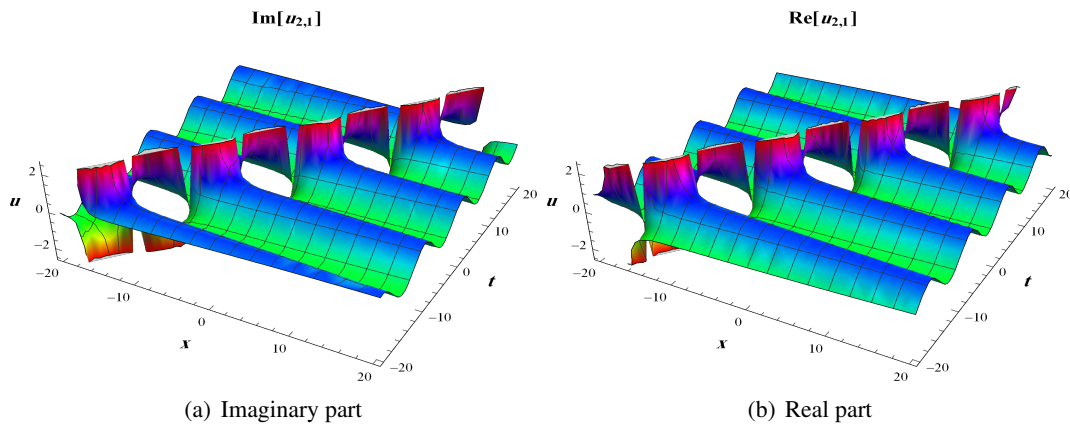


Fig. 6 Three-dimensional figures of equation (33).

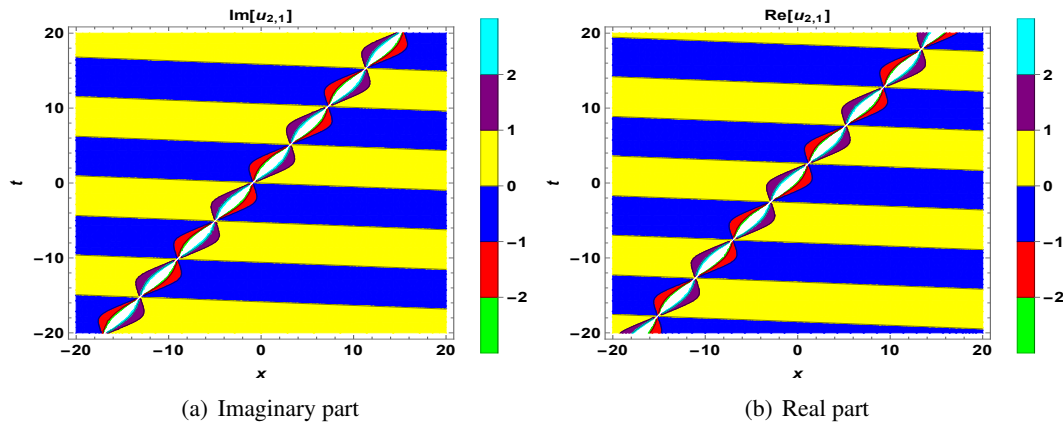


Fig. 7 Contour surface plots of equation (33).

**Case 2.2.** One will obtain the subsequent coefficients:

$$\beta_2 = -\frac{\frac{\sqrt{2}(\omega_4^2\lambda - 9v_1^2\beta^2\delta_1^2)\sqrt{\omega_4^2\gamma^2\delta_1^2\lambda(\omega_4^2\alpha^2\gamma^2\lambda - 2v_1^2\beta^2)}}{\omega_4^2v_1^2\beta^2\delta_1^2\lambda}}{6\gamma^3} + 12\gamma, \quad \omega_2 = 0, \quad \omega_0 = \frac{\omega_4\alpha v_0}{2v_1\beta}, \quad \omega_1 = \frac{\omega_4\alpha}{2\beta}, \quad (34)$$

$$\beta_1 = \frac{1}{\gamma} - \frac{3\sqrt{\omega_4^2\gamma^2\delta_1^2\lambda(\omega_4^2\alpha^2\gamma^2\lambda - 2v_1^2\beta^2)}}{\sqrt{2}\omega_4^2\gamma^2\lambda}, \quad \omega_3 = \frac{\omega_4 v_0}{v_1}, \quad \delta_2 = -\frac{\omega_4^2\lambda}{9v_1^2\gamma\beta^2\delta_1} - \frac{\delta_1}{\gamma},$$

with the specified values in equation (34) one gets the following solution

$$u_{2,2}(x,t) = \frac{\alpha \left( 2\mathcal{C} \alpha v_1^2 \delta_1 (\gamma \delta_2 + \delta_1) - \omega_3 \mathcal{Z} \sqrt{\lambda} e^{2\alpha(\delta_1 x - \delta_2 t)} \right) e^{\frac{i(\mathcal{B} \gamma \delta_2 t + \delta_1 (\mathcal{B} + \sqrt{\gamma \delta_2 + \delta_1})(2t + \gamma x))}{\gamma^2 \delta_1 \sqrt{\gamma \delta_2 + \delta_1}}}}{2\mathcal{C} \alpha v_1 \mathcal{Z} \sqrt{\lambda} - \omega_3 v_1 \lambda e^{2\alpha(\delta_1 x - \delta_2 t)}}, \quad (35)$$

where

$$\mathcal{B} = \sqrt{\delta_1 (2\alpha^2 \gamma^2 \delta_1 (\gamma \delta_2 + \delta_1) + 1)}, \quad \mathcal{Z} = \sqrt{-v_1^2 \delta_1 (\gamma \delta_2 + \delta_1)}.$$

**Case 2.3.** One gets the following coefficients:

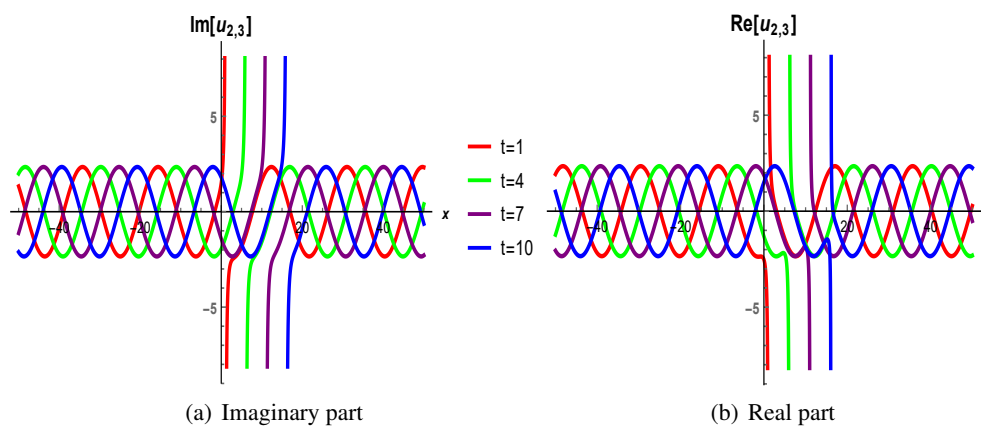
$$\beta_1 = \frac{2\gamma \delta_2 + 2\delta_1 - \sqrt{2}\mathcal{Q}}{2\gamma(\gamma \delta_2 + \delta_1)}, \quad \beta_2 = \frac{-4\delta_1^2 + 2\delta_1 (\sqrt{2}\mathcal{Q} - 2\gamma \delta_2) + \sqrt{2}\gamma \delta_2 \mathcal{Q}}{2\gamma^2 \delta_1 (\gamma \delta_2 + \delta_1)}, \quad \omega_0 = 0, \quad \omega_1 = \frac{\omega_4 \alpha}{2\beta}, \quad (36)$$

$$\omega_2 = 0, \quad \omega_3 = 0, \quad v_0 = 0, \quad \lambda = -\frac{9v_1^2 \beta^2 \delta_1 (\gamma \delta_2 + \delta_1)}{\omega_4^2}, \quad \mathcal{Q} = \sqrt{\delta_1 (\gamma \delta_2 + \delta_1) (9\alpha^2 \gamma^2 \delta_1 (\gamma \delta_2 + \delta_1) + 2)},$$

with the known parameters mentioned in equation (36) we will obtain the following solution:

$$u_{2,3}(x,t) = \frac{\omega_4 \alpha (\mathcal{C} \alpha + \beta e^{3\alpha(\delta_1 x - \delta_2 t)}) e^{-\frac{i(\sqrt{2}\mathcal{Q} \gamma \delta_2 t + \delta_1 (\sqrt{2}\mathcal{Q} - 2\gamma \delta_2)(2t + \gamma x) - 2\delta_1^2(2t + \gamma x))}{2\gamma^2 \delta_1 (\gamma \delta_2 + \delta_1)}}}{2v_1 \beta (\mathcal{C} \alpha - \beta e^{3\alpha(\delta_1 x - \delta_2 t)})}. \quad (37)$$

Profiles of the obtained solution in equation (37) where  $\mathcal{C} = \frac{1}{4}$ ,  $\beta = \frac{2}{3}$ ,  $v_1 = \frac{2}{5}$ ,  $\delta_2 = \frac{4}{3}$ ,  $\gamma = \frac{3}{4}$ ,  $\delta_1 = \frac{4}{5}$ ,  $\alpha = \frac{5}{6}$ ,  $\omega_4 = \frac{3}{2}$ , and  $-10 \leq x \leq 10$ ,  $-10 \leq t \leq 10$ , are outlined in the following:



**Fig. 8** Two dimensional time-evolution graphs of equation (37) where  $t$  values are existed in the legend.

**Case 3.** In equation (15) suppose  $m = 2$  and  $M = 3$ , and  $n = 4$ , thereafter equations (5) and (6) are taking the following styles:

$$V(\eta) = \frac{\omega_0 + \omega_1 \mathcal{T} + \omega_2 \mathcal{T}^2 + \omega_3 \mathcal{T}^3 + \omega_4 \mathcal{T}^4}{v_0 + v_1 \mathcal{T} + v_2 \mathcal{T}^2}, \quad (38)$$

and

$$\mathcal{T}' = \alpha \mathcal{T} + \beta \mathcal{T}^3, \quad \alpha \neq 0, \beta \neq 0, \quad (39)$$

hence,

$$\begin{aligned} V' &= -\frac{(\omega_4 \mathcal{T}^4 + \omega_3 \mathcal{T}^3 + \omega_2 \mathcal{T}^2 + \omega_1 \mathcal{T} + \omega_0)(2v_2 \mathcal{T} + v_1) \mathcal{T}'}{(v_2 \mathcal{T}^2 + v_1 \mathcal{T} + v_0)^2} \\ &\quad + \frac{(4\omega_4 \mathcal{T}^3 + 3\omega_3 \mathcal{T}^2 + 2\omega_2 \mathcal{T} + \omega_1) \mathcal{T}'}{v_2 \mathcal{T}^2 + v_1 \mathcal{T} + v_0} \\ &= \frac{\Psi(\mathcal{T})}{\Omega(\mathcal{T})}, \end{aligned} \quad (40)$$

there should be  $\omega_4 \neq 0$ ,  $v_2 \neq 0$ , one can reach the following:

$$V'' = \frac{\Omega(\mathcal{T})\Psi'(\mathcal{T}) - \Psi(\mathcal{T})\Omega'(\mathcal{T})}{(\Omega(\mathcal{T}))^2}. \quad (41)$$

By restoring equations (38)-(41) into equation (12) and equating to zero all the coefficients of the same powers of  $\mathcal{T}^i$  for  $i = 0, 1, 2, \dots$ , we obtain an algebraic system of equations as follows:

$$\mathcal{C}_i = 0, \text{ where } \mathcal{C}_i \text{ is the coefficient of } \mathcal{T}^i, \text{ for any } i = 0, \dots, 13. \quad (42)$$

The following sub-cases have been attained using computer software packages to solve (42) when  $\alpha \neq \beta$ .

**Case 3.1.** We obtain the following parameters:

$$\begin{aligned} \beta_1 &= \frac{\sqrt{\delta_1(\gamma\delta_2 + \delta_1)(2\alpha^2\gamma^2\delta_1(\gamma\delta_2 + \delta_1) + 1)} + \gamma\delta_2 + \delta_1}{\gamma(\gamma\delta_2 + \delta_1)}, \omega_1 = \frac{\omega_4\alpha v_1}{2v_2\beta}, \\ \beta_2 &= -\frac{\frac{(\gamma\delta_2 + 2\delta_1)\sqrt{\delta_1(\gamma\delta_2 + \delta_1)(2\alpha^2\gamma^2\delta_1(\gamma\delta_2 + \delta_1) + 1)}}{\delta_1(\gamma\delta_2 + \delta_1)} + 2}{\gamma^2}, \omega_0 = \frac{\omega_4\alpha v_0}{2v_2\beta}, \omega_3 = \frac{\omega_4 v_1}{v_2}, \\ \omega_2 &= \frac{1}{2}\omega_4\left(\frac{\alpha}{\beta} + \frac{2v_0}{v_2}\right), \lambda = -\frac{4v_2^2\beta^2\delta_1(\gamma\delta_2 + \delta_1)}{\omega_4^2}. \end{aligned} \quad (43)$$

By utilizing the declared parameter values in equation (43), the subsequent solution is constructed:

$$u_{3,1}(x, t) = \frac{\left(\frac{\omega_4 v_1}{\mathcal{A}^{3/2} v_2} + \frac{\omega_4}{\mathcal{A}^2} + \frac{\omega_4\left(\frac{\alpha}{\beta} + \frac{2v_0}{v_2}\right)}{2\mathcal{A}} + \frac{\omega_4\alpha v_1}{2\sqrt{\mathcal{A}} v_2 \beta} + \frac{\omega_4\alpha v_0}{2v_2 \beta}\right) e^{i\left(\frac{\mathcal{B}t}{\gamma^2} + \frac{x\mathcal{D}}{\gamma(\gamma\delta_2 + \delta_1)}\right)}}{\frac{v_1}{\sqrt{\mathcal{A}}} + \frac{v_2}{\mathcal{A}} + v_0}, \quad (44)$$

where

$$\mathcal{B} = \frac{(\gamma\delta_2 + 2\delta_1)\sqrt{\delta_1(\gamma\delta_2 + \delta_1)(2\alpha^2\gamma^2\delta_1(\gamma\delta_2 + \delta_1) + 1)}}{\delta_1(\gamma\delta_2 + \delta_1)} + 2,$$

$$\mathcal{D} = \sqrt{\delta_1(\gamma\delta_2 + \delta_1)(2\alpha^2\gamma^2\delta_1(\gamma\delta_2 + \delta_1) + 1)} + \gamma\delta_2 + \delta_1, \quad \text{and} \quad \mathcal{A} = \mathcal{C}e^{-2\alpha(\delta_1 x - \delta_2 t)} - \frac{\beta}{\alpha}.$$

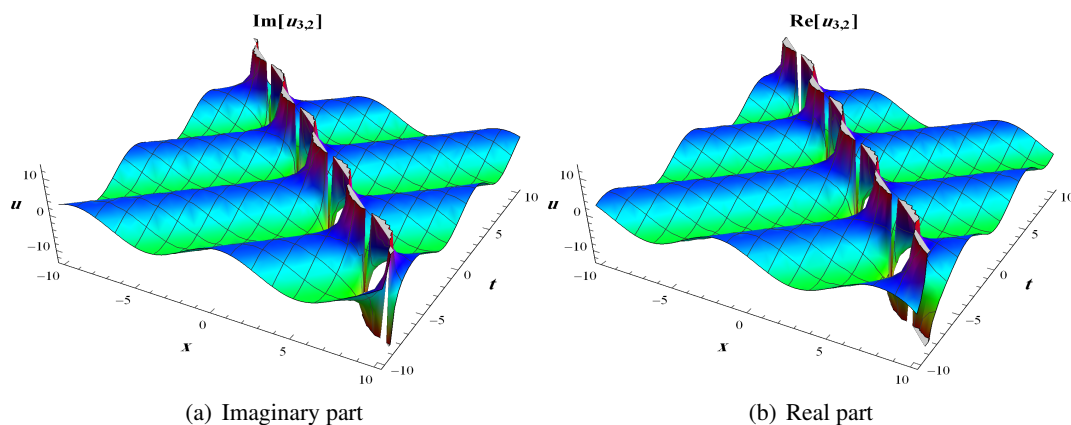
**Case 3.2.** We can obtain the following coefficients:

$$\begin{aligned}\omega_0 &= \frac{\omega_4 \alpha v_0}{2v_2 \beta}, \omega_1 = \frac{\omega_4 \alpha v_1}{2v_2 \beta}, \omega_2 = \frac{1}{2} \omega_4 \left( \frac{\alpha}{\beta} + \frac{2v_0}{v_2} \right), \gamma = \frac{\frac{\omega_4^2 \alpha^2 \lambda}{v_2^2 \beta^2} - 2\beta_1^2}{2\beta_1} + 2, \\ \delta_1 &= -\frac{\omega_4 \beta_1 \sqrt{\lambda} \sqrt{\omega_4^2 \alpha^2 \lambda - 2v_2^2 \beta_1^2 \beta^2}}{\sqrt{2\omega_4^4 \alpha^4 \lambda^2 - 4\omega_4^2 \alpha^2 v_2^2 (\beta_1^2 - 2\beta_2) \beta^2 \lambda + 8v_2^4 \beta_2^2 \beta^4}}, \omega_3 = \frac{\omega_4 v_1}{v_2}, \\ \delta_2 &= \frac{\omega_4 \beta_2 \sqrt{\lambda} (\omega_4^2 \alpha^2 \lambda + 2v_2^2 (\beta_1^2 + \beta_2) \beta^2)}{\sqrt{2\omega_4^2 \alpha^2 \lambda - 4v_2^2 \beta_1^2 \beta^2} \sqrt{\omega_4^4 \alpha^4 \lambda^2 - 2\omega_4^2 \alpha^2 v_2^2 (\beta_1^2 - 2\beta_2) \beta^2 \lambda + 4v_2^4 \beta_2^2 \beta^4}}.\end{aligned}\quad (45)$$

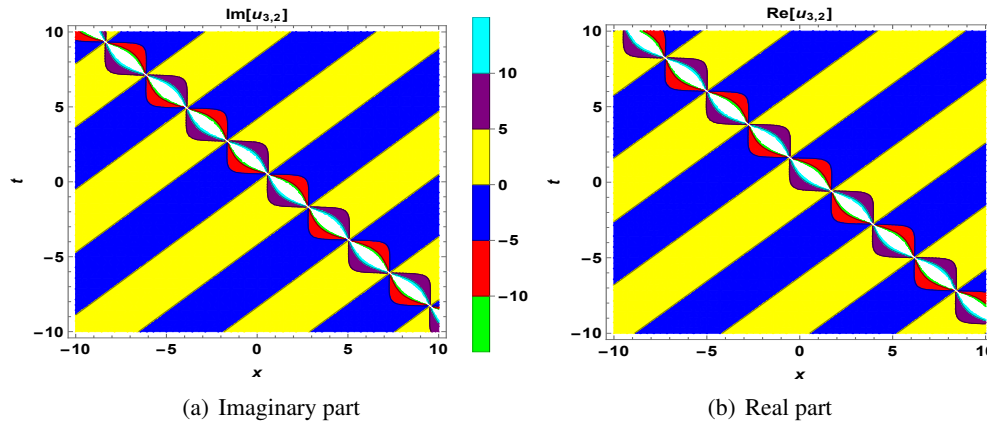
The provided quantities in equation (45) lead to the following solution:

$$u_{3,2}(x,t) = \frac{\omega_4 \alpha}{2v_2} \left( \frac{1}{\beta} - \frac{2}{\beta - \mathcal{C} \alpha e^{\frac{\sqrt{2}\omega_4 \alpha \sqrt{\lambda} (\omega_4^2 \alpha^2 \lambda (\beta_2 t + \beta_1 x) + 2v_2^2 \beta^2 (\beta_2 \beta_1^2 t + \beta_2^2 t + \beta_1^3 (-x)))}{\sqrt{\omega_4^2 \alpha^2 \lambda - 2v_2^2 \beta_1^2 \beta^2} \sqrt{\omega_4^4 \alpha^4 \lambda^2 - 2\omega_4^2 \alpha^2 v_2^2 (\beta_1^2 - 2\beta_2) \beta^2 \lambda + 4v_2^4 \beta_2^2 \beta^4}}}} \right) e^{i(\beta_1 x - \beta_2 t)}. \quad (46)$$

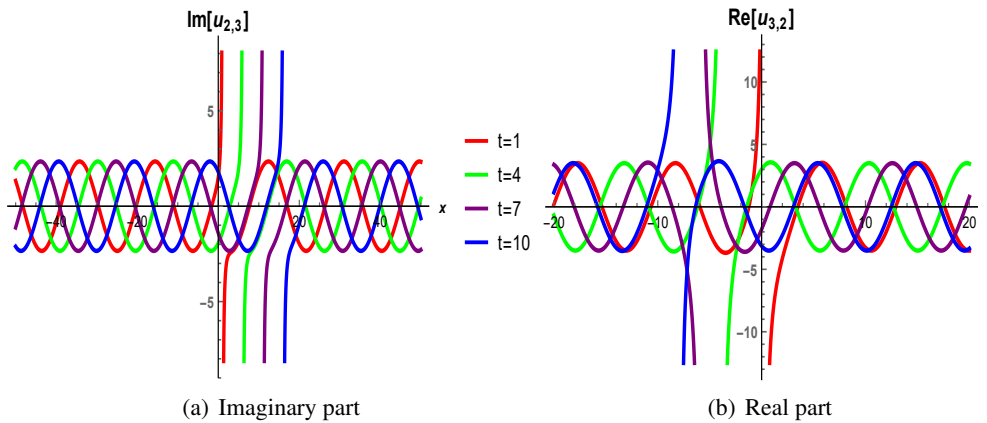
Profile of the obtained solution in equation (46) where  $\alpha = \frac{5}{3}, \beta = \frac{1}{3}, \beta_1 = \frac{2}{3}, \beta_2 = \frac{3}{4}, \lambda = -\frac{2}{3}, v_2 = \frac{5}{2}, \omega_4 = \frac{7}{2}, \mathcal{C} = \frac{3}{5}$ , and  $-10 \leq x \leq 10, -10 \leq t \leq 10$ , are figured out in the following:



**Fig. 9** Three-dimensional figures of equation (46).



**Fig. 10** Contour surface plots of equation (46).



**Fig. 11** Two-dimensional time-evolution graphs of equation (46) where  $-20 \leq x \leq 20$ , and  $t$  values are mentioned in the legend.

**Case 3.3.** The following coefficients are gained.

$$\beta_2 = \frac{-2\delta_1^2 + 2\delta_1(\mathcal{L} - \gamma\delta_2) + \gamma\delta_2\mathcal{L}}{\gamma^2\delta_1(\gamma\delta_2 + \delta_1)}, \beta_1 = \frac{\gamma\delta_2 + \delta_1 - \mathcal{L}}{\gamma(\gamma\delta_2 + \delta_1)}, \omega_0 = \frac{\omega_4\alpha^2}{4\beta^2}, \omega_1 = 0, \omega_2 = \frac{\omega_4\alpha}{\beta}, \omega_3 = 0, \quad (47)$$

$$v_0 = \frac{\alpha v_2}{2\beta}, v_1 = 0, \lambda = -\frac{4v_2^2\beta^2\delta_1(\gamma\delta_2 + \delta_1)}{\omega_4^2}, \mathcal{L} = \sqrt{\delta_1(\gamma\delta_2 + \delta_1)(2\alpha^2\gamma^2\delta_1(\gamma\delta_2 + \delta_1) + 1)},$$

the declared values of parameters in equation (47) give the following solution

$$u_{3,3}(x, t) = \frac{\omega_4 \left( \frac{2\beta}{\mathcal{L} e^{-2\alpha(\delta_1 x - \delta_2 t)} - \frac{\beta}{\alpha}} + \alpha \right)}{2v_2\beta} e^{i \left( \frac{x(-\mathcal{L} + \gamma\delta_2 + \delta_1)}{\gamma(\gamma\delta_2 + \delta_1)} - \frac{t(2\delta_1(\mathcal{L} - \gamma\delta_2) + \mathcal{L}\gamma\delta_2 - 2\delta_1^2)}{\gamma^2\delta_1(\gamma\delta_2 + \delta_1)} \right)}. \quad (48)$$

#### 4 Result and discussion

Here, we give a summary of the dynamic behaviors of the figures, in general, the three-dimensional plots: demonstrate how the solution becomes over both space and time, providing a complete view of the solution's

dynamics in three dimensions. Two-dimensional time evolution graphs concentrate on the material evolution at typical spatial points, emphasizing how the solution modifies over time. Three-dimensional revolving plots offer a dynamic perspective by allowing the visualization of the solution from different angles. While, contour figures provide a top-down view of the solution's surface, emphasizing regions with similar values and gradients. These visualizations are essential in order to comprehend the complex demeanor of solutions to differential equations, presenting perspicuity in their spatial and material dynamics. For the Multi-periodic wave see Figure 1, periodic time evolution in two dimension Figure 2, breather wave in three dimensions and its revolving plots are in Figures 3 and 4, the piece-wise periodic traveling-waves Figure 5, multi-hole traveling waves in three dimensions and contour surfaces are presented in Figures 6 and 7, the periodic time-evolution waves with one growth up point and one going down point Figures 8 and 11, multi-hole traveling waves, multiple breather waves in three dimensions, and contour surfaces are presented in Figures 9 and 10.

## 5 Conclusion

This paper provides comprehensive and accurate solutions for a nonlinear second-order modified unstable Schrödinger model in a (1+1)-dimensional space. This has been accomplished by using IBSEFM. This is the first implementation of the approach to the specified model. Our study's findings are innovative and significant compared to previous investigations since we introduced many unexplored analytically outcomes. The solutions created are complicated exponential rational functions and their combinations. In addition, the results are presented via both two- and three-dimensional graphical representations, as well as two-dimensional visualizations showing the effects of temporal changes. Contour plots and 3D rotating plots have been shown to enhance the understanding of the dynamic characteristics of the identified solutions. Moreover, each result has been validated by re-entering them into the corresponding equation using computational software tools. The efficacy and reliability of the method for examining the exact solutions of the given equation have been validated by the research findings, which will be used in future mathematical models. The results show that the unrestricted parameters of the gathered solutions have a significant influence on the waveform and its dynamics, potentially representing many complex and unknown aspects seen in different scientific disciplines. That being said, there have been reports of certain renewed, previously unreported solution types.

## 6 Declarations

### 6.1 Competing interests:

The author affirms that there is no conflict of interest related to the publishing of this work.

### 6.2 Funding:

Not applicable.

### 6.3 Author's contributions:

K.A.M.-Supervision, Conceptualization, Material Preparation, Design, Methodology, Formal Analysis, Validation, Software, Writing-Review Editing. The author read and approved the final submitted version of this manuscript.

### 6.4 Acknowledgement:

We appreciate the comments from reviewers that improved the statement of our paper.

### 6.5 Data availability statement:

Every data and detail that supports the outcomes of this investigation is available in the publication.

## 6.6 Using of AI tools:

The author affirms that Artificial Intelligence (AI) techniques were not used in creating this content.

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